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INTER-RELATIONS AMONG THE FOUR PRINCIPAL TYPES OF ORDER*

BY
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The four types of order whose inter-relations are considered in this paper may be called, for brevity, (1) serial order; (2) betweenness; (3) cyclic order; and (4) separation.

We first recapitulate the known sets of postulates which define each of these types as an abstract system, and recall the usual geometric interpretation of each type; we then develop the way in which each of these four types may be defined in terms of each of the other three. (For convenience of reference, the numbering of the postulates in earlier publications has been retained.)

1. **Serial order.** A system (K, R), where K is a class of elements A, B, C, $\cdots$, and $R(AB)$, or simply AB, is a dyadic relation, is called a "system of serial order" when and only when the following four postulates are satisfied.

In each of these postulates it is understood that distinct letters represent distinct elements of K. [The notation "$=0$" means "is false"; the "horseshoe", $\supset$, means "If . . . then"; the "wedge," $\vee$, means "or" (in the sense of "at least one"); and the "dot," . , means "and." Dots, singly or in groups, serve also as punctuation marks.]

POSTULATE D. $AA=0$. ("Irreflexiveness.")
POSTULATE I. $AB \vee BA$. ("Connexity.")
POSTULATE II. $AB.BA:=0$. ("Asymmetry.")
POSTULATE IV. $AB. \supset .XB \vee AX$. ("Inclusiveness.")

From properties II and IV, the following property is deducible as a theorem:

POSTULATE III. $AB.BC. \supset .AC$. ("Transitivity for distinct elements.")

(Proof. By IV, $AB. \supset .CB \vee AC$. But CB conflicts with BC, by II. Hence AC.)

Also, IV is a consequence of III and I.

(Proof. By I, $XB \vee BX$. But if BX, then by III, $AB.BX. \supset .AX$. Hence $XB \vee AX$.)

Hence the following alternative sets of postulates are equivalent, and

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either of them may be taken as a set of independent postulates for serial order*:

(1) D, I, II, III; (2) D, I, II, IV.

Geometrically speaking, this "abstract" system (K, R) of serial order may be represented by a "concrete" system in which K is the class of points on a directed straight line (that is, a straight line having a definite "sense" indicated by an arrow), and $R(AB)$ means "the point A precedes the point B" when the line is traversed in the direction of the arrow. Or briefly: serial order is the order of points on a *directed straight line*.

2. **Betweenness.** A system (K, R), where K is a class of elements A, B, $C, \cdots$, and $R(ABC)$, or simply ABC, is a triadic relation, is called a system of "betweenness" when and only when the following five postulates are satisfied. (In each of these postulates, beyond D, it is understood that distinct letters represent distinct elements of K.)

POSTULATE D. If ABC is true, then A, B, C are distinct.
POSTULATE B. $BAC \vee CAB \vee ABC \vee CBA \vee ACB \vee BCA$.
POSTULATE A. $ABC . \supset . CBA$.
POSTULATE C. $ABC . ACB : = 0$.
POSTULATE 9. $ABC . \supset . ABX \vee XBC$.

From these five properties, the following eight properties are deducible as theorems:

POSTULATE 1. $XAB . ABY . \supset . XAY$.
POSTULATE 2. $XAB . AYB . \supset . XAY$.
POSTULATE 3. $XAB . AYB . \supset . XYB$.

POSTULATE 4. $AXB . AYB . \supset . AXY \vee AYX$.
POSTULATE 5. $AXB . AYB . \supset . (AXY \vee YXB) . (AYX \vee XYB)$.

POSTULATE 6. $XAB . YAB . \supset . XYB \vee YXB$.
POSTULATE 7. $XAB . YAB . \supset . XYA \vee YXA$.
POSTULATE 8. $XAB . YAB . \supset . (XYA \vee YXB) . (YXA \vee XYB)$.

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* Sets of postulates for serial order date back at least as far as the early work of G. Peano and B. Russell. [Partial references will be found in E. V. Huntington's *A complete set of postulates for the theory of absolute continuous magnitude*, these Transactions, vol. 3 (1902), pp. 264–279; or, *Complete existential theory of the postulates for serial order*, Bulletin of the American Mathematical Society, vol. 23 (1917), pp. 276–280; or *The Continuum and Other Types of Serial Order*, second edition, Harvard University Press, 1917.] In Set 1, if Postulate III (the law of transitivity for distinct elements) is replaced by a law of transitivity for *all* elements, then Postulate II becomes redundant, being a consequence of this extended law of transitivity and Postulate D. In the present paper, Postulate IV is introduced for the sake of its analogy with Postulates 9 and 10, below.

The following twelve sets of postulates are equivalent, and any one of them may be taken as a set of independent postulates for betweenness*:

(1) A, B, C, D, 1, 2.
(2) A, B, C, D, 1, 5.
(3) A, B, C, D, 1, 6.
(4) A, B, C, D, 1, 7.
(5) A, B, C, D, 1, 8.
(6) A, B, C, D, 2, 4.
(7) A, B, C, D, 2, 5.
(8) A, B, C, D, 3, 5.
(9) A, B, C, D, 3, 4, 6.
(10) A, B, C, D, 3, 4, 7.
(11) A, B, C, D, 3, 4, 8.
(12) A, B, C, D, 9.

The most familiar concrete example of this abstract system of betweenness is the system (K, R) in which K is the class of points on an undirected straight line, and $R(ABC)$ means "B is between A and C" in the geometric sense. (This is the concrete example from which the abstract system takes its name.) In brief, "betweenness" is the order of points on an *undirected straight line.*

3. **Cyclic order.** A system (K, S), where K is a class of elements A, B, $C, \cdots$, and $S(ABC)$, or simply ABC, is a triadic relation, is called a system of "cyclic order" when and only when the following five postulates are satisfied. (In each of these postulates, beyond D, it is understood that distinct letters represent distinct elements of K.)

Postulate D. If ABC is true, then A, B, C are distinct.
Postulate B′. The system contains at least one true triad, say XYZ.
Postulate E. $ABC. \supset .BCA$.
Postulate C. $ABC.ACB: =0$.
Postulate 9. $ABC. \supset .ABX \vee XBC$.

From these five properties, the following three properties are deducible as theorems:

Postulate B. $ABC \vee BCA \vee CAB \vee CBA \vee BAC \vee ACB$.
Postulate 2. $XAB.AYB. \supset .XAY$.
Postulate 3. $XAB.AYB. \supset .XYB$.

The following four sets of postulates are equivalent, and any one of them

* Sets 1–11 were given by E. V. Huntington and J. R. Kline, *Sets of independent postulates for betweenness*, these Transactions, vol. 18 (1917), pp. 301–325. Set 12 was given by E. V. Huntington, *A new set of postulates for betweenness with proof of complete independence*, ibid., vol. 26 (1924), pp. 257–282. (This latter paper includes a discussion of certain peculiarities of Postulates 5 and 8, and an analysis of the significance of E. H. Moore's concept of complete independence.) W. E. Van de Walle, *On the complete independence of the postulates for betweenness*, ibid., vol. 26 (1924), pp. 249–256, shows that each of the Sets 1–10 is completely independent, and that Set 11 is not.

may be taken as a set of independent postulates for cyclic order:*

(1) B, C, D, E, 2. (3) B, C, D, E, 9.
(2) B, C, D, E, 3. (4) B′, C, D, E, 9.

Cyclic order is represented geometrically by a class K of points on a directed closed line, with $S(ABC)$ meaning "the arc running from A through B to C, in the direction of the arrow, is less than one complete circuit." In brief, cyclic order is the order of points on a *directed closed line*.

When necessary to distinguish between the two triadic relations, $R(ABC)$ for betweenness, and $S(ABC)$ for cyclic order, the prefixes R and S will be retained.

4. **Separation.** A system (K, R), where K is a class of elements A, B, $C, \cdots$, and $R(ABCD)$, or simply $ABCD$, is a tetradic relation, is called a system of "separation of pairs," or simply a system of "separation," when and only when the following six postulates are satisfied. (In each of these postulates, beyond D, it is understood that distinct letters represent distinct elements of K.)

POSTULATE D. If $ABCD$ is true, then A, B, C, D are distinct elements of K.

POSTULATE F′. The system contains at least one true tetrad, say $XYZW$.

POSTULATE G. $ABCD. \supset .BCDA$.

POSTULATE H. $ABCD.ABDC: =0$.

POSTULATE R′. At least one true tetrad is reversible; that is, if there is any true tetrad, then there is at least one true tetrad $ABCD$ such that $DCBA$ is also true.

POSTULATE 10. $ABCD. \supset .AXCD \vee ABCX$.

* See E. V. Huntington, *Sets of completely independent postulates for cyclic order*, Proceedings of the National Academy of Sciences [Washington], vol. 10 (1924), pp. 74–78. Another definition of cyclic order (in terms of a tetradic relation) is referred to in the next footnote.

Set (4) is new, and requires a proof of B from B′, E, C, 9, which proceeds as follows:

By B′, there is one true triad, say UVW. By 9, $UVW. \supset .AVW \vee UVA$.

If UVA, then by E, VAU, whence by 9, $VAU. \supset .WAU \vee VAW$, whence by E, $UWA \vee AWV$. Therefore $AVW \vee UWA \vee AWV$.

But if UWA, then by 9, $UWA. \supset .VWA \vee UWV$, where UWV conflicts with UVW by C, and $VWA. \supset .AVW$ by E. Therefore $AVW \vee AWV$, whence by E, $VWA \vee WVA$. We have thus proved

(a) $$UVW. \supset .VWA \vee WVA,$$

where UVW represents any true triad.

Case 1. If VWA, then by (a), $VWA. \supset .WAB \vee AWB$, whence by E, $WAB \vee WBA$. But if WAB, then by (a), $WAB. \supset .ABC \vee BAC$, and if WBA, then by (a), $WBA. \supset .BAC \vee ABC$.

Case 2. If WVA, then by (a), $WVA. \supset .VAB \vee AVB$, whence by E, $VAB \vee VBA$. But if VAB, then by (a), $VAB. \supset .ABC \vee BAC$, and if VBA, then by (a), $VBA. \supset .BAC \vee ABC$.

Hence in any case, $ABC \vee BAC$. Hence by E, $ABC.BCA.CAB: \vee :BAC.ACB.CBA$, from which B follows at once.

From these six properties the following eleven properties are deducible as theorems:

POSTULATE F. If A, B, C, D are distinct elements of K, then at least one of the twenty-four tetrads $ABCD$, $ABDC$, $\cdots$, $DCBA$ is true.

POSTULATE R. Every true tetrad is reversible; that is, $ABCD. \supset .DCBA$.

POSTULATE 11. $ABXC.ABCY. \supset .ABXY$.
POSTULATE 12. $ABXC.ABCY. \supset .BXCY$.
POSTULATE 13. $ABXC.ABCY. \supset .AXCY$.

POSTULATE 14. $ABCX.ABCY. \supset .ABXY \vee ABYX$.
POSTULATE 15. $ABCX.ABCY. \supset .ACXY \vee ACYX$.
POSTULATE 16. $ABCX.ABCY. \supset .BCXY \vee BCYX$.

POSTULATE 17. $ABCX.ABCY. \supset .(ABXY \vee ACYX).(ABYX \vee ACXY)$.
POSTULATE 18. $ABCX.ABCY. \supset .(ABXY \vee BCYX).(ABYX \vee BCXY)$.
POSTULATE 19. $ABCX.ABCY. \supset .(ACXY \vee BCYX).(ACYX \vee BCXY)$.

The following ten sets of postulates are equivalent, and any one of these may be taken as a set of independent postulates for separation*:

(1) D, F, G, H, R, 10.
(2) D, F, G, H, R, 12.
(3) D, F, G, H, R, 13.
(4) D, F, G, H, R, 11, 14.
(5) D, F, G, H, R, 11, 15.
(6) D, F, G, H, R, 11, 16.
(7) D, F, G, H, R, 11, 17.
(8) D, F, G, H, R, 11, 18.
(9) D, F, G, H, R, 11, 19.
(10) D, F′, G, H, R′, 10.

A geometrical example of a system of "separation" is the system (K, R) in which K is a class of points on an undirected closed line, and $R(ABCD)$ means "the pair of points A, C is separated by the pair B, D." In brief, "separation" treats of the order of points on an *undirected closed line*.

In the language of modern geometry, "separation" is the theory of order on the "projective line" (the so-called "straight line" of projective geometry). It may be suggested, in passing, that the concept of the "fourth harmonic point" determined by three given points on such a projective line is a concept which it would be interesting to define by the postulational method.

* See E. V. Huntington and K. E. Rosinger, *Postulates for separation of point-pairs (reversible order on a closed line)*, Proceedings of the American Academy of Arts and Sciences [Boston], vol. 67 (1932), pp. 61–145. On p. 70 of this paper the following corollary is established: *In every system which satisfies Postulates* D, F′, G, H, 10, *we have either* (R) *Every true tetrad is reversible*; *or else* (S) *Every true tetrad is non-reversible*; and on p. 63 it is noted that if we introduce

POSTULATE S'. At least one true tetrad is non-reversible,

then *Postulates* D, F′, G, H, 10, S′ *will define the theory of non-reversible order on a closed line* (just as Postulates D, F′, G, H, 10, R′ define the theory of reversible order on a closed line). This theory of non-reversible order on a closed line is essentially the same as the theory of cyclic order, expressed in terms of a tetradic instead of a triadic relation.

The inter-relations among these four types of order may be classified under four headings.

§§1.1–1.5

Under the first heading, we show that each of the other three types may be defined directly in terms of serial order; and also that separation may be defined directly in terms of each of the other three types. The details of the proofs require nothing more than a checking up of all the possible cases for each of the postulates involved, and will be left to the reader.

1.1. **Betweenness defined in terms of serial order.** In a given system of serial order, three elements A, B, C will stand in the "betweenness" relation ABC when AB and BC are true, and also when CB and BA are true, but not otherwise. That is, in the system of serial order we may define the relation of betweenness as follows:

$$ABC::=::AB.BC:\vee:CB.BA.$$

The triadic relation thus defined is readily shown to satisfy all the postulates A, B, C, D, 9 for betweenness.

1.2. **Cyclic order defined in terms of serial order.** Similarly, in a given system of serial order the relation of cyclic order may be defined as follows:

$$ABC::=::AB.BC:\vee:BC.CA:\vee:CA.AB.$$

The triadic relation thus defined satisfies all the postulates B, C, D, E, 9 for cyclic order.

1.3. **Separation defined in terms of serial order.** Again, in a given system of serial order, four elements A, B, C, D will stand in the "separation" relation, $ABCD$, under conditions expressed by the following definition:

$$\begin{aligned}ABCD::=::&AB.BC.CD:\vee:BC.CD.DA:\vee:CD.DA.AB:\vee\\ &:DA.AB.BC:\vee:DC.CB.BA:\vee:AD.DC.CB:\vee\\ &:BA.AD.DC:\vee:CB.BA.AD.\end{aligned}$$

The tetradic relation thus defined satisfies all the postulates D, F′, G, H, R′, 10 for separation.

1.4. **Separation defined in terms of betweenness.** Suppose now we have a given system of betweenness. The relation of separation may be defined in this system as follows:

$$ABCD::=::ABC.BCD:\vee:BCD.CDA:\vee:CDA.DAB:\vee:DAB.ABC.$$

1.5. **Separation defined in terms of cyclic order.** Again, if we have a given system of cyclic order, the relation of separation may be defined in that system as follows:

$$ABCD::=::ABC.CDA.\vee.ADC.CBA.$$

§§2.1–2.3

Under the second heading we consider definitions which are not absolute, but involve a reference to an arbitrarily selected element of the given system, say Z.

2.1. **Serial order defined in terms of cyclic order, with respect to Z.** In a given system of cyclic order, if we exclude any arbitrarily chosen element Z, the remaining elements may be arranged in serial order (with respect to Z) by the following definition:

$$AB:=:ZAB.$$

The element Z itself may then be brought into the series, if desired, by defining AZ as true and ZA as false.

2.2. **Betweenness, $R(ABC)$, defined in terms of cyclic order, with respect to Z.** In a given system of cyclic order, if we exclude any arbitrarily chosen element Z, we may define the betweenness relation $R(ABC)$ among the remaining elements as follows:

$$R(ABC)::=::ZAB.BCZ:\vee:ZCB.BAZ.$$

The element Z itself may then be brought into the betweenness system, if desired, by defining $R(ABZ)$ and $R(ZBC)$ as true and $R(AZC)$ as false.

2.3. **Betweenness defined in terms of separation, with respect to Z.** Suppose now the given system is a system of separation. Then if we exclude an arbitrary element Z, we may define the betweenness relation among the remaining elements as follows:

$$ABC:=:ZABC.$$

The element Z itself may then be brought into the betweenness system, if desired, by defining ABZ and ZBC as true and AZC as false.

§§3.1–3.2

Under the third heading, the definitions are also not absolute, but involve a reference to an arbitrarily chosen pair of elements, say U and V, in the given system.

3.1. **Serial order defined in terms of betweenness, with respect to U, V.** In a given betweenness system, let U, V be any two selected elements. Then all the elements may be arranged in serial order (with respect to U, V) by the following definitions (where, as usual, distinct letters denote distinct elements):

UV true; and VU false.

$AU:=:AUV.$ $\quad AV:=:AUV\vee UAV.$ $\quad UA:=:UAV\vee UVA.$

$VA:=:UVA.$

$AB::=::AUV.ABV:\vee:AUV.AVB:\vee:UAV.UAB:\vee:UVA.UAB.$

3.2. **Cyclic order, $S(ABC)$, defined in terms of betweenness, with respect to U, V.** In a given betweenness system, let U, V again be any two arbitrarily chosen elements. Then we may define the relation of cyclic order among the elements of this system (with respect to U, V) as follows*:

$$S(AUV).S(UVA).S(VAU)::=::AUV \vee UVA.$$
$$S(AVU).S(VUA).S(UAV)::=::UAV.$$
$$S(ABU).S(BUA).S(UAB)::$$
$$=::ABU.BUV:\vee:BUV.BUA:\vee:UAV.UAB:\vee:UVA.UAB.$$
$$S(AUB).S(UBA).S(BAU)::$$
$$=::AUV.AUB:\vee:UBV.UBA:\vee:UVB.UBA:\vee:UAB.AUV.$$
$$S(ABV).S(BVA).S(VAB)::$$
$$=::AUV.ABV:\vee:UAB.ABV:\vee:AVB.AVU:\vee:UVA.UAB.$$
$$S(AVB).S(VBA).S(BAV)::$$
$$=::AVB.UVB:\vee:UVB.UBA:\vee:BUV.BAV:\vee:UBA.BAV.$$
$$S(ABC).S(BCA).S(CAB)::$$
$$=::BCA.BCU.BUV:\vee:BCA.BUC.BUV:\vee:UCA.UBC.UBV:$$
$$\vee:UCA.UAB.UCV:\vee:UAB.UBC.UAV:\vee:AUB.ABC.AUV:$$
$$\vee:ABU.AUV.ABC:\vee:CUA.CAB.CUV:\vee:CAU.CUV.CAB:$$
$$\vee:UVB.VBC.BCA:\vee:UVA.VAB.ABC:\vee:UVC.VCA.CAB.$$

§§4.1–4.2

Under the fourth heading, the definitions involve reference to three arbitrary elements of the given system. In §4.1, one of the three reference elements, say Z, is distinguished from the other two, say U, V. In §4.2, the three reference elements, say U, V, W, are coordinate.

4.1. **Serial order defined in terms of separation, with respect to Z and U, V.** Given a separation system, in which two elements U and V are arbitrarily selected as reference elements. Then if a third arbitrary element Z is excluded from the system, all the remaining elements may be arranged in serial order (with respect to U, V) by the following definitions:

UV true; and VU false.

$$AU:=:ZAUV. \quad UA:=:ZUAV \vee ZUVA.$$
$$AV:=:ZAUV \vee ZUAV. \quad VA:=:ZUVA.$$
$$AB::=::ZAUV.ZABU:\vee:ZAUV.ZAUB:\vee:ZUAV.ZUAB:$$
$$\vee:ZUVA.ZUAB.$$

The element Z itself may then be brought into the series, if desired, by defining UZ, VZ, AZ as true and ZU, ZV, ZA as false.

* For assistance in formulating and verifying the definitions under §§3.2 and 4.2, I am indebted to Mr. B. Notcutt.

4.2. **Cyclic order defined in terms of separation, with respect to U, V, W.** Given a separation system, in which three elements, U, V, W, are arbitrarily selected as reference elements ("anchorage points"). The relation of cyclic order (with respect to U, V, W) may be thus defined in this system:

$UVW.VWU.WUV$ true; and $WVU.UWV.VUW$ false.

$AUV.UVA.VAU:=:AUVW \vee AWUV.$
$AVW.VWA.WAV:=:AVWU \vee AUVW.$
$AWU.WUA.UAW:=:AWUV \vee AVWU.$

$AVU.VUA.UAV:=:AVWU.$
$AWV.WVA.VAW:=:AWUV.$
$AUW.UWA.WAU:=:AUVW.$

$UAB.ABU.BUA:=:AVWU.ABVU: \vee :AVWU.AVBU:$
$\qquad \vee :AWUV.ABUV: \vee :AUVW.ABUV.$
$VAB.ABV.BVA:=:AWUV.ABWV: \vee :AWUV.AWBV:$
$\qquad \vee :AUVW.ABVW: \vee :AVWU.ABVW.$
$WAB.ABW.BWA:=:AUVW.ABUW: \vee :AUVW.AUBW:$
$\qquad \vee :AVWU.ABWU: \vee :AWUV.ABWU.$

$ABC::=::AUVW.ABVW.ABCW: \vee :BUVW.BCVW.BCAW:$
$\qquad \vee :CUVW.CAVW.CABW: \vee :AUVW.ABVW.ABWC:$
$\qquad \vee :BUVW.BCVW.BCWA: \vee :CUVW.CAVW.CAWB.$

Summary. These results may be summarized in the following table, in which Z, U, V, W denote arbitrarily selected elements of the given system.

	The relation of	may be defined within a given system of
1.1	Betweenness	serial order.
1.2	Cyclic order	serial order.
1.3	Separation	serial order.
1.4	Separation	betweenness.
1.5	Separation	cyclic order.
2.1	Serial order	cyclic order, with respect to Z.
2.2	Betweenness	cyclic order, with respect to Z.
2.3	Betweenness	separation, with respect to Z.
3.1	Serial order	betweenness, with respect to U, V.
3.2	Cyclic order	betweenness, with respect to U, V.
4.1	Serial order	separation, with respect to Z and U, V.
4.2	Cyclic order	separation, with respect to U, V, W.

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CONTRIBUTION À L'ÉTUDE DU SAUT D'UNE FONCTION DONNÉE PAR SON DÉVELOPPEMENT EN SÉRIE D'HERMITE OU DE LAGUERRE*

PAR
ERVAND KOGBETLIANTZ

Introduction

Dans l'intervalle infini $(-\infty, +\infty)$ on peut développer une fonction donnée $f(x)$ en série d'Hermite,

$$f(x) \sim \sum_{n=0}^{\infty} \frac{H_n(x)}{2^n n! \pi^{1/2}} \int_{-\infty}^{+\infty} e^{-u^2} H_n(u) f(u) du \qquad (-\infty < x < \infty), \tag{1}$$

où le $n$ième polynome d'Hermite $H_n(x)$ est défini par

$$e^{-x^2} H_n(x) = \frac{d^n(e^{-x^2})}{dx^n}.$$

De même, dans l'intervalle $(0, \infty)$ on a pour $\alpha > -1$† la série de Laguerre:

$$f(x) \sim \sum_{0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}(x) \int_0^{\infty} e^{-u} u^{\alpha} L_n^{(\alpha)}(u) f(u) du, \tag{2}$$

le polynome de Laguerre $L_n^{(\alpha)}(x)$ étant défini par

$$n! x^{\alpha} e^{-x} L_n^{(\alpha)}(x) = \frac{d^n}{dx^n} [x^{n+\alpha} e^{-x}].$$

La série d'Hermite (1) dérivée terme à terme par rapport à x donne une série procédant également suivant les polynomes d'Hermite car

$$H_n'(x) = -2nH_{n-1}(x).$$

Cette nouvelle série

$$-\sum_{n=0}^{\infty} \frac{H_n(x)}{2^n n! \pi^{1/2}} \int_{-\infty}^{+\infty} e^{-u^2} H_{n+1}(u) f(u) du \tag{3}$$

n'est autre chose que le développement formel en série d'Hermite de la dérivée $f'(x)$, si $f(x)$ en possède une. En effet, dans l'hypothèse que $f(x)$ et $f'(x)$, sommables (L) dans tout intervalle fini, vérifient à l'infini, pour $|x| \to \infty$,

* Presented to the Society, April 20, 1935; received by the editors September 8, 1933.

† Pour $\alpha \leqq -1$ les polynomes de Laguerre ne forment plus un système orthogonal dans $(0, \infty)$.

la condition d'être $O(e^{qx^2})$ avec $q<1$, on peut faire tendre A vers l'infini dans la relation

$$\int_{-A}^{A} e^{-u^2}f'(u)H_{n-1}(u)du = e^{-A^2}\,|\, f(u)H_{n-1}(u)\,|_{-A}^{A} - \int_{-A}^{A} e^{-u^2}H_n(u)f(u)du$$

donc

$$\int_{-\infty}^{\infty} f'(u)e^{-u^2}H_{n-1}(u)du = - \int_{-\infty}^{\infty} e^{-u^2}H_n(u)f(u)du$$

ce qui prouve notre assertion.

La série (3) diverge en tout point $x=x_0$ où $f(x)$ possède une discontinuité caractérisée par un saut *fini*

$$D(x_0) = f(x_0+0) - f(x_0-0),$$

les nombres $f(x_0 \pm 0)$ existant par hypothèse. La divergence de la série (3) pour $x=x_0$ est essentielle, c'est à dire sa somme partielle $f_n(x_0)$ tend vers ∞ avec n. Dans ces conditions la série (3) n'est sommable pour $x=x_0$ par aucun procédé de sommation régulier à coefficients positifs.

Néanmoins, $f_n(x_0)$ peut servir pour déterminer à partir de la série divergente (3) le saut $D(x_0)$ de $f(x)$. Ainsi M. Jacob* a prouvé que l'on a

$$D(x_0) = \frac{\pi}{2^{1/2}} \lim_{n=\infty} \frac{f_n(x_0)}{n^{1/2}} \tag{4}$$

sous des conditions très restrictives imposées à $f(x)$; M. Jacob suppose que $f(x)$ est à variation bornée dans $(-\infty, +\infty)$ et qu'elle vérifie en outre la condition d'existence des deux intégrales suivantes:

$$\int_{-\infty}^{-a} e^{-u^2/2} \left| \frac{df(u)}{u} \right| < G \text{ et } \int_{a}^{+\infty} e^{-u^2/2} \frac{|\, df(u)\,|}{u} < G. \tag{5}$$

Au §2 nous démontrons que le résultat (4) subsiste sous des conditions beaucoup plus larges, à savoir:

(I) $f(x)$ est sommable (L) dans tout intervalle fini,

(II) le produit $|x^{-1}f(x)|e^{-x^2/2}$ est intégrable à l'infini, c'est à dire les intégrales

$$\int_{-\infty}^{-a} e^{-u^2/2}\,|\, f(u)\,|\, \frac{du}{|\,u\,|} \quad \text{et} \quad \int_{a}^{\infty} e^{-u^2/2}\,|\, f(u)\,|\, \frac{du}{u}$$

existent, et

* Giornale dell'Istituto Italiano degli Attuari, vol. 2 (1931), pp. 100–106, 356–368.

(III) l'intégrale définie

$$\int_{-\epsilon}^{\epsilon} \left| f(x_0 + t) - f(x_0 + o \operatorname{sgn} t) \right| \frac{dt}{|t|} < G, \tag{6}$$

où ϵ est aussi petit qu'on veut, mais fixe, existe.

La condition (6) relative à l'allure de $f(x)$ au voisinage immédiat du point $x = x_0$ peut être omise si au lieu des sommes partielles $f_n(x_0)$ de la série (3) on considère leurs moyennes arithmétiques $f_n^{(\delta)}(x_0)$ d'ordre positif δ, définies pour tout $\delta > -1$ ainsi:

$$f_n^{(\delta)}(x_0) = \delta \sum_{m=0}^{n} \frac{n(n-1)\cdots(n-m+1)}{(n+\delta)(n+\delta-1)\cdots(n+\delta-m)} f_m(x_0).$$

On a en effet, le théorème suivant:

THÉORÈME I. *Si* $|f(x)|$ *et* $e^{-u^2/2}|u^{-2\delta-1}f(u)|$ *sont intégrables dans les intervalles* $|x| \leqq a_1$ *et* $a_2 \leqq |u| \leqq \infty$ *respectivement, les nombres positifs* a_1, a_2 *étant aussi grands qu'on veut mais fixes, on a pour tout* $\delta > 0$

$$\lim_{n=\infty} \frac{f_n^{(\delta)}(x_0)}{n^{1/2}} = \frac{\Gamma(\delta+1)}{(2\pi)^{1/2}\Gamma\left(\delta + \frac{3}{2}\right)} D(x_0) \tag{7}$$

et ce résultat subsiste aussi pour $\delta = 0$ *pourvu que* $f(x)$ *vérifie au voisinage du point* $x = x_0$ *la condition* (6), *c'est à dire pourvu que l'expression* $|u^{-1}\{f(x_0+u) - f(x_0+o \operatorname{sgn} u)\}|$ *soit intégrable dans l'intervalle* $(-\epsilon, \epsilon)$, ϵ *étant aussi petit qu'on veut, mais fixe.*

On constate ainsi que, quant à l'allure de $f(x)$ à l'infini, la condition qui assure la possibilité de déduire la valeur $D(x_0)$ de son saut au point de discontinuité $x = x_0$ à partir de sa série d'Hermite dérivée terme à terme est exactement la même que celle qui concerne la sommabilité (C, δ) de la série d'Hermite (1) de $f(x)$.* Cette condition

$$\begin{gathered} \int_{-\infty}^{-a} e^{-x^2/2} |x|^{-(2\delta+1)} |f(x)| \, dx < G, \\ \int_{a}^{+\infty} e^{-x^2/2} x^{-(2\delta+1)} |f(x)| \, dx < G \end{gathered} \tag{8}$$

devient pour $\delta = 0$

* E. Kogbetliantz, Annales de l'Ecole Normale Supérieure, (3), vol. 49 (1932), p. 141.

$$\int_{-\infty}^{-a} e^{-x^2/2} \, | f(x) | \, \frac{dx}{|x|} < G, \qquad \int_{a}^{+\infty} e^{-x^2/2} \, | f(x) | \, \frac{dx}{x} < G.$$

En la comparant à la condition correspondante (5) de M. Jacob on constate que la classe de fonctions $f(x)$ auxquelles est applicable le résultat (4) est élargie considérablement.

La condition (6) est vérifiée si $f(x)$ par exemple est à variation bornée dans l'intervalle $|x-x_0| \leqq \epsilon$. Elle ne concerne que le voisinage immédiat du point $x=x_0$ et est vérifiée en particulier, si l'on a pour $u \to 0$

$$f(x_0 + u) - f(x_0 + o \operatorname{sgn} u) = O\left[\left(\log \frac{1}{|u|}\right)^{-(1+\eta)}\right]$$

quelque petit que soit le nombre positif fixe η. On peut la remplacer (voir §2) par une autre. Posons à cet effet pour $|t| \leqq \epsilon$

$$\psi(t) = t[f(x_0 + t) - f(x_0 + o \operatorname{sgn} t)]$$

et soit

$$\psi(t) = \int_0^t \chi(t) dt.$$

Le résultat (4) *subsiste si l'on remplace la condition* (8) *par la suivante:*

(9) $$\int_0^h | \chi(t) | \, dt = O(h) \qquad \text{pour } h \to 0.$$

Considérons maintenant la série de Laguerre (2). Dérivée terme à terme elle devient

(10) $$-\sum_{n=0}^{\infty} \frac{\Gamma(n+2)}{\Gamma(n+\alpha+2)} L_n^{(\alpha+1)}(x) \int_0^{\infty} e^{-u} u^{\alpha} L_{n+1}^{(\alpha)}(u) f(u) du$$

car $dL_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x)dx$. D'autre part on a

$$n e^{-u} u^{\alpha} L_n^{(\alpha)}(u) du = d\{e^{-u} u^{\alpha+1} L_{n-1}^{(\alpha+1)}(u)\},$$

ce qui permet d'écrire, en supposant l'existence de la dérivée $f'(x)$,

$$-n \int_{\epsilon}^{A} e^{-u} u^{\alpha} L_n^{(\alpha)}(u) f(u) du = - \left| e^{-u} u^{\alpha+1} L_{n-1}^{(\alpha+1)}(u) f(u) \right|_{\epsilon}^{A}$$
$$+ \int_{\epsilon}^{A} e^{-u} u^{\alpha+1} L_{n-1}^{(\alpha+1)}(u) f'(u) du.$$

Dans les hypothèses $f(u) = O(e^{qu})$, $f'(u) = O(e^{qu})$ pour $u \to \infty$ avec $q<1$ et $u^{\alpha+1} f(u) = o(1)$ pour $u \to 0$ on en déduit

$$-n\int_0^\infty e^{-u}u^\alpha L_n^{(\alpha)}(u)f(u)du = \int_0^\infty e^{-u}u^{\alpha+1}L_{n-1}^{(\alpha+1)}(u)f'(u)du,$$

ce qui prouve que (10) n'est autre chose que le développement formel de $f'(x)$ suivant les polynomes de Laguerre $L_n^{(\alpha+1)}(x)$.

Ceci dit, soit $f_n^{(\delta)}(x_0)$ la moyenne arithmétique de sommes partielles de la série (10) considérée au point $x=x_0$ où $f(x)$ admet un saut fini $D(x_0)$. On a pour $x_0>0$ le résultat analogue à celui obtenu pour la série d'Hermite:

THÉORÈME II. *Si* $x^{(\alpha+\delta)/2-1/4}|f(x)|$, $|f(x)|$ *et* $e^{-x/2}x^{\alpha/2-\delta-3/4}|f(x)|$ *sont intégrables dans les intervalles* $(0, \epsilon)$, (ϵ, a) *et* (a, ∞) *respectivement, les nombres positifs* ϵ *et* a^{-1} *étant aussi petits qu'on veut mais fixes, on a pour tout* $\delta>0$

$$\lim_{n=\infty}\frac{f_n^{(\delta)}(x_0)}{n^{1/2}} = \frac{\Gamma(\delta+1)}{2\pi^{1/2}\Gamma\left(\delta+\dfrac{3}{2}\right)}\cdot\frac{D(x_0)}{x_0{}^{1/2}}, \tag{11}$$

et ce résultat subsiste aussi pour $\delta=0$, *pourvu qu'au voisinage du point* $x=x_0$, $f(x)$ *vérifie les conditions* (6) *ou* (9).

En étudiant pour $r\to 1$ l'allure des intégrales

$$P_H(r, x) = -\int_{-\infty}^{+\infty} e^{-u^2}f(u)\left\{\sum_{n=0}^\infty \frac{H_n(x)H_{n+1}(u)}{2^n n!\pi^{1/2}}r^n\right\}du, \tag{12}$$

$$P_L(r, x) = -\int_0^\infty e^{-u}u^\alpha f(u)\left\{\sum_{n=0}^\infty \frac{\Gamma(n+2)r^n}{\Gamma(n+\alpha+2)}L_n^{(\alpha+1)}(x)L_{n+1}^{(\alpha)}(u)\right\}du, \tag{13}$$

obtenues en appliquant aux séries (3) et (10) la méthode de sommation d'Abel-Poisson et en intervertissant les signes $\int$ et $\sum$, on constate que le saut $D(x_0)$ est lié aux limites des produits

$$P_H(r, x)(1-r)^{1/2} \quad \text{et} \quad P_L(r, x)(1-r)^{1/2}$$

par les relations suivantes:

$$D(x_0) = (2\pi)^{1/2}\lim_{r=1}\left\{P_H(r, x)(1-r)^{1/2}\right\}, \tag{14}$$

$$D(x_0) = 2(\pi x_0)^{1/2}\lim_{r=1}\left\{P_L(r, x)(1-r)^{1/2}\right\}. \tag{15}$$

Ces formules si simples sont valables sous l'unique hypothèse de l'intégrabilité du produit $e^{-u^2}|f(u)|$ dans $(-\infty, +\infty)$ pour (14) et de celui $e^{-u}|f(u)|$ dans $(0, \infty)$ pour la formule (15), cette dernière exigeant aussi l'intégrabilité du produit $u^\alpha|f(u)|$ dans $(0, \epsilon)$. On constate ainsi qu'au point de vue de l'allure à l'infini la classe des fonctions auxquelles sont applicables les résultats (14) et (15) est beaucoup plus vaste que celle des fonctions dont

l'allure à l'infini assure la validité des formules (7) et (11). Néanmoins au point de vue de la détermination effective du nombre $D(x_0)$ la simplicité des formules (14) et (15) n'est qu'apparente. En réalité, étant donné un développement d'Hermite (1) ou un développement de Laguerre (2) dont on se propose d'extraire, en le dérivant d'abord terme à terme, la valeur du saut $D(x_0)$ de la fonction développée en un point déterminé $x=x_0$, on ne peut tenir compte dans les calculs à réaliser avec $r<1$ que d'un nombre *fini* de termes des séries (12) et (13) supposées convergentes et dont les sommes sont désignées par $P_H(r, x_0)$ et $P_L(r, x_0)$. Ce nombre de termes que l'on doit calculer pour connaître une valeur approchée de P_H ou de P_L avec une précision donnée d'avance en fonction de r croît extrêmement vite quand r tend vers l'unité. Or, $D(x_0)$ n'est représenté (aux facteurs numériques près) que par la *limite* du produit $P(r, x_0)\,(1-r)^{1/2}$ pour $r\rightarrow 1$. C'est à dire en réalité on a représenté dans (14) et (15) le nombre $D(x_0)$ par deux passages à la limite superposés

$$\lim_{r=1}\left\{(1 - r)^{1/2}\left[\lim_{n=\infty} S_n(r, x_0)\right]\right\}$$

où $S_n(r, x_0)$ désigne la $n$ième somme partielle de la série (12) ou (13). Ceci explique les énormes difficultés de calcul que l'on rencontre dès que l'on essaye de calculer la valeur numérique du saut $D(x_0)$ à l'aide des formules élégantes (14) et (15), difficultés que l'on peut qualifier sans exagération d'insurmontables.

En outre, la liberté beaucoup plus grande que laissent les conditions suffisantes de (14) et (15) à l'allure de la fonction développée à l'infini s'explique par le fait que les résultats (14) et (15) concernent les *intégrales* de Poisson formées dans les systèmes orthogonaux d'Hermite et de Laguerre et dérivées par rapport à x. Si l'on veut parler de la sommation d'Abel-Poisson des *séries* (3) et (10) elles-mêmes on doit tenir compte des résultats obtenus par E. Hille* d'après lesquels les séries obtenues en intervertissant les signes $\int$ et $\sum$ dans les seconds membres de (12) et (13) ne convergent que si l'allure de $f(x)$ à l'infini assure l'intégrabilité à l'infini des produits $e^{-ku^2}|f(u)|$ et $e^{-ku}|f(u)|$ respectivement pour toute valeur de k supérieure à un demi, $k>\frac{1}{2}$.

Notre méthode nous a permis en outre de donner au §1 la démonstration d'un résultat énoncé sans démonstration par E. Hille et qui concerne la sommabilité du développement de Laguerre (2) en un point $x>0$ par le procédé d'Abel-Poisson.

* E. Hille, Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 261–269, Annals of Mathematics, (2), vol. 27 (1926), pp. 427–464, et Mathematische Zeitschrift, vol. 32 (1930), pp. 422–425.

Ce résultat s'énonce ainsi: en tout point x, où existent les deux nombres $f(x\pm 0)$, l'intégrale de Poisson relative à la série (2) tend pour $r\to 1$ vers l'expression $\frac{1}{2}[f(x+0)+f(x-0)]$, pourvu que les produits $u^\alpha|f(u)|$ et $e^{-u}|f(u)|$ soient intégrables dans les intervalles $(0, \epsilon)$ et (ϵ, ∞).

Tous ces résultats (7), (11), (14), et (15) ont été publiés dans une Note* insérée aux Comptes Rendus de l'Académie des Sciences de Paris.

1. Méthode de Poisson

En dérivant par rapport à x les fonctions génératrices des séries noyaux des développements (1) et (2)

$$\sum_0^\infty \frac{H_n(x)H_n(u)}{2^n n!\pi^{1/2}} r^n = \pi^{-1/2}(1 - r^2)^{-1/2} \exp\left[-(x^2r^2 - 2xur + u^2r^2)/(1 - r^2)\right],$$

$$\sum_0^\infty \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}(x)L_n^{(\alpha)}(u)r^n$$
$$= \frac{(uxr)^{-\alpha/2}}{1-r} \exp\left[-(x+u)r/(1-r)\right] I_\alpha\left(\frac{2(uxr)^{1/2}}{1-r}\right),$$

on trouve celles des séries-noyaux de (12) et de (13), à savoir:

$$\begin{aligned} &-\sum_0^\infty \frac{H_n(x)H_{n+1}(u)}{2^n n!\pi^{1/2}} r^n \\ (16)\qquad &= \frac{\exp\left[-d^2r/(1-r) + s^2r/(1+r)\right]}{\pi^{1/2}(1-r^2)^{1/2}} \left\{\frac{u-x}{1-r} + \frac{u+x}{1+r}\right\} \end{aligned}$$

où l'on a posé $(x-u)^2 = 2d^2$ et $(x+u)^2 = 2s^2$, ainsi que

$$\begin{aligned} &-\sum_0^\infty \frac{\Gamma(n+2)L_n^{(\alpha+1)}(x)L_{n+1}^{(\alpha)}(u)}{\Gamma(n+\alpha+2)} r^n \\ (17)\qquad &= \frac{\exp\left[-(x+u)r/(1-r)\right]\left\{I_{\alpha+1}(\tau)u^{1/2} - I_\alpha(\tau)(xr)^{1/2}\right\}}{u^{\alpha/2}(xr)^{(\alpha+1)/2}(1-r)^2} \end{aligned}$$

en posant $\tau = 2(uxr)^{1/2}/(1-r)$. La fonction de Bessel $I_\alpha(\tau)$ désigne comme toujours celle

$$i^{-\alpha}J_\alpha(i\tau) = I_\alpha(\tau).$$

Grâce aux relations (16) et (17) on trouve les expressions suivantes:

* E. Kogbetliantz, Comptes Rendus, vol. 196 (1933), pp. 464–466.

$$(18)\quad \begin{aligned}(1-r)^{1/2}P_H(r,x) &= \frac{2}{(\pi(1+r))^{1/2}}\\ &\times \int_{-\infty}^{\infty} \exp\left[-u^2 - \frac{u^2r^2 - 2uxr + x^2r^2}{1-r^2}\right] f(u)\,\frac{u-xr}{1-r^2}\,du,\end{aligned}$$

$$(19)\quad \begin{aligned}(1-r)^{1/2}P_L(r,x) &= (rx)^{-(\alpha+1)/2}(1-r)^{-3/2}\\ &\times \int_0^{\infty} \exp\left[-u-(u+x)r/(1-r)\right] f(u)\left\{I_{\alpha+1}(\tau)u^{1/2} - I_\alpha(\tau)(xr)^{1/2}\right\}du.\end{aligned}$$

Posons pour étudier la limite du second membre de (18) quand r tend vers l'unité

$$\phi_r(u) = \frac{2}{(\pi(1+r))^{1/2}}\,\frac{u-xr}{1-r^2}\exp\left[-(u^2r^2-2xur+x^2r^2)/(1-r^2)\right].$$

En dérivant par rapport à u, on trouve

$$\begin{aligned}(1-r^2)^2 \exp\left[(u^2r^2-2xur+x^2r^2)/(1-r^2)\right]\phi_r'(u)(\pi(1+r))^{1/2}\\ = -2\left\{2r^2u^2 - 2rx(1+r^2)u + r^2(2x^2+1) - 1\right\}.\end{aligned}$$

Les deux racines de l'équation $\phi_r'(u)=0$ sont évidemment

$$(u_1 < u_2)\qquad u_{1,2} = \frac{x(1+r^2) \pm (1-r^2)^{1/2}(2+x^2(1-r^2))^{1/2}}{2r},$$

et l'on a $0<u_1<xr<u_2$, car

$$u_{1,2} - xr = \pm\frac{(1-r^2)^{1/2}}{2r}\left\{[2+x^2(1-r^2)]^{1/2} \pm x(1-r^2)^{1/2}\right\}.$$

On constate que $\phi_r'(u)$ est négative dans les intervalles $(-\infty, u_1)$ et $(u_2, +\infty)$.

Les extrema de $\phi_r(u)$ tendent vers l'infini quand r tend vers l'unité. Plus précisément on a pour $r\to 1$

$$\phi_r(u_{1,2}) = \pm\frac{\exp[x^2-1/2]}{2(1-r)^{1/2}} + O(1)$$

tandis que pour ϵ fixe et positif, on peut écrire

$$\phi_r(xr\pm\epsilon) = \pm\,\epsilon\,\frac{\exp\left[-\epsilon^2/(1-r^2)+(xr\pm\epsilon)^2\right]}{1-r^2} = O\left\{\frac{\exp\left[-\epsilon^2/(1-r^2)\right]}{1-r^2}\right\}.$$

Par conséquent, vu que $\phi_r'(u)<0$ pour $u<u_1$ et $u>u_2$ et que $\phi_r(u)\gtrless 0$ suivant que $u\gtrless xr$, on trouve

$$|\phi_r(u)| \leqq |\phi_r(xr \pm \epsilon)| = O\left\{\frac{\exp[-\epsilon^2/(1-r^2)]}{1-r^2}\right\}$$

uniformément en u pourvu que l'on ait $|u-xr| \geqq \epsilon$.

On peut maintenant conclure. Supposons que l'allure à l'infini de la fonction $f(x)$, intégrable (L) dans tout intervalle fini, vérifie la condition d'intégrabilité du produit $e^{-x^2}|f(x)|$ dans l'intervalle $(-\infty, +\infty)$. Soit, par conséquent,

$$\int_{-\infty}^{\infty} e^{-x^2}|f(x)|\,dx < G. \tag{20}$$

En décomposant l'intervalle d'intégration $(-\infty, +\infty)$ en trois: $u \leqq xr-\epsilon$, $|u-xr| \leqq \epsilon$ et $u \geqq xr+\epsilon$, on trouve

$$(1-r)^{1/2}P_H(r\ \ x) = \int_{-\infty}^{\infty} e^{-u^2} f(u)\phi_r(u)du = i_1 + i_2 + i_3,$$

où, pour $r \to 1$,

$$\begin{aligned}|i_1| &\leqq \int_{-\infty}^{xr-\epsilon} |\phi_r(u)|\, e^{-u^2}|f(u)|\,du \\ &= O\left\{\frac{\exp[-\epsilon^2/(1-r^2)]}{1-r^2}\int_{\infty}^{xr-\epsilon} e^{-u^2}|f(u)|\,du\right\} = o(1).\end{aligned}$$

et de même $i_3 = o(1)$.

Quant à l'intégrale

$$i_2 = \int_{xr-\epsilon}^{xr+\epsilon} \phi_r(u)e^{-u} f(u)du = \int_{xr-\epsilon}^{xr} + \int_{xr}^{xr+\epsilon},$$

transformons la par les substitutions $u = xr \pm t^{1/2}(1-r^2)^{1/2}$ qui donnent

$$\phi_r(u)e^{-u^2}du = \frac{e^{-t}dt}{(\pi(1+r))^{1/2}},$$

en

$$\begin{aligned}i_2 &= \frac{1}{(\pi(1+r))^{1/2}}\left\{\int_{\epsilon^2/(1-r^2)}^{0} + \int_0^{\epsilon^2/(1-r^2)}\right\} \\ &= \frac{1+o(1)}{(2\pi)^{1/2}}\int_0^{\epsilon^2/(1-r^2)} e^{-t}[f(xr + t^{1/2}(1-r^2)^{1/2}) - f(xr - t^{1/2}(1-r^2)^{1/2})]dt.\end{aligned}$$

Or, pour $r \to 1$ et $\epsilon \leqq \epsilon_0(\eta)$, on a quelque petite que soit la quantité fixe η donnée d'avance,

$$|f(xr+t^{1/2}(1-r^2)^{1/2})-f(xr-t^{1/2}(1-r^2)^{1/2})-D(x)|<\eta(2\pi)^{1/2}$$

où l'on a posé comme toujours

$$D(x)=f(x+0)-f(x-0),$$

l'existence des deux nombres $f(x\pm 0)$ au point x étant assurée par hypothèse. On a donc pour $r\to 1$

$$\left| i_2-\frac{D(x)}{(2\pi)^{1/2}}(1-\exp[-\epsilon^2/(1-r^2)])\right|<\eta,$$

ce qui achève la preuve de la formule

$$\lim_{r=1}[(2\pi(1-r))^{1/2}P_H(r,x)]=D(x)$$

valable sous l'unique hypothèse (20) de l'intégrabilité du produit $e^{-x^2}|f(x)|$ dans l'intervalle infini $(-\infty,+\infty)$.

Le même raisonnement s'applique au second membre de (19). On peut l'écrire ainsi:

$$(1-r)^{1/2}P_L(r,x)=\int_0^\infty \Psi_r(u)e^{-u}f(u)du,$$

en posant pour abréger

$$\Psi_r(u)=(1-r)^{\alpha-3/2}(2xr)^{-\alpha-1}\psi_r(u)$$

et

$$\psi_r(u)=\exp[-(x+u)r/(1-r)][(1-r)\tau^{\alpha+1}I_{\alpha+1}(\tau)-2xr\tau^\alpha I_\alpha(\tau)]$$

où comme toujours $\tau=2(xur)^{1/2}/(1-r)$.

Etudions la fonction $\psi_r(u)$ pour x fixe et positif et $u\geqq 1-r$. Grâce au fait que

$$\frac{d}{dx}[x^\alpha I_\alpha(x)]=x^\alpha I_{\alpha-1}(x)$$

on trouve facilement que la dérivée $\psi_r'(u)$ est le produit d'une fonction, qui reste positive et ne s'annule pas pour $u\geqq 1-r$ et $r\leqq 1$, par l'expression suivante:

$$\vartheta_r(u)=-uI_{\alpha+1}(\tau)+\frac{1+r}{r^{1/2}}I_\alpha(\tau)(xu)^{1/2}-xI_{\alpha-1}(\tau).$$

En annulant $\psi_r'(u)$, on trouve l'équation $\vartheta_r(u)=0$, dont les racines se laissent exprimer par des formules approchées étant donné que pour $u\geqq 1-r$

et $r\to 1$ la variable τ tend vers l'infini et que l'on peut, par conséquent, utiliser les expressions approchées bien connues

$$(21)\quad I_\alpha(\tau) = \frac{e^\tau}{(2\pi\tau)^{1/2}}\left\{\sum_{k=0}^{m-1}(-1)^k\frac{\Gamma(\alpha+k+\frac{1}{2})}{\Gamma(\alpha-k+\frac{1}{2})}\frac{(2\tau)^{-k}}{k!}+O(|\tau|^{-m})\right\}.$$

Calculs faits, on constate que dans l'intervalle $(1-r,\ \infty)$ l'équation $\psi_r'(u)=0$ n'a que deux racines réelles u_1 et u_2 situées de part et d'autre du nombre positif x: $1-r<u_1<x<u_2$. Plus précisément, on a

$$u_{1,2}^{1/2} = x^{1/2}\left\{1\mp\left(\frac{1-r}{2}\right)^{1/2}+\frac{(\alpha-1)(1-r)}{2}+O[(1-r)^{3/2}]\right\},$$

ce qui permet d'écrire

$$u_{1,2} = x[1\mp(2(1-r))^{1/2}+O(1-r)].$$

Dans l'intervalle $u_1<u<u_2$ la dérivée $\psi_r'(u)$ est positive et elle reste négative quand u varie dans les intervalles $(1-r,\ u_1)$ et $(u_2,\ \infty)$.

Vu que l'expression approchée de $\psi_r(u)$ est égale à

$$\psi_r(u) = \frac{(4rux)^{\alpha/2}x^{1/4}\exp[2(uxr)^{1/2}/(1+r^{1/2})-(u^{1/2}-x^{1/2})^2r/(1-r)]}{\pi^{1/2}(ru)^{1/4}(1-r)^{\alpha-1}}$$
$$\times\left\{\frac{u^{1/2}-x^{1/2}}{(1-r)^{1/2}}+O[(u(1-r))^{1/2}]\right\},$$

on calcule aisément les extrema $\Psi(u_1)$ et $\Psi(u_2)$ de la fonction $\Psi(u)$ dans l'intervalle $(\epsilon,\ \infty)$:

$$\Psi_{1,2} = \Psi(u_{1,2}) = \frac{\mp e^{x-1/2}}{2x(2\pi(1-r))^{1/2}}+O(1)\qquad (r\to 1).$$

On constate que ces extrema tendent vers $-\infty$ et $+\infty$ quand r tend vers l'unité. En outre, il est facile de vérifier à l'aide de (21) que $\psi_r(u)<0$ pour $1-r<u<u_1$. Par conséquent, étant donné que l'expression approchée de $\Psi(u)$ à savoir

$$(22)\quad \Psi_r(u) = \frac{u^{\alpha/2-1/4}\exp[(ux)^{1/2}-(u^{1/2}-x^{1/2})^2r/(1-r)]}{2x^{\alpha/2+3/4}\pi^{1/2}(1-r)}\{u^{1/2}-x^{1/2}+O[(1-r)u^{1/2}]\}$$

entraîne pour $u^{1/2}=(1\pm\delta)x^{1/2}$, où δ est positive, fixe et aussi petite qu'on veut, celle:

$$\Psi[x(1\pm\delta)^2] = \pm\frac{\delta(1\pm\delta)^{\alpha+1/2}}{2\pi^{1/2}}e^{x(1\pm\delta)}\frac{\exp[-rx\delta^2/(1-r)]}{1-r}\{1+O(1-r)\},$$

on voit que les résultats acquis permettent d'écrire

(23) $\Psi_r(u) = O\{\exp[-x\delta^2/(1-r)]/(1-r)\}$ $(u \geqq 1-r, |u^{1/2} - x^{1/2}| \geqq \delta x^{1/2})$

et cela uniformément en u dans les intervalles $\epsilon \leqq u \leqq x(1-\delta)^2$ et $x(1+\delta)^2 \leqq u < \infty$, où ϵ et δ sont deux quantités positives aussi petites qu'on veut, choisies d'avance.

Nous pouvons conclure ainsi:

$$(1-r)^{1/2}P_L(r, x) = \int_0^\infty \Psi_r(u)e^{-u}f(u)du$$

$$= \int_0^\epsilon + \int_\epsilon^{x(1-\delta)^2} + \int_{x(1-\delta)^2}^{x(1+\delta)^2} + \int_{x(1+\delta)^2}^\infty = \sum_{k=1}^4 j_k,$$

où d'après (23) on a, quelque petit que soit ϵ,

$$\lim_{r=1} j_2 = \lim_{r=1} j_4 = 0$$

pourvu que l'allure de $f(x)$ à l'infini assure l'existence de l'intégrale définie suivante:

$$\int_\epsilon^\infty e^{-u}|f(u)|du. \tag{24}$$

Considérons maintenant l'intégrale j_1 étendue à l'intervalle $(0, \epsilon)$. Nous allons prouver que l'on a de même $\lim_{r=1} j_1 = 0$, si l'on suppose que le produit $u^\alpha|f(u)|$ est intégrable à l'origine. Soit, en effet,

$$\int_0^\epsilon u^\alpha|f(u)|du < G. \tag{25}$$

Il est facile de prouver que la fonction $\Psi_r(u)$ tend vers zéro (quand $r \to 1$) plus vite qu'une puissance quelconque de $1-r$. On a montré ailleurs* que l'on a, quelque soit $u > 0$,

$$\sigma_n^{(\gamma)} = O\{u^{-(\alpha/2+1/4)}n^{(1+\gamma)/2}\},$$

$\sigma_n^{(\gamma)}$ désignant la $n$ième sigma-somme d'ordre γ de la série-noyau du développement (10):

$$\sigma_n^{(\gamma)} = -\sum_{m=0}^n \frac{\Gamma(n-m+\gamma+1)}{\Gamma(n-m+1)\Gamma(\gamma+1)} \frac{\Gamma(m+2)}{\Gamma(m+\alpha+2)} L_m^{(\alpha+1)}(x)L_{m+1}^{(\alpha)}(u).$$

Supposons que l'on limite les valeurs de u par l'inégalité $nu \geqq 1$. Dans ces conditions on aura pour $\alpha < -\frac{1}{2}$

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* Voir E. Kogbetliantz, Journal of Mathematics and Physics, sous presse, §5, (BL).

$$\sigma_n^{(\gamma)} = O(n^{(1+\gamma)/2}) \qquad (\alpha < -\tfrac{1}{2}, n^{-1} \leqq u \leqq \epsilon)$$

tandis que pour $\alpha \geqq -\frac{1}{2}$,

$$\sigma_n^{(\gamma)} = O(n^{(\gamma+\alpha)/2+3/4}) \qquad (\alpha \geqq -\tfrac{1}{2}).$$

Posons

$$\mu = \max\left[\frac{1+\gamma}{2}, \quad \frac{\gamma+\alpha}{2} + \frac{3}{4}\right].$$

On a par conséquent, pour $nu \geqq 1$,

$$\sigma_n^{(\gamma)} = O(n^\mu) \qquad \left(u \geqq \frac{1}{n}\right)$$

et cela quelque soit $\alpha > -1$.

Pour $nu \leqq 1$ nous allons utiliser la relation*

$$\sigma_n^{(\gamma)} = \sum_{m=0}^{n-1} \frac{(xu)^m}{m!} \left\{ \frac{uL_{n-m-1}^{(2m+\alpha+\gamma+3)}(x+u)}{\Gamma(m+\alpha+2)} - \frac{L_{n-m-1}^{(2m+\alpha+\gamma+2)}(x+u)}{\Gamma(m+\alpha+1)} \right\}$$

ainsi que l'inégalité†

$$L_n^{(\alpha)}(x) = O\left[e^{x/2} n^{\alpha/2-1/4} x^{-\alpha/2-1/4}\right].$$

Par conséquent, on a l'évaluation suivante, le second terme entre les parenthèses étant en valeur absolue supérieur au premier terme grâce à l'hypothèse $nu \leqq 1$:

$$\sigma_n^{(\gamma)} = O\left\{ \sum_0^{n-1} \frac{(xu)^m}{m!} \frac{(n-m)^{m+(\alpha+\gamma)/2+3/4}}{(x+u)^{m+(\alpha+\gamma)/2+5/4}\Gamma(m+\alpha+1)} \right\}$$

$$= O\left\{ \left(\frac{n}{x+u}\right)^{(\alpha+\gamma)/2+3/4} \frac{1}{(x+u)^{1/2}} \sum_0^\infty \frac{\left(\frac{x}{x+u}\right)^m}{m!\Gamma(m+\alpha+1)} \right\}$$

$$= O(n^{(\alpha+\gamma)/2+3/4}),$$

car $x+u \geqq x > 0$. Par conséquent, quelque soient $\alpha > -1$ et $u \geqq 0$ on a

$$\sigma_n^{(\gamma)} = O(n^\mu) = O\left[\frac{\Gamma(n+\mu+1)}{n!\,\Gamma(\mu+1)}\right] = O[A_n^{(\mu)}]$$

d'où, en désignant le terme général de la série-noyau du développement (10) par w_n et vu que

* Voir E. Kogbetliantz, Journal of Mathematics and Physics, loc. cit., §5, (F).

† E. Kogbetliantz, Annales de l'Ecole Normale Supérieure, (3), vol. 49 (1932), p. 149, (27).

$$\Phi(x, u; r) = \sum_0^\infty w_n r^n = (1 - r)^{\gamma+1} \sum_0^\infty \sigma_n^{(\gamma)} r^n:$$

$$| \Phi(x, u; r) | = O\left\{(1 - r)^{\gamma+1} \sum_0^\infty A_n^{(\mu)} r^n\right\} = O[(1 - r)^{\gamma-\mu}].$$

Or,

$$\gamma - \mu = \min\left[\frac{\gamma - 1}{2}, \frac{\gamma - \alpha}{2} - \frac{3}{4}\right] > N$$

car quelque grand que soit le nombre N, on peut toujours choisir γ supérieur au plus grand des deux nombres $2N+1$ et $2N+\alpha+3/2$.

Etant donné que l'on a

$$P_L(r, x) = \int_0^\infty e^{-u} u^\alpha \Phi(u, x; r) f(u) du,$$

on trouve immédiatement le résultat cherché relatif à l'intégrale j_1:

$$| j_1 | < (1 - r)^{1/2} \int_0^\epsilon e^{-u} u^\alpha | \Phi(u, x; r) | | f(u) | du$$

$$= O\left\{ (1 - r)^{N+1/2} \int_0^\epsilon u^\alpha | f(u) | du\right\} = o(1) \qquad (r \to 1)$$

sous l'hypothèse de l'existence de l'intégrale (25).

Il ne nous reste qu'à prouver que la limite de j_3 existe et donne le saut $D(x) = f(x+0) - f(x-0)$ de la fonction $f(x)$. Il est évident que la différence

$$\phi(u) = f(u) - f(x + o \operatorname{sgn} (u - x)) \qquad (u \gtrless x)$$

tend vers zéro quand $u \to x$. Ensuite l'orthogonalité des polynomes de Laguerre entraîne la relation

$$\int_0^\infty e^{-u} u^\alpha \Phi(u, x; r) du = - \sum_0^\infty \frac{\Gamma(n + 2) r^n}{\Gamma(n + \alpha + 2)} L_n^{(\alpha+1)}(x) \int_0^\infty e^{-u} u^\alpha L_{n+1}^{(\alpha)}(u) du = 0$$

et l'on a par conséquent

$$\int_0^\infty \Psi_r(u) e^{-u} du = (1 - r)^{1/2} \int_0^\infty e^{-u} u^\alpha \Phi(u, x; r) du = 0,$$

ce qui permet de poser

$$\int_x^\infty e^{-u} \Psi_r(u) du = \lambda_r(x) = - \int_0^x e^{-u} \Psi_r(u) du$$

$$= - (1 - r)^{1/2} \int_0^x e^{-u} u^\alpha \phi(u, x; r) du.$$

Il nous faut maintenant calculer la fonction $\lambda_r(x)$. On a

$$(n+1)!\int_0^x e^{-u}u^{\alpha}L_{n+1}^{(\alpha)}(u)du = n!e^{-x}x^{\alpha+1}L_n^{(\alpha+1)}(x)$$

car $\alpha > -1$. Par conséquent:

$$\begin{aligned}\lambda_r(x) &= -(1-r)^{1/2}\int_0^x e^{-u}u^{\alpha}\left\{-\sum_0^{\infty}\frac{(n+1)!}{\Gamma(n+\alpha+2)}L_n^{(\alpha+1)}(x)L_{n+1}^{(\alpha)}(u)r^n\right\}du\\ &= e^{-x}x^{\alpha+1}(1-r)^{1/2}\sum_0^{\infty}\frac{n!r^n}{\Gamma(n+\alpha+2)}\left\{L_n^{(\alpha+1)}(x)\right\}^2\\ &= \frac{e^{-x}x^{\alpha+1}e^{-2xr/(1-r)}}{(1-r)^{1/2}(xr^{1/2})^{\alpha+1}}I_{\alpha}\left(\frac{2xr^{1/2}}{1-r}\right).\end{aligned}$$

La formule (21) donne à ce résultat la forme définitive suivante:

$$\lambda_r(x) = \frac{\exp\left[-(1-r^{1/2})x/(1+r^{1/2})\right]}{2(\pi x)^{1/2}}\left[1+O(1-r)\right],$$

d'où

$$\lim_{r=1}\lambda_r(x) = \frac{1}{2(\pi x)^{1/2}}.$$

Ceci posé, nous avons

$$\begin{aligned}j_3 - \lambda_r(x)D(x) &= \int_{x(1-\delta)^2}^{x(1+\delta)^2} e^{-u}\Psi_r(u)f(u)du\\ &\quad - f(x+0)\int_x^{\infty}e^{-u}\Psi_r(u)du - f(x-0)\int_0^x e^{-u}\Psi_r(u)du\\ &= \int_{x(1-\delta)^2}^{x(1+\delta)^2} e^{-u}\phi(u)\Psi_r(u)du - f(x+0)\int_{x(1+\delta)^2}^{\infty}e^{-u}\Psi_r(u)du\\ &\quad - f(x-0)\int_0^{x(1-\delta)^2}e^{-u}\Psi_r(u)du = \sum_{k=1}^{3} i_k.\end{aligned}$$

Vu que la fonction $f(u)\equiv 1$ vérifie les conditions (24) et (25), nous avons immédiatement

$$\lim_{r=1} i_2 = \lim_{r=1} i_3 = 0$$

donc

$$\lim_{r=1}\left\{j_3 - \lambda_r(x)D(x)\right\} = \lim_{r=1}\left\{j_3 - \frac{D(x)}{2(\pi x)^{1/2}}\right\} = \lim_{r=1} i_1$$

où

$$i_1 = \int_{x(1-\delta)^2}^{x(1+\delta)^2} e^{-u}\phi(u)\Psi_r(u)du$$

avec $\phi(u) = f(u) - f[x + o\,\mathrm{sgn}\,(u-x)] \to 0$ pour $u \to x$. Par conséquent, en choisissant δ suffisamment petit on peut rendre la borne supérieure de $|\phi(u)|$ dans l'intervalle $[x(1-\delta)^2, x(1+\delta)^2]$ aussi petite qu'on veut et cela prouve que l'intégrale i_1 est aussi petite qu'on veut en valeur absolue si l'on a

$$\int_{x(1-\delta)^2}^{x(1+\delta)^2} |\Psi_r(u)|\, du = O(1).$$

Or, la formule approchée (22) prouve que pour x fixe et u compris dans l'intervalle $[x(1-\delta)^2, x(1+\delta)^2]$, on a

$$\Psi_r(u) = O\left\{\exp\left[-(u^{1/2} - x^{1/2})^2/(1-r)\right]\left[1 + \frac{|u^{1/2} - x^{1/2}|}{1-r}\right]\right\},$$

ce qui permet de conclure ainsi: posons $t = u^{1/2} - x^{1/2}$; alors

$$\int_{x(1-\delta)^2}^{x(1+\delta)^2} |\Psi_r(u)|\, du = O\left\{\int_0^{\delta x^{1/2}} \exp\left[-t^2/(1-r)\right]\left(1 + \frac{t}{1-r}\right)dt\right\} = O(1).$$

On a démontré ainsi que

$$\lim_{r=1} i_1 = 0,$$

et par conséquent

$$\lim_{r=1} \left\{(1-r)^{1/2} P_L(u, x; r)\right\} = \lim_{r=1} j_3 = \frac{D(x)}{2(\pi x)^{1/2}}.$$

Il est intéressant d'observer que la même méthode nous fournit une preuve facile du résultat énoncé par Einar Hille,* relatif à la sommabilité

* E. Hille, Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 261, 265, 348. S. Wigert, Arkiv för Matematik, vol. 15 (1921), No. 25, a donnée une démonstration pour le cas $\alpha = 0$ en supposant que $f(x)$ soit continue et satisfasse à une condition semblable à (24) avec e^{-u} remplacée par e^{-au} pour tout $a > \frac{1}{2}$. Hille remarque que le procédé de Wigert s'étend au cas général et il donne les formules nécessaires pour cette extension. Le noyau $F_r(u, x)$ se trouve aussi dans une note de G. H. Hardy, Journal of the London Mathematical Society, vol. 7 (1932), pp. 138, 192.

Il faut encore remarquer qu'il s'agit d'une généralisation du procédé d'Abel-Poisson dans (26). En effet, l'hypothèse (24) n'assure l'analyticité de l'intégrale de Poisson que dans le cercle $|r - \frac{1}{2}| < \frac{1}{2}$. Il s'ensuit que la série d'Abel-Laguerre, obtenue en appliquant à la série (2) la méthode de sommation d'Abel-Poisson, ne peut être convergente pour aucune valeur de $r \neq 0$. Quand on remplace dans (24) e^{-u} par e^{-au}, l'analyticité est assurée dans $|r - 1 + 1/(2a)| < 1/(2a)$, et, si $a < 1$, la somme de la série est donnée par l'intégrale de Poisson pour $|r| < \min(1, 1/a - 1)$. C'est donc seulement pour $a \leqq \frac{1}{2}$ qu'on peut parler de la sommabilité Abel-Poisson au sens ordinaire. Pour la situation correspondante dans la théorie de la série d'Hermite voir les travaux de Hille dans Annals of Mathematics, (2), vol. 27 (1926), p. 427, et Mathematische Zeitschrift, vol. 32 (1930), p. 422.

du développement de Laguerre (2) en un point $x>0$ par le procédé d'Abel-Poisson et dont la démonstration, semble-t-il, n'a pas été publiée par l'auteur. Il s'agit de prouver que l'on a

$$\lim_{r=1}\int_0^\infty e^{-u}F_r(u,\,x)f(u)du=\tfrac{1}{2}[f(x+0)+f(x-0)] \tag{26}$$

où

$$F_r(u,\,x)=\frac{u^{\alpha/2}\exp\left[-(x+u)r/(1-r)\right]}{(1-r)(xr)^{\alpha/2}}I_\alpha\left(\frac{2(uxr)^{1/2}}{1-r}\right)$$
$$=\left(\frac{1-r}{2xr}\right)^\alpha\frac{\exp\left[-(x+u)r/(1-r)\right]}{1-r}\tau^\alpha I_\alpha(\tau).$$

Or, on trouve pour la dérivée de $F_r(u,\,x)$ par rapport à u l'expression suivante:

$$\frac{d}{du}F_r(u,\,x)=\frac{F_r(u,\,x)}{1-r}\left\{I_{\alpha-1}(\tau)x^{1/2}-I_\alpha(\tau)(ru)^{1/2}\right\}\left(\frac{r}{u}\right)^{1/2}$$

d'où pour $u\geqq\epsilon$ la conclusion suivante: dans l'intervalle infini $(\epsilon,\,\infty)$ la fonction $F_r(u,\,x)$ est positive et ne possède qu'un seul maximum, dont l'abscisse est $u=u_0=x+(2x+2\alpha-1)(1-r)+O[(1-r)^2]$.

Ce maximum $F_r(u_0,\,x)$ tend vers $+\infty$ quand r tend vers l'unité, mais on a

$$F_r[x(1\pm\delta)^2,\,x]=O\left\{\frac{\exp\left[-xr\delta^2/(1-r)\right]}{(x(1-r))^{1/2}}\right\},$$

d'où immédiatement

$$\lim_{r=1}\int_\epsilon^{x(1-\delta)^2}e^{-u}F_r(u,\,x)f(u)du=\lim_{r=1}\int_{x(1+\delta)^2}^\infty e^{-u}F_r(u,\,x)f(u)du=0$$

sous l'unique hypothèse (24). De même l'hypothèse (25) assure

$$\lim_{r=1}\int_0^\epsilon e^{-u}F_r(u,\,x)f(u)du=0,$$

car pour $0\leqq u\leqq\epsilon$ on a $u^{-\alpha}F_r(u,\,x)e^{-u}=O[(1-r)^N]$, le nombre fixe N étant aussi grand qu'on veut. Enfin, quant à l'intégrale

$$\int_{x(1-\delta)^2}^{x(1+\delta)}e^{-u}F_r(u,\,x)f(u)du,$$

la formule approchée

$$e^{-u}F_r(u,\,x)=\frac{u^{\alpha/2-1/4}\exp\left[-(u^{1/2}-x^{1/2})^2/(1-r)\right]}{2(xr)^{\alpha/2+1/4}(\pi(1-r))^{1/2}}\left\{1+O\left(\frac{1-r}{u^{1/2}}\right)\right\}$$

prouve que l'intégrale du produit $e^{-u}|F_r(u, x)|$ est bornée:

$$\begin{aligned}\int_{x(1-\delta)^2}^{x(1+\delta)^2} & e^{-u}\,|F_r(u, x)|\,du \\ &= \frac{1+\eta}{2\pi^{1/2}}\int_{x(1-\delta)^2}^{x(1+\delta)^2}\exp\left[-\,(u^{1/2}-x^{1/2})^2/(1-r)\right]\frac{du}{(u(1-r))^{1/2}} \\ &= \frac{2(1+\eta)}{\pi^{1/2}}\int_0^{\delta(x/(1-r))^{1/2}} e^{-\xi^2}d\xi = O(1) \qquad (r\to 1)\end{aligned}$$

ce qui achève la preuve de (26).

2. Sommation (C, δ) de la série (3)

Pour démontrer la relation

$$\text{(7)}\qquad \lim_{n=\infty}\frac{f_n^{(\delta)}(x_0)}{n^{1/2}} = \frac{\Gamma(\delta+1)}{(2\pi)^{1/2}\Gamma\left(\delta+\dfrac{3}{2}\right)}D(x_0),$$

où $f_n^{(\delta)}(x_0)$ désigne la $n$ième moyenne arithmétique d'ordre δ de la série divergente

$$\text{(3)}\qquad -\sum_{n=0}^{\infty}\frac{H_n(x)}{2^n n!\pi^{1/2}}\int_{-\infty}^{\infty}e^{-u^2}H_{n+1}(u)f(u)du,$$

observons que l'on a

$$\text{(27)}\qquad f_n^{(\delta)}(x_0) = \int_{-\infty}^{\infty}e^{-u^2}S_n^{(\delta)}(u, x_0)f(u)du,$$

$S_n^{(\delta)}(u, x)$ désignant la moyenne arithmétique d'ordre δ de la série-noyau de (3), c'est à dire de la série

$$\text{(28)}\qquad -\sum_0^{\infty}\frac{H_n(x)H_{n+1}(u)}{2^n n!\pi^{1/2}} \sim 0 \qquad (u \neq x).$$

Cette moyenne $S_n^{(\delta)}(u, x)$ vérifie* les deux inégalités suivantes:

$$\text{(29)}\qquad S_n^{(\delta)}(u, x) = O\left\{\frac{e^{(x^2+u^2)/2}n^{(1-\delta)/2}}{|u-x|^{\delta+1}}\right\} \qquad (0 \leqq u < \infty;\ \delta \geqq 0;\ |x| \leqq a),$$

$$\text{(30)}\qquad S_n^{(\delta)}(u, x) = O\left\{\frac{e^{u^2/2}n^{1/2}}{|u|^{2\delta+1}}\right\} \qquad (|u| \geqq en^{1/2};\ \delta \geqq 0,\ |x| \leqq a),$$

* Voir E. Kogbetliantz, Annales de l'Ecole Normale Supérieure, (3), vol. 49 (1932), p. 172, (62), p. 173, (65).

ainsi que celle:

$$S_n^{(\delta)}(u, x) = O(ne^{(x^2+u^2)/2}) \tag{31}$$

valable quels que soient u et x et dont la preuve qui suit est basée sur l'inégalité

$$H_n(x) = O[n^{-1/4}e^{x^2/2}(2^n n!)^{1/2}].$$

En effet, en posant

$$n!A_n^{(\delta)}\Gamma(\delta + 1) = \Gamma(n + \delta + 1)$$

on a

$$S_n^{(\delta)}(u, x) = \frac{-1}{A_n^{(\delta)}} \sum_0^n A_{n-m}^{(\delta)} \frac{H_m(x)H_{m+1}(u)}{2^m m! \pi^{1/2}}$$
$$= O\left[n^{-\delta} e^{(x^2+u^2)/2} \sum_0^n A_{n-m}^{(\delta)}\right] = O(ne^{(x^2+u^2)/2}).$$

Ceci posé, soit d'abord $\delta > 0$, le cas $\delta = 0$ étant écarté pour le moment. On peut écrire d'après (27)

$$\begin{aligned}\frac{f_n^{(\delta)}(x)}{n^{1/2}} &= \frac{1}{n^{1/2}}\int_{-\infty}^{\infty} = \frac{1}{n^{1/2}}\int_{-\infty}^{-A} + \frac{1}{n^{1/2}}\int_{-A}^{x-\epsilon} \\ &+ \frac{1}{n^{1/2}}\int_{x-\epsilon}^{x+\epsilon} + \frac{1}{n^{1/2}}\int_{x+\epsilon}^{A} + \frac{1}{n^{1/2}}\int_{A}^{\infty} = \sum_{k=1}^{5} J_k,\end{aligned} \tag{32}$$

la fonction sous les signes somme étant

$$e^{-u^2}S_n^{(\delta)}(u, x)f(u)du.$$

Les inégalités (29) et (30) entraînent:

$$\begin{aligned}|J_1| &= \left| \frac{1}{n^{1/2}}\int_{-\infty}^{-\epsilon n^{1/2}} + \frac{1}{n^{1/2}}\int_{-\epsilon n^{1/2}}^{-A} e^{-u^2}S_n^{(\delta)}(u, x)f(u)du \right| \\ &= O\left[\int_{-\infty}^{-\epsilon n^{1/2}} e^{-u^2/2}|f(u)| \frac{du}{|u|^{2\delta+1}}\right] \\ &\quad + O\left[\int_{-\epsilon n^{1/2}}^{-A} e^{-u^2/2}|f(u)| \frac{du}{n^{\delta/2}|u|^{\delta+1}}\right] \\ &= O\left[\int_{-\infty}^{-A} e^{-u^2/2}|f(u)| \frac{du}{|u|^{2\delta+1}}\right] < \eta\end{aligned}$$

quelque petite que soit la quantité positive η choisie d'avance, pourvu que

le produit $e^{-u^2/2}|f(u)|\,|u|^{-(2\delta+1)}$ soit intégrable à l'infini, le nombre fixe A étant suffisamment grand: $A \geqq A_0 = A_0(\eta)$. On trouve également $|J_5| < \eta$, ces deux résultats relatifs à J_1 et J_5 étant rendus possibles par l'hypothèse de l'existence des deux intégrales définies que voici:

$$(33) \qquad \int_{-\infty}^{-a} e^{-u^2/2}|f(u)| \frac{du}{|u|^{2\delta+1}} < G, \qquad \int_a^{\infty} e^{-u^2/2}|f(u)| \frac{du}{u^{2\delta+1}} < G.$$

Ensuite l'inégalité (29) nous donne

$$\begin{aligned} |J_4| &\leqq \frac{1}{n^{1/2}} \int_{x+\epsilon}^{A} e^{-u^2} |S_n^{(\delta)}(u, x)|\,|f(u)|\,du \\ &= O\left\{\frac{1}{\epsilon^{\delta+1} n^{\delta/2}} \int_{x+\epsilon}^{A} |f(u)|\,du\right\} = O(n^{-\delta/2}), \end{aligned}$$

la fonction $f(u)$ étant par hypothèse sommable (L) dans tout intervalle fini. De même pour J_2 et par conséquent on a, si $\delta > 0$,

$$\lim_{n=\infty} J_2 = \lim_{n=\infty} J_4 = 0.$$

On constate ainsi que pour $\delta > 0$ la différence

$$\frac{f_n^{(\delta)}(x)}{n^{1/2}} - \frac{1}{n^{1/2}} \int_{x-\epsilon}^{x+\epsilon} e^{-u^2} S_n^{(\delta)}(u, x) f(u) du$$

peut être rendue aussi petite qu'on veut en valeur absolue en choisissant d'abord A ensuite n suffisamment grands, ce qui veut dire que l'on a pour tout $\delta > 0$

$$\lim_{n=\infty} \frac{f_n^{(\delta)}(x)}{n^{1/2}} = \lim_{n=\infty} J_3 = \lim_{n=\infty} \frac{1}{n^{1/2}} \int_{x-\epsilon}^{x+\epsilon} e^{-u^2} S_n^{(\delta)}(u, x) f(u) du.$$

Posons, pour $u = x$, $\phi(u) = 0$, et pour $u \neq x$,

$$\phi(u) = f(u) - f(x + o \operatorname{sgn}(u - x))$$

et soit $\theta(\epsilon)$ la borne supérieure de $|\phi(u)|$ dans l'intervalle $(x - \epsilon, x + \epsilon)$. On a évidemment

$$\lim_{\epsilon=0} \theta(\epsilon) = 0.$$

On peut écrire

$$(34) \qquad \begin{aligned} J_3 = \frac{1}{n^{1/2}} \int_{x-\epsilon}^{x+\epsilon} e^{-u^2} S_n^{(\delta)}(u, x) \phi(u) du &+ \frac{f(x+0)}{n^{1/2}} \int_x^{\infty} e^{-u^2} S_n^{(\delta)}(u, x) du \\ &+ \frac{f(x-0)}{n^{1/2}} \int_{-\infty}^{x} e^{-u^2} S_n^{(\delta)}(u, x) du - J_6 \end{aligned}$$

où

$$J_6 = \frac{f(x+0)}{n^{1/2}} \int_{x+\epsilon}^{\infty} e^{-u^2} S_n^{(\delta)}(u, x) du + \frac{f(x-0)}{n^{1/2}} \int_{-\infty}^{x-\epsilon} e^{-u^2} S_n^{(\delta)}(u, x) du.$$

L'inégalité (29) nous donne pour $\delta > 0$

$$J_6 = O\left\{ \frac{|f(x+0)|}{n^{\delta/2}} \int_{x+\epsilon}^{\infty} e^{-u^2/2} \frac{du}{(u-x)^{\delta+1}} + \frac{|f(x-0)|}{n^{\delta/2}} \int_{-\infty}^{x-\epsilon} e^{-u^2/2} \frac{du}{(x-u)^{\delta+1}} \right\}$$

c'est à dire $J_6 = O(n^{-\delta/2})$, ce qui prouve que l'on a

$$\lim_{n=\infty} J_6 = 0.$$

Ensuite

$$\left| \frac{1}{n^{1/2}} \int_{x-\epsilon}^{x+\epsilon} \phi(u) e^{-u^2} S_n^{(\delta)}(u, x) du \right| \leqq \frac{\theta(\epsilon)}{n^{1/2}} \int_{x-\epsilon}^{x+\epsilon} | S_n^{(\delta)}(u, x) | \, du.$$

Soit

$$i_n = \frac{1}{n^{1/2}} \int_{x-\epsilon}^{x+\epsilon} | S_n^{(\delta)}(u, x) | \, du$$

$$= \frac{1}{n^{1/2}} \int_{x-\epsilon}^{x-n^{-1/2}} + \frac{1}{n^{1/2}} \int_{x-n^{-1/2}}^{x+n^{-1/2}} + \frac{1}{n^{1/2}} \int_{x+n^{-1/2}}^{x+\epsilon} = \sum_{k=1}^{3} i_{nk}.$$

On peut appliquer aux intégrales i_{n1} et i_{n3} l'inégalité (29) tandis que i_{n2} exige l'application de l'inégalité (31):

$$i_{n2} = O\left[n^{1/2} \int_{x-n^{-1/2}}^{x+n^{-1/2}} du \right] = O(1),$$

et de même $i_{n1} = O(1)$, $i_{n3} = O(1)$, car par exemple

$$i_{n1} = \frac{1}{n^{1/2}} O\left[n^{(1-\delta)/2} \int_{n^{-1/2}}^{\infty} \frac{du}{u^{\delta+1}} \right] = O(1).$$

Par conséquent, on a établi que $i_n = O(1)$, d'où

$$\frac{1}{n^{1/2}} \int_{x-\epsilon}^{x+\epsilon} e^{-u^2} \phi(u) S_n^{(\delta)}(u, x) du = O[i_n \theta(\epsilon)] = O[\theta(\epsilon)] \leqq \eta$$

pourvu que ϵ soit assez petit: $\epsilon \leqq \epsilon_0 = \epsilon_0(\eta)$.

Observons encore que grâce à l'orthogonalité des polynomes d'Hermite on obtient

$$A_n^{(\delta)} \int_{-\infty}^{\infty} e^{-u^2} S_n^{(\delta)}(u, x)du = - \sum_0^n \frac{A_{n-m}^{(\delta)} H_m(x)}{2^m m! \pi^{1/2}} \int_{-\infty}^{\infty} e^{-u^2} H_{m+1}(u)du = 0,$$

ce qui permet de poser pour $\delta \geqq 0$

$$j_n^{(\delta)}(x) = \int_x^{\infty} e^{-u^2} S_n^{(\delta)}(u, x)du = - \int_{-\infty}^{x} e^{-u^2} S_n^{(\delta)}(u, x)du.$$

Les remarques faites conduisent à la conclusion suivante:

$$\lim_{n=\infty} I_3 = [f(x+0) - f(x-0)] \lim_{n=\infty} \frac{j_n^{(\delta)}(x)}{n^{1/2}}.$$

Pour calculer la limite figurant au second membre il suffit d'utiliser la définition même du polynome d'Hermite:

$$- \int_x^{\infty} e^{-u^2} H_{m+1}(u)du = - \left| \frac{d^m e^{-u^2}}{dx^m} \right|_x^{\infty} = e^{-x^2} H_m(x),$$

et par conséquent

$$j_n^{(\delta)}(x) = \frac{e^{-x^2}}{A_n^{(\delta)}} \sum_0^n A_{n-m}^{(\delta)} \frac{H_m^2(x)}{2^m m! \pi^{1/2}} = \frac{e^{-x^2} \sigma_n^{(\delta)}(x, x)}{A_n^{(\delta)}}$$

où $\sigma_n^{(\delta)}(u, x)$ désigne la $n$ième sigma-somme d'ordre δ de la série-noyau de développement (1). Or, on a

$$\sum_0^{\infty} \frac{H_m(x)H_m(u)}{2^m m! \pi^{1/2}} z^m = \frac{\exp[-(u^2z^2 - 2uxz + x^2z^2)/(1-z^2)]}{\pi^{1/2}(1-z^2)^{1/2}}$$

d'où

$$\pi^{1/2} \sum_0^{\infty} \sigma_n^{(\delta)}(x, x) z^n = \frac{\exp[2x^2z/(1+z)]}{(1-z)^{\delta+3/2}(1+z)^{1/2}}.$$

Vu que l'allure du second membre pour $z \to 1$ est celle de la fonction $(2\pi)^{-1/2}(1-z)^{-\delta-3/2}e^{x^2}$ tandis que pour $z \to -1$ cette allure est celle de la fonction $\pi^{-1/2}2^{-\delta-3/2}(1+z)^{-1/2} \exp[2x^2z/(1+z)]$ on peut obtenir l'expression approchée de $\sigma_n^{(\delta)}(x, x)$ en développant suivant les puissances de z la fonction auxiliaire:

$$\frac{1}{(1-z)^{\delta+3/2}} \{\phi(1) - (1-z)\phi'(1) + \cdots\}$$
$$+ \frac{\exp[2x^2z/(1+z)]}{(1+z)^{1/2}} \{\psi(-1) + (1+z)\psi'(-1) + \cdots\},$$

où l'on a posé

$$\phi(z)\pi^{1/2} = \frac{\exp\left[2x^2z/(1+z)\right]}{(1+z)^{1/2}},$$

$$\psi(z)\pi^{1/2} = \frac{1}{(1-z)^{\delta+3/2}}.$$

On trouve ainsi la formule approchée, valable pour $\delta > -1$:

$$A_n^{(\delta)} j_n^{(\delta)}(x)\pi^{1/2} = e^{-x^2}\sigma_n^{(\delta)}(x, x)\pi^{1/2} = e^{-x^2}\pi^{1/2}\{A_n^{(\delta+1/2)}\phi(1) - A_n^{(\delta-1/2)}\phi'(1)$$
$$+ \cdots + (-1)^n L_n^{(-1/2)}(2x^2)\psi(-1) + \cdots\}.$$

L'erreur étant inférieure en valeur absolue au premier terme rejeté, on en déduit pour $\delta \geqq 0$ et grâce à l'inégalité vérifiée par le polynome de Laguerre

$$j_n^{(\delta)}(x) \quad \frac{A_n^{(\delta+1/2)}}{A_n^{(\delta)}(2\pi)^{1/2}}\left\{1 + O\left(\frac{x^2}{n}\right) + O(n^{-3/4})\right\},$$

d'où enfin

$$\lim_{n=\infty} \frac{j_n^{(\delta)}(x)}{n^{1/2}} = \frac{\Gamma(\delta+1)}{\Gamma\left(\delta+\frac{3}{2}\right)(2\pi)^{1/2}} \qquad (\delta \geqq 0),$$

et cela achève la preuve de la relation (7) pour $\delta > 0$.

Pour donner un exemple considérons la série d'Hermite de la fonction $f(x) = \frac{1}{2}\operatorname{sgn}(x-a)$* qui fait le saut $D(a) = 1$, étant égale à ± 1 suivant que $x \gtrless a$:

$$\frac{1}{2}\operatorname{sgn}(x-a) = -\frac{1}{\pi^{1/2}}\int_0^a e^{-u^2}du - \frac{e^{-a^2}}{\pi^{1/2}}\sum_0^\infty \frac{H_n(a)H_{n+1}(x)}{2^{n+1}(n+1)!}.$$

Dérivée terme à terme par rapport à x elle nous donne la série-noyau du développement (1) et en désignant la somme partielle de cette série-noyau par $\sigma_n^{(0)}(u, x)$, on obtient pour $x = a$ l'expression

$$f_n^{(0)}(a) = e^{-a^2}\sigma_n^{(0)}(a, a) = e^{-a^2}\sum_0^n \frac{H_m^2(a)}{2^m m!\pi^{1/2}}.$$

La formule approchée pour $e^{-x^2}\sigma_n^{(\delta)}(x, x)$, que nous venons d'écrire, devient pour $\delta = 0$, $x = a$,

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* E. Kogbetliantz, Annales de l'Ecole Normale Supérieure, vol. 49 (1932), p. 197, (81).

$$A_n^{(1/2)}\phi(1) - A_n^{(-1/2)}\phi'(1) + \cdots + (-1)^n L_n^{(-1/2)}(2a^2)\psi(1) + \cdots$$

d'où, vu que $e^{-a^2}\phi(1)(2\pi)^{1/2}=1$,

$$f_n^{(0)}(a) = \frac{A_n^{(1/2)}}{(2\pi)^{1/2}}\left[1 + O\left(\frac{a^2}{n}\right) + O\left(n^{-3/4}\right)\right]$$

et par conséquent

$$\lim_{n=\infty} \frac{f_n^{(0)}(a)}{n^{1/2}} = \frac{1}{(2\pi)^{1/2}} \lim_{n=\infty} \frac{A_n^{(1/2)}}{n^{1/2}} = \frac{2^{1/2}}{\pi}. \tag{35}$$

Appliquons la relation (7) avec $\delta=0$. On trouve:

$$\lim_{n=\infty} \frac{f_n^{(0)}(a)}{n^{1/2}} = \frac{\Gamma(1)D(a)}{\Gamma\left(\frac{3}{2}\right)(2\pi)^{1/2}} = \frac{2^{1/2}}{\pi} D(a) \tag{36}$$

et l'on voit en comparant la relation particulière (35) à celle générale (36), qu'on a déterminé bien exactement $D(a)=1$ dans le cas $\delta=0$ pour la fonction

$$f(x) = \tfrac{1}{2}\operatorname{sgn}(x-a).$$

Nous allons maintenant étudier le cas général, où $\delta=0$. On a, en décomposant $f_n^{(0)}(x)$ d'après (32) en cinq intégrales I_k, $k=1, 2, 3, 4, 5$, le même résultat que pour $\delta>0$ en ce qui concerne les intégrales I_1 et I_5 pourvu que le produit $|u|^{-1}e^{-u^2/2}|f(u)|$ soit intégrable dans les intervalles $a\leqq|u|\leqq\infty$. De même on va voir que I_2 et I_4 tendent vers zéro quand $n\to\infty$ aussi pour $\delta=0$, si $f(x)$ est sommable (L) dans tout intervalle fini. Seulement dans ce cas, $\delta=0$, nous devons utiliser la formule approchée:

$$S_n^{(0)}(u, x) = -\frac{e^{(x^2+u^2)/2}n^{1/2}}{(u-x)\pi}\left\{\cos\left[(u-x)(2n)^{1/2}\right] + O\left(\frac{1}{n^{1/2}}\right)\right\} \tag{37}$$

valable pour toutes les valeurs finies de u et de x.

Cette formule est facile à déduire de l'expression approchée:

$$\begin{aligned} g_n(\alpha, \beta) = \frac{e^{(x^2+u^2)/2}}{\pi^{1/2}}\Bigg\{&\frac{n^{\alpha-3/4}}{d^{2\alpha-1/2}}\left[4^{-\beta}\cos\left(2dn^{1/2} - (\alpha-\tfrac{1}{4})\pi\right) + O(n^{-1/2})\right] \\ &+ (-1)^n \frac{n^{\beta-3/4}}{s^{2\beta-1/2}}\left[4^{-\alpha}\cos\left(2sn^{1/2} - (\beta-\tfrac{1}{4})\pi\right) + O(n^{-1/2})\right]\Bigg\} \end{aligned}$$

démontrée ailleurs* pour le coefficient $g_n=g_n(\alpha, \beta)$ du développement

* E. Kogbetliantz, Journal of Mathematics and Physics, loc. cit., §3, (G).

$$G_\beta^{(\alpha)}(z) = (1 - z)^{-2\alpha}(1 + z)^{-2\beta} \exp\left[- d^2z/(1 - z) + s^2z/(1 + z)\right] = \sum_0^\infty g_n z^n.$$

En effet, on a, en comparant la fonction génératrice de la suite $S_n^{(0)}(u, x)$

$$\sum_0^\infty z^n S_n^{(0)}(u, x) = \frac{2(u - xz) \exp\left[- d^2z/(1 - z) + s^2z/(1 + z)\right]}{(1 - z)^{5/2}(1 + z)^{3/2}\pi^{1/2}}$$

où $2^{1/2}d = |u - x|$, $2^{1/2}s = u + x$, à la fonction $G_\beta^{(\alpha)}(z)$, on trouve

$$\pi^{1/2} \sum_0^\infty z^n S_n^{(0)}(u, x) = (u - x)G_{1/4}^{(5/4)}(z) + (u + x)G_{3/4}^{(3/4)}(z)$$

d'où la formule (37), car

$$\pi^{1/2} S_n^{(0)}(u, x) = (u - x)g_n\left(\frac{5}{4}, \frac{1}{4}\right) + (u + x)g_n\left(\frac{3}{4}, \frac{3}{4}\right).$$

On a, en effet, grâce à (37) et pour $\delta = 0$,

$$I_4 = \frac{1}{n^{1/2}} \int_{x+\epsilon}^{A} e^{-u^2} S_n^{(0)}(u, x) f(u) du$$

$$= - \frac{e^{x^2/2}}{\pi} \int_{x+\epsilon}^{A} e^{-u^2/2} f(u) \cos\left[(u - x)(2n)^{1/2}\right] \frac{du}{u - x}$$

$$+ O\left[\frac{1}{n^{1/2}} \int_{x+\epsilon}^{A} |f(u)|\, du\right],$$

d'où

$$\lim_{n=\infty} I_4 = 0,$$

vu que $f(u)$ est sommable (L) dans tout intervalle fini. On trouve de même

$$\lim_{n=\infty} I_2 = 0$$

et il ne nous reste qu'à étudier I_3, car

$$\lim_{n=\infty} \frac{f_n^{(0)}(x)}{n^{1/2}} = \lim_{n=\infty} I_3 = \lim_{n=\infty} \frac{1}{n^{1/2}} \int_{x-\epsilon}^{x+\epsilon} e^{-u^2} S_n^{(0)}(u, x) f(u) du.$$

En décomposant I_3 d'après (34), occupons nous d'abord du premier terme de cette décomposition, à savoir

$$\tau_1 = \frac{1}{n^{1/2}} \int_{x-\epsilon}^{x+\epsilon} \phi(u) e^{-u^2} S_n^{(0)}(u, x) du$$

où dans l'intervalle $|u - x| \leqq \epsilon$ on a pour $\epsilon \to 0$

$$|\phi(u)| = |f(u) - f(x + o \operatorname{sgn}(u - x))| \leqq \theta(\epsilon) \to 0.$$

L'inégalité (29) qui pour $\delta = 0$ s'écrit

$$S_n^{(0)}(u, x) = O\left(\frac{e^{(u^2+x^2)/2}n^{1/2}}{|u - x|}\right)$$

prouve que l'on a pour ϵ suffisamment petit

$$|\tau_1| \leqq O\left\{\int_{x-\epsilon}^{x+\epsilon} |\phi(u)| \frac{du}{|u - x|}\right\} < \eta$$

si le produit $|(u-x)^{-1}\phi(u)|$ est intégrable au voisinage du point $u=x$. Cette condition que l'on peut noter,

$$\int_{-\epsilon}^{+\epsilon} |f(x + u) - f(x + o \operatorname{sgn} u)| \frac{du}{|u|} < G, \tag{38}$$

et qui est suffisante pour le moment, est vérifiée, par exemple, si l'on a pour $h \to 0$

$$f(x + h) - f\left(x + o \frac{h}{|h|}\right) = O\left[\left(\log \frac{1}{|h|}\right)^{-(1+\rho)}\right]$$

le nombre fixe ρ étant aussi petit qu'on veut, mais positif.

Etudions ensuite le dernier terme I_6 dans (34). Pour $\delta = 0$ ce terme s'écrit

$$I_6 = \frac{f(x + 0)}{n^{1/2}} \int_{x+\epsilon}^{\infty} e^{-u^2} S_n^{(0)}(u, x) du + \frac{f(x - 0)}{n^{1/2}} \int_{-\infty}^{x-\epsilon} e^{-u^2} S_n^{(0)}(u, x) du.$$

Les inégalités (29) et (30) éliminent immédiatement les parties infinies $(-\infty, -A)$ et (A, ∞) tandis que pour $|u| \leqq A$, $|u-x| \geqq \epsilon$ on peut utiliser la formule approchée (37). On démontre ainsi que l'on a

$$\lim_{n=\infty} I_6 = 0.$$

On a ainsi

$$\lim_{n=\infty} \frac{f_n^{(0)}(x)}{n^{1/2}} = [f(x + 0) - f(x - 0)] \lim_{n=\infty} \frac{j_n^{(0)}(x)}{n^{1/2}} = \frac{2^{1/2}}{\pi} D(x),$$

ce résultat étant acquis dans les hypothèses suivantes:

(I) $f(x)$ est sommable (L) dans tout intervalle fini;

(II) $f(x)$ vérifie la condition (33) avec $\delta = 0$;

(III) $f(x)$ vérifie la condition (38).

La condition (38) peut d'ailleurs être remplacée par une autre, si l'allure

de $f(x)$ au voisinage immédiat du point x est telle qu'en posant pour $u=x+t$, $|t|\leqq\epsilon$,

$$t\phi(x+t) = t\psi(t) = t[f(x+t) - f(x+o\operatorname{sgn} t)] = \int_0^t \chi(u)du$$

on puisse définir une fonction $\chi(t)$ vérifiant la condition

$$\int_0^t |\chi(u)|\, du \leqq At \tag{39}$$

où A est une constante positive aussi grande qu'on veut mais fixe. Supposant remplie la condition (39), donnons nous un nombre fixe θ aussi petit qu'on veut et positif et soit N assez grand pour avoir $3A\cdot 2^{1/2}<N\theta$.

La formule approchée (37) et l'inégalité

$$|S_n^{(0)}(u, x)| \leqq \sum_0^n \frac{|H_m(x)H_{m+1}(u)|}{2^m m!\pi^{1/2}} = O(ne^{(u^2+x^2)/2}) \tag{40}$$

permettent d'écrire en décomposant τ_1 en trois termes:

$$\begin{aligned}\tau_1 &= \frac{1}{n^{1/2}}\left\{\int_{x-\epsilon}^{x-N/n^{1/2}} + \int_{x-N/n^{1/2}}^{x+N/n^{1/2}} + \int_{x+N/n^{1/2}}^{x+\epsilon}\right\}\phi(u)e^{-u^2}S_n^{(0)}(u, x)du \\ &= \tau' + \tau'' + \tau'''.\end{aligned}$$

Grâce à (40) on a immédiatement

$$|\tau''| \leqq \frac{1}{n^{1/2}} \max_{|u-x|\leqq N/n^{1/2}} |\phi(u)|\, O(n)\, \frac{2N}{n^{1/2}} = O\left\{\max_{|u-x|\leqq N/n^{1/2}} |\phi(u)|\right\}$$

donc

$$\lim_{n=\infty} \tau'' = 0,$$

car N est fixe. Ensuite la substitution $u=x-t$ nous donne

$$\begin{aligned}\tau''' &= -\frac{1}{\pi}\int_{N/n^{1/2}}^{\epsilon} e^{xt-t^2/2}\psi(t)\cos(t(2n)^{1/2})\frac{dt}{t} + O\left[\frac{\log n}{n^{1/2}}\right] \\ &= -\frac{1}{\pi}\int_{N/n^{1/2}}^{\epsilon} \cos(t(2n)^{1/2})\psi(t)\frac{dt}{t} + o(1) + o(\epsilon)\end{aligned}$$

et il suffit de prouver que le premier terme peut être rendu aussi petit qu'on veut en valeur absolue si $\chi(t)$ vérifie la condition (39). Intégrons le par parties:

$$-\frac{1}{\pi}\int_{N/n^{1/2}}^{\epsilon} \frac{\cos(t(2n)^{1/2})}{t^2}\, t\psi(t)dt = \frac{N}{n^{1/2}}\,\psi\left(\frac{N}{n^{1/2}}\right)\vartheta_n\left(\frac{N}{n^{1/2}}\right) - \int_{N/n^{1/2}}^{\epsilon} \vartheta_n(t)\chi(t)dt$$

en posant

$$\vartheta_n(t) = \int_t^{\epsilon} u^{-2} \cos (u(2n)^{1/2})du.$$

Mais, il est évident que la fonction $\vartheta_n(t)$ vérifie l'inégalité $|\vartheta_n(t)| \leqq n^{-1/2}t^{-2}2^{1/2}$, car

$$\vartheta_n(t) = \frac{1}{t^2}\int_t^{\xi} \cos (t(2n)^{1/2})dt = \frac{\sin (\xi(2n)^{1/2}) - \sin (t(2n)^{1/2})}{t^2(2n)^{1/2}}.$$

Par conséquent, tenant compte du fait que $\psi(Nn^{-1/2})$ tend vers zéro quand $n\to\infty$, on trouve que la partie intégrée est $o(1)$ pour $n\to\infty$:

$$\frac{N}{n^{1/2}}\psi\left(\frac{N}{n^{1/2}}\right)\vartheta_n\left(\frac{N}{n^{1/2}}\right) = o\left(\frac{Nn2^{1/2}}{n^{1/2}N^2n^{1/2}}\right) = o\left(\frac{1}{N}\right) = o(1).$$

Quant à l'intégrale, on peut écrire

$$\begin{aligned}\left|\int_{N/n^{1/2}}^{\epsilon}\vartheta_n(t)\chi(t)dt\right| &\leqq \left(\frac{2}{n}\right)^{1/2}\int_{N/n^{1/2}}^{\epsilon} t^{-2}\,|\,\chi(t)\,|\,dt \\ &= \left(\frac{2}{n}\right)^{1/2}\left\{\left|\frac{1}{t^2}\int_0^t |\,\chi(u)\,|\,du\right|_{N/n^{1/2}}^{\epsilon}\right. \\ &\qquad \left. + 2\int_{N/n^{1/2}}^{\epsilon}\frac{dt}{t^3}\int_0^t |\,\chi(u)\,|\,du\right\} \\ &\leqq O(n^{-1/2}) + A\left(\frac{2}{n}\right)^{1/2}\frac{n^{1/2}}{N} + 2\left(\frac{2}{n}\right)^{1/2}A\int_{N/n^{1/2}}^{\epsilon}\frac{dt}{t^2} \\ &= o(1) + \frac{3A2^{1/2}}{N} < o(1) + \theta < 2\theta,\end{aligned}$$

pour n suffisamment grand.

3. Sommation (C, δ) de la série (10)

Considérons la série-noyau de (10),

$$(41)\qquad -\sum_0^{\infty}\frac{\Gamma(n+2)}{\Gamma(n+\alpha+2)}L_n^{(\alpha+1)}(x)L_{n+1}^{(\alpha)}(u) \sim 0 \qquad (u \neq x),$$

et désignons par $S_n^{(\delta)}(u, x)$ ses moyennes arithmétiques d'ordre δ. Celles $f_n^{(\delta)}(x)$ de la série

$$(10)\qquad -\sum_0^{\infty}\frac{\Gamma(n+2)L_n^{(\alpha+1)}(x)}{\Gamma(n+\alpha+2)}\int_0^{\infty}L_{n+1}^{(\alpha)}(u)e^{-u}u^{\alpha}f(u)du$$

s'expriment à l'aide des $S_n^{(\delta)}(u, x)$:

$$f_n^{(\delta)}(x) = \int_0^\infty e^{-u} u^\alpha S_n^{(\delta)}(u, x) f(u) du. \tag{42}$$

Les moyennes $S_n^{(\delta)}(u, x)$ vérifient* les inégalités

$$S_n^{(\delta)}(u, x) = O\left\{\frac{e^{(x+u)/2} n^{(1-\delta)/2}}{(ux)^{\alpha/2+1/4} \mid u^{1/2} - x^{1/2} \mid^{\delta+1} x^{1/2}}\right\} \tag{43}$$

$$(0 \leqq u \leqq \infty,\ \delta \geqq 0,\ 0 \leqq x \leqq a),$$

$$S_n^{(\delta)}(u, x) = O\left(\frac{e^{(u+x)/2} n^{1/2}}{x^{\alpha/2+3/4} u^{\alpha/2+3/4+\delta}}\right) \quad (u \geqq e^2 n,\ \delta \geqq 0,\ 0 \leqq x \leqq a), \tag{44}$$

ainsi que celle

$$S_n^{(\delta)}(u, x) = O\left(\frac{e^{(u+x)/2} n}{(ux)^{\alpha/2+1/4} x^{1/2}}\right) \quad (\delta \geqq 0), \tag{45}$$

dont voici la preuve pour u et x quelconques et $\delta \geqq 0$:

$$\begin{aligned} | S_n^{(\delta)}(u, x) | &\leqq [A_n^{(\delta)}]^{-1} \sum_0^n A_{n-m}^{(\delta)} \frac{\Gamma(m+2)}{\Gamma(m+\alpha+2)} | L_m^{(\alpha+1)}(x) L_{m+1}^{(\alpha)}(u) | \\ &= O\left\{ n^{-\delta} \sum_0^n A_{n-m}^{(\delta)} (m+1)^{-\alpha} e^{(x+u)/2} (m+1)^\alpha x^{-\alpha/2-3/4} u^{-\alpha/2-1/4} \right\} \\ &= O\{ n^{-\delta} e^{(x+u)/2} (xu)^{-\alpha/2-3/4} u^{1/2} A_n^{(\delta+1)} \} = O\left\{ \frac{e^{(x+u)/2} n}{(ux)^{\alpha/2+1/4} x^{1/2}} \right\}. \end{aligned}$$

Décomposons dans (42) l'intervalle d'intégration $(0, \infty)$ ainsi:

$$\begin{aligned} \frac{f_n^{(\delta)}(x)}{n^{1/2}} = \frac{1}{n^{1/2}} \int_0^\infty &= (0, \infty) = \sum_{k=1}^7 J_k = \left(0, \frac{1}{n}\right) + \left(\frac{1}{n}, \eta\right) \\ &+ [\eta, x(1-\epsilon)^2] + [x(1-\epsilon)^2, x(1+\epsilon)^2] + [x(1+\epsilon)^2, A] \\ &+ [A, e^2 n] + [e^2 n, \infty] \end{aligned} \tag{46}$$

et supposons que $f(x)$, sommable (L) dans tout intervalle fini, vérifie en outre les deux conditions suivantes:

$$\int_a^\infty e^{-u/2} u^{\alpha/2-3/4-\delta} | f(u) | du < G, \tag{47}$$

$$\int_0^\epsilon u^\beta | f(u) | du < G, \tag{48}$$

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* Voir E. Kogbetliantz, Journal of Mathematics and Physics, loc. cit., §5, (BL) et (F), dont on déduit facilement (44).

où le nombre β est égal au plus petit des deux nombres α et $(\alpha+\delta)/2-\frac{1}{4}$, donc

$$\beta = \begin{cases} \dfrac{\alpha+\delta}{2} - \dfrac{1}{4} & \text{pour } \delta \leqq \alpha + \dfrac{1}{2}, \\ \alpha & \text{pour } \delta \geqq \alpha + \dfrac{1}{2}. \end{cases}$$

On a vu déjà au §1 que la sigma-somme d'ordre δ, $\sigma_n^{(\delta)}(u, x)$, de la série-noyau (41) vérifie pour $0 \leqq u \leqq 1/n$ l'inégalité

$$\sigma_n^{(\delta)}(u, x) = O(n^{(\alpha+\delta)/2+3/4}) \qquad \left(0 \leqq u \leqq \frac{1}{n}\,;\ \delta \geqq 0\right).$$

On en déduit pour $S_n^{(\delta)}(u, x)$ l'inégalité correspondante:

$$S_n^{(\delta)}(u, x) = \frac{\sigma_n^{(\delta)}}{A_n^{(\delta)}} = O(n^{(\alpha-\delta)/2+3/4}),$$

ce qui permet de conclure pour $\delta \leqq \alpha + \frac{1}{2}$ ainsi:

$$\begin{aligned} J_1 &= O\left\{\frac{1}{n^{1/2}} \int_0^{1/n} e^{-u} u^\alpha \,|\, f(u) \,|\, n^{(\alpha-\delta)/2+3/4} du\right\} \\ &= O\left\{\int_0^{1/n} (nu)^{(\alpha-\delta)/2+1/4} u^{(\alpha+\delta)/2-1/4} \,|\, f(u) \,|\, du\right\} \\ &= O\left\{\int_0^{1/n} u^{(\alpha+\delta)/2-1/4} \,|\, f(u) \,|\, du\right\}. \end{aligned}$$

Au contraire, pour $\delta > \alpha + \frac{1}{2}$ on trouve:

$$J_1 = O\left\{n^{(\alpha-\delta)/2+1/4} \int_0^{1/n} u^\alpha \,|\, f(u) \,|\, du\right\} = o\left\{\int_0^{1/n} u^\alpha \,|\, f(u) \,|\, du\right\}.$$

Dans l'intervalle $(1/n, \eta)$ nous avons d'après (43):

$$S_n^{(\delta)}(u, x) = O\left\{\frac{n^{(1-\delta)/2}}{(ux)^{\alpha/2+1/4} x^{(\delta+1)/2}}\right\} \qquad \left(\frac{1}{n} \leqq u \leqq \eta\right),$$

d'où, x étant fixe et positif:

$$\begin{aligned} J_2 &= O\left\{\frac{n^{(1-\delta)/2}}{n^{1/2}} \int_{1/n}^{\eta} u^{\alpha/2-1/4} \,|\, f(u) \,|\, du\right\} \\ &= O\left\{\int_{1/n}^{\eta} (nu)^{-\delta/2} u^{(\alpha+\delta)/2-1/4} \,|\, f(u) \,|\, du\right\} = O\left\{\int_{1/n}^{\eta} u^{(\alpha+\delta)/2-1/4} \,|\, f(u) \,|\, du\right\}. \end{aligned}$$

Par conséquent, quelque soit $\delta \geqq 0$ on trouve que

$$J_1 + J_2 = O\left\{\int_0^\eta u^\beta \,|\, f(u) \,|\, du\right\}$$

et l'hypothèse (48) entraine la possibilité d'obtenir l'inégalité

$$(49) \qquad |J_1 + J_2| \leqq \omega$$

quelque petit que soit $\omega > 0$, en choisissant η suffisamment petit: $\eta \leqq \eta_0 = \eta_0(\omega)$.

Pour évaluer les intégrales J_3 et J_5 nous allons distinguer deux cas: $\delta > 0$ et $\delta = 0$. Dans le premier cas l'inégalité (43) donne immédiatement

$$J_3 = O\left\{\frac{1}{n^{1/2}}\int_\eta^{(1-\epsilon)^2 x} n^{(1-\delta)/2} \,|\, f(u) \,|\, du\right\} = O(n^{-\delta/2})$$

et de même pour J_5; donc pour $\delta > 0$ on a

$$(50) \qquad \lim_{n=\infty} J_3 = \lim_{n=\infty} J_5 = 0 \qquad (\delta \geqq 0).$$

Dans le second cas, $\delta = 0$, on est obligé d'employer la formule approchée*

$$(51) \qquad S_n^{(0)}(u, x) = -\frac{e^{(x+u)/2} n^{1/2}[\cos(2(u^{1/2} - x^{1/2})n^{1/2}) + O(n^{-1/2})]}{2\pi u^{\alpha/2+1/4} x^{\alpha/2+3/4}(u^{1/2} - x^{1/2})}.$$

Il suffit de considérer J_3 et grâce à cette formule on a, après la substitution $u^{1/2} = (1-t)x^{1/2}$:

$$J_3 = \frac{1}{\pi x^{1/2}}\int_\epsilon^{1-(\eta/x)^{1/2}} \Psi(x, t) \cos(2t(nx)^{1/2})dt + O(n^{-1/2}),$$

où

$$\Psi(x, t) = e^{xt - t^2/2}(1 - t)^{\alpha+1/2} t^{-1} f[x(1 - t)^2]$$

est sommable (L) dans l'intervalle $[\epsilon, 1 - (\eta/x)^{1/2}]$.

On justifie, par conséquent, le résultat (50) aussi pour $\delta = 0$.

Dans l'intervalle $(A, e^2 n)$, on obtient à l'aide de l'inégalité (43)

$$J_6 = O\left\{n^{-\delta/2}\int_A^{e^2 n} e^{-u/2} u^{(\alpha-\delta)/2-3/4} \,|\, f(u) \,|\, du\right\}$$

$$= O\left\{\int_A^{e^2 n} e^{-u/2} u^{\alpha/2-\delta-3/4} \,|\, f(u) \,|\, du\right\}$$

tandis que l'inégalité (44) entraîne

* Voir E. Kogbetliantz, Journal of Mathematics and Physics, loc. cit., §6, (21), où $S_n^{(\delta)}$ est désigné par $\partial L_n^{(\delta)}/\partial x$.

$$J_7 = O\left\{ \int_{\epsilon^2 n}^{\infty} e^{-u/2} u^{\alpha/2-\delta-3/4} \mid f(u) \mid du \right\}.$$

L'hypothèse (47) permet ainsi d'obtenir l'inégalité

$$J_6 + J_7 = O\left\{ \int_A^{\infty} e^{-u/2} u^{\alpha/2-\delta-3/4} \mid f(u) \mid du \right\} \leqq \omega$$

quelque petit que soit $\omega > 0$, en choisissant le nombre fixe A suffisamment grand: $A \geqq A_0 = A_0(\omega)$.

Ce dernier résultat joint à (49) et (50) permet de conclure

$$\lim_{n=\infty} \frac{f_n^{(\delta)}(x)}{n^{1/2}} = \lim_{n=\infty} J_4 = \lim_{n=\infty} \frac{1}{n^{1/2}} \int_{x(1-\epsilon)^2}^{x(1+\epsilon)^2} e^{-u} u^{\alpha} S_n^{(\delta)}(u, x) f(u) du.$$

Soit

$$\phi(u) = f(u) - f(x + o \text{ sgn } (u - x)).$$

Pour évaluer la limite de l'intégrale J_4 nous la présentons ainsi:

$$J_4 = \frac{f(x+0)}{n^{1/2}} \int_x^{\infty} e^{-u} u^{\alpha} S_n^{(\delta)}(u, x) du + \frac{f(x-0)}{n^{1/2}} \int_0^x e^{-u} u^{\alpha} S_n^{(\delta)}(u, x) du$$
$$+ \frac{1}{n^{1/2}} \int_{x(1-\epsilon)^2}^{x(1+\epsilon)^2} \phi(u) e^{-u} u^{\alpha} S_n^{(\delta)}(u, x) du - J_8,$$

où

$$J_8 = \frac{f(x+0)}{n^{1/2}} \int_{x(1+\epsilon)^2}^{\infty} e^{-u} u^{\alpha} S_n^{(\delta)}(u, x) du + \frac{f(x-0)}{n^{1/2}} \int_0^{x(1-\epsilon)^2} e^{-u} u^{\alpha} S_n^{(\delta)}(u, x) du.$$

Observons qu'en vertu de l'orthogonalité des polynomes de Laguerre on a

$$\int_0^{\infty} e^{-u} u^{\alpha} S_n^{(\delta)}(u, x) du$$
$$= \sum_0^n \frac{\Gamma(m+2)}{\Gamma(m+\alpha+2)} \cdot \frac{A_{n-m}^{(\delta)}}{A_n^{(\delta)}} \cdot L_m^{(\alpha+1)}(x) \int_0^{\infty} e^{-u} u^{\alpha} L_{m+1}^{(\alpha)}(u) du = 0,$$

ce qui permet de définir $\lambda_n^{(\delta)}(x)$ ainsi:

$$\lambda_n^{(\delta)}(x) = -\frac{1}{n^{1/2}} \int_0^x e^{-u} u^{\alpha} S_n^{(\delta)}(u, x) du = \frac{1}{n^{1/2}} \int_x^{\infty} e^{-u} u^{\alpha} S_n^{(\delta)}(u, x) du.$$

La preuve du résultat cherché, à savoir:

$$\lim_{n=\infty} \frac{f_n^{(\delta)}(x)}{n^{1/2}} = \frac{\Gamma(\delta+1)}{\Gamma\left(\delta+\frac{3}{2}\right)} \cdot \frac{D(x)}{2(\pi x)^{1/2}} \qquad (\delta \geqq 0), \tag{11}$$

exige la détermination de la limite

$$l = \lim_{n=\infty} \lambda_n^{(\delta)}(x)$$

et l'étude des deux intégrales J_8 et J_9, la dernière étant définie par

$$J_9 = \frac{1}{n^{1/2}} \int_{x(1-\epsilon)^2}^{x(1+\epsilon)^2} e^{-u} u^{\alpha} S_n^{(\delta)}(u, x)\phi(u)du.$$

Vu que la borne supérieure $\theta(\epsilon)$ de la valeur absolue de $\phi(u)$ dans l'intervalle $[x(1-\epsilon)^2, x(1+\epsilon)^2]$ tend vers zéro avec ϵ, on a dans le cas $\delta>0$ et grâce aux inégalités (43) et (45) le résultat

$$J_9 = O[\theta(\epsilon)] < \omega \tag{52}$$

si ϵ est suffisamment petit: $\epsilon \leqq \epsilon_0 = \epsilon_0(\omega)$.

En effet, décomposons l'intervalle d'intégration en trois intervalles:

$$\begin{aligned} J_9 = \frac{1}{n^{1/2}} \int_{x(1-\epsilon)^2}^{x(1+\epsilon)^2} = [x(1-\epsilon)^2, x(1+\epsilon)^2] &= \left[x(1-\epsilon)^2, x\left(1-\frac{1}{n^{1/2}}\right)^2\right] \\ &+ \left[x\left(1-\frac{1}{n^{1/2}}\right)^2, x\left(1+\frac{1}{n^{1/2}}\right)^2\right] \\ &+ \left[x\left(1+\frac{1}{n^{1/2}}\right)^2, x(1+\epsilon)^2\right] \\ &= \sum_1^3 i_k. \end{aligned}$$

L'inégalité (45) nous donne, étant donné que $|\phi(u)| \leqq \theta(\epsilon)$,

$$|i_2| \leqq \frac{\theta(\epsilon)}{n^{1/2}} \int_{x(1-n^{-1/2})^2}^{x(1+n^{-1/2})^2} O(n)du = O[\theta(\epsilon)],$$

tandis que celle (43) appliquée aux intégrales i_1 et i_3 conduit à écrire, en employant la substitution $u^{1/2} = (1+t)x^{1/2}$,

$$i_1 + i_3 = O\left[n^{-\delta/2} \int_{n^{-1/2}}^{\epsilon} \theta(\epsilon) \frac{dt}{t^{\delta+1}}\right] = O[\theta(\epsilon)] \qquad (\delta > 0),$$

ce qui achève la preuve de (52) pour $\delta>0$.

De même le terme J_8 est facile à évaluer, si $\delta>0$. L'inégalité (43) donne en effet pour $\delta>0$

$$J_8 = O\left[n^{-\delta/2}\int_0^\infty e^{-u/2}u^{\alpha/2-1/4}du\right] = O(n^{-\delta/2})$$

donc on a démontré le résultat cherché,

$$\lim_{n=\infty} J_8 = 0 \qquad (\delta \geqq 0) \tag{53}$$

pour $\delta>0$. C'est le cas $\delta=0$ qui exige une analyse plus approfondie basée sur la formule approchée (51). Soit donc $\delta=0$ et supposons qu'au voisinage immédiat $|u-x| \leqq \gamma$ du point $u=x$ la fonction $f(u)$ vérifie la condition (38), c'est à dire l'intégrale définie

$$\int_{x-\gamma}^{x+\gamma} |\phi(u)| \frac{du}{|u|} < G \tag{38}$$

existe. Dans cette hypothèse (38) et grâce à l'inégalité (43) on obtient immédiatement

$$J_9 = O\left\{\int_{x(1-\epsilon)^2}^{x(1+\epsilon)^2} |\phi(u)| \frac{du}{|u|}\right\} < \omega,$$

quelque petit que soit $\omega>0$ pourvu que ϵ soit assez petit: $\epsilon \leqq \epsilon_0 = \epsilon_0(\omega)$. Quant au terme J_8 on trouve grâce à (51), en posant $(u^{1/2}-x^{1/2})\xi(u)e^{u/2} = u^{\alpha/2-1/4}$:

$$\begin{aligned} 2\pi J_8 = &- x^{-\alpha/2-3/4}e^{x/2}\left\{f(x+0)\int_{x(1+\epsilon)^2}^{A} \xi(u)\cos\left[2(u^{1/2}-x^{1/2})n^{1/2}\right]du\right. \\ &\left. + f(x-0)\int_0^{x(1-\epsilon)^2} \xi(u)\cos\left[2(u^{1/2}-x^{1/2})n^{1/2}\right]du\right\} + O(n^{-1/2}) \\ &+ O\left[n^{-1/2}\int_A^\infty e^{-u}u^{\alpha}|S_n^{(0)}(u,x)|\,du\right]. \end{aligned}$$

Vu que $\xi(u)$ est sommable (L) dans les intervalles $[0,\ x(1-\epsilon)^2]$ et $[x(1+\epsilon)^2,\ A]$ on constate que les termes entre les parenthèses tendent vers zéro quand $n\to\infty$.

Quant au dernier terme, les inégalités (43) et (44) permettent de l'écrire ainsi:

$$O\left\{n^{-1/2}\int_A^\infty e^{-u}u^{\alpha}|S_n^{(0)}(u,x)|\,du\right\} = O\left\{\int_A^\infty e^{-u/2}u^{\alpha/2-\delta-3/4}du\right\} < \omega$$

pourvu que le nombre fixe A soit assez grand: $A \geqq A_0(\omega)$.

On parvient ainsi dans le cas $\delta=0$ sous l'unique condition (38) au résultat:

$$\lim_{n=\infty} \frac{f_n^{(\delta)}(x)}{n^{1/2}} = [f(x+0) - f(x-0)] \lim_{n=\infty} \lambda_n^{(\delta)}(x) \qquad (\delta \geqq 0) \tag{54}$$

démontré déjà pour $\delta > 0$.

Observons que la condition (38) peut être remplacée par celle (39). Supposons que l'allure de $f(u)$ au voisinage du point x permet de définir une fonction $\chi(t)$ vérifiant la condition

$$\int_0^t |\chi(u)| du \leqq At \tag{39}$$

et telle que

$$t\phi(x+t) = t\left\{f(x+t) - f\left(x + o\frac{t}{|t|}\right)\right\} = \int_0^t \chi(u)du.$$

Décomposons l'intervalle d'intégration dans

$$J_9 = \frac{1}{n^{1/2}} \int_{x(1-\epsilon)^2}^{x(1+\epsilon)^2} e^{-u} u^{\alpha} S_n^{(\delta)}(u, x)\phi(u)du$$

en trois intervalles partiels ainsi:

$$\begin{aligned} J_9 &= [x(1-\epsilon)^2, x(1+\epsilon)^2] \\ &= \left[x(1-\epsilon)^2, x\left(1 - \frac{N}{n^{1/2}}\right)^2\right] + \left[x\left(1 - \frac{N}{n^{1/2}}\right)^2, x\left(1 + \frac{N}{n^{1/2}}\right)^2\right] \\ &\quad + \left[x\left(1 + \frac{N}{n^{1/2}}\right)^2, x(1+\epsilon)^2\right] = j_1 + j_2 + j_3, \end{aligned}$$

le nombre N étant suffisamment grand pour avoir $4A < \pi x \omega N$, où ω est une quantité positive choisie d'avance et aussi petite qu'on veut. L'intégrale j_2 est facile à évaluer à l'aide de l'inégalité (45). On obtient ainsi, tenant compte du fait que la borne supérieure $\theta(\epsilon)$ de $|\phi(u)|$ dans l'intervalle $[x(1-\epsilon)^2, x(1+\epsilon)^2]$ tend vers zéro avec ϵ,

$$j_2 = O\left[n^{-1/2} n\theta\left(\frac{N}{n^{1/2}}\right) \int_{x(1-Nn^{-1/2})^2}^{x(1+Nn^{-1/2})^2} du\right] = O\left[\theta\left(\frac{N}{n^{1/2}}\right)\right] = o(1),$$

c'est à dire

$$\lim_{n=\infty} j_2 = 0.$$

Il suffit de considérer ensuite j_3, car le même raisonnement s'applique à j_1. A l'aide de la formule approchée (51) et de la substitution $u^{1/2} = (1+t)x^{1/2}$ on donne à j_3 la forme suivante:

$$j_3 = -\frac{1}{\pi x^{1/2}}\int_{Nn^{-1/2}}^{\epsilon} [1 + O(t)]\phi[x(1+t)^2]\cos(2t(nx)^{1/2})\frac{dt}{t} + O\left(\frac{\log n}{n}\right)$$

$$= -\frac{1}{\pi x^{1/2}}\int_{Nn^{-1/2}}^{\epsilon} \phi[x(1+t)^2]\cos(2t(nx)^{1/2})\frac{dt}{t} + O\left(\frac{\log n}{n}\right) + O[\epsilon\theta(\epsilon)].$$

Or,

$$xt(2+t)\phi(x + 2xt + xt^2) = \int_0^{xt(2+t)} \chi(u)du$$

entraîne grâce à la condition (39) que nous supposons remplie

$$\phi[x(1+t)^2] = \frac{1}{2xt}\int_0^{xt(2+t)} \chi(u)du + O(t)$$

d'où par conséquent

$$-\frac{1}{\pi x^{1/2}}\int_{Nn^{-1/2}}^{\epsilon} \phi[x(1+t)^2]\cos(2t(nx)^{1/2})\frac{dt}{t}$$

$$= -\frac{1}{2\pi x^{3/2}}\int_{Nn^{-1/2}}^{\epsilon} \cos(2t(nx)^{1/2})\frac{dt}{t^2}\int_0^{xt(2+t)} \chi(u)du + O(\epsilon).$$

Posons

$$\vartheta_n(x, t) = -\int_t^{\epsilon} \cos(2\tau(nx)^{1/2})\frac{d\tau}{\tau^2}$$

et intégrons par parties. On trouve ainsi

$$j_3 = O\left(\frac{\log n}{n}\right) + O(\epsilon) + \frac{1}{2\pi x^{3/2}}\int_{Nn^{-1/2}}^{\epsilon} d\vartheta_n(x, t)\int_0^{xt(2+t)} \chi(u)du$$

$$= o(1) + O(\epsilon) + \frac{1}{2\pi x^{3/2}}\left\{\left|\vartheta_n(x, t)\int_0^{xt(2+t)} \chi(u)du\right|_{Nn^{-1/2}}^{\epsilon}\right.$$

$$\left. - 2x\int_{Nn^{-1/2}}^{\epsilon} \vartheta_n(x, t)\chi[xt(2+t)](1+t)dt\right\}.$$

Or, pour $t \to 0$ on a

$$\int_0^{xt(2+t)} \chi(u)du = 2xt\phi[x(1+t)^2] + O(t^2) = o(xt) \qquad (t \to 0),$$

donc pour $n \to \infty$

$$\left|\frac{\vartheta_n(x, t)}{2\pi x^{3/2}}\int_0^{xt(2+t)} \chi(u)du\right|_{t=Nn^{-1/2}} = o\left\{\frac{N}{(nx)^{1/2}}\vartheta_n\left(x, \frac{N}{n^{1/2}}\right)\right\} \qquad (n \to \infty).$$

D'ailleurs

$$|\vartheta_n(x,t)| = \frac{1}{t^2}\left|\int_t^\xi \cos(2\tau(nx)^{1/2})d\tau\right| \leqq \frac{1}{t^2(nx)^{1/2}},$$

ce qui entraîne pour $n\to\infty$

$$|j_3| \leqq O(\epsilon) + o(1) + o\left(\frac{N}{(nx)^{1/2}}\cdot\frac{n}{N^2(nx)^{1/2}}\right)$$

$$+ \frac{1+\epsilon}{\pi x^{1/2}}\int_{Nn^{-1/2}}^{\epsilon} \frac{|\chi[xu(2+u)]|\,du}{u^2(nx)^{1/2}} = O(\epsilon) + o(1) + \frac{1+\epsilon}{\pi x n^{1/2}} j_4.$$

Une nouvelle intégration par parties transforme l'intégrale j_4 en

$$j_4 = \left| t^{-2}\int_0^t |\chi[xu(2+u)]|\,du\right|_{Nn^{-1/2}}^{\epsilon} + 2\int_{Nn^{-1/2}}^{\epsilon}\frac{dt}{t^3}\int_0^t |\chi[xu(2+u)]|\,du.$$

On a, en vertu de (39) et en posant $xu(2+u)=\tau$,

$$\int_0^t |\chi[xu(2+u)]|\,du = \int_0^{xt(2+t)} |\chi(\tau)|\frac{d\tau}{2x(1+u)}$$

$$< \frac{Axt(2+t)}{2x} < (1+\epsilon)At.$$

Par conséquent:

$$|j_4| \leqq \frac{A}{\epsilon}(1+\epsilon) + \frac{1+\epsilon}{N}An^{1/2} + 2(1+\epsilon)A\int_{Nn^{-1/2}}^{\epsilon}\frac{dt}{t^2} \leqq \frac{3A(1+\epsilon)n^{1/2}}{N},$$

ce qui entraîne

$$|j_3| \leqq O(\epsilon) + o(1) + \frac{3A(1+\epsilon)^2}{\pi x N} < 2\omega$$

quelque petit que soit $\omega>0$, pourvu que ϵ soit assez petit et n assez grand.

Ce raisonnement prouve qu'au cas, où $\delta=0$, la condition (39) est suffisante pour assurer le résultat (54).

Pour achever la démonstration de (11) il suffit de calculer la limite de $\lambda_n^{(\delta)}(x)$ quand $n\to\infty$. L'identité

$$m!e^{-u}u^{\alpha}L_m^{(\alpha)}(u) = \frac{d^m}{du^m}(e^{-u}u^{m+\alpha})$$

entraîne évidemment

$$-\Gamma(m+2)\int_x^\infty e^{-u}u^{\alpha}L_{m+1}^{(\alpha)}(u)du = m!e^{-x}x^{\alpha+1}L_m^{(\alpha+1)}(x),$$

et cette relation permet d'écrire

$$\lambda_n^{(\delta)}(x) = -\frac{1}{A_n^{(\delta)}n^{1/2}}\sum_0^n A_{n-m}^{(\delta)}\frac{\Gamma(m+2)L_m^{(\alpha+1)}(x)}{\Gamma(m+\alpha+2)}\int_x^\infty e^{-u}u^\alpha L_{m+1}^{(\alpha)}(u)du$$

$$= \frac{e^{-x}x^{\alpha+1}}{A_n^{(\delta)}n^{1/2}}\sum_0^n A_{n-m}^{(\delta)}\frac{\Gamma(m+1)[L_m^{(\alpha+1)}(x)]^2}{\Gamma(m+\alpha+2)} = \frac{\mu_n^{(\delta)}}{A_n^{(\delta)}n^{1/2}}.$$

Or, la relation

$$\sum_0^\infty \frac{\Gamma(m+1)}{\Gamma(m+\alpha+2)}[L_m^{(\alpha+1)}(x)]^2 z^m = \frac{e^{-2xz/(1-z)}I_{\alpha+1}\left(\frac{2xz^{1/2}}{1-z}\right)}{(xz^{1/2})^{\alpha+1}(1-z)}$$

entraîne cette autre relation:

$$\sum_0^\infty \mu_n^{(\delta)}z^n = M_\delta(z) = z^{-(\alpha+1)/2}(1-z)^{-\delta-2}e^{-x(1+z)/(1-z)}I_{\alpha+1}\left(\frac{2xz^{1/2}}{1-z}\right).$$

L'allure du second membre pour $z\to 1$ est facile à préciser à l'aide de la formule approchée (21) et l'on obtient ainsi:

$$M_\delta(z) = \frac{\exp[-x(1-z^{1/2})/(1+z^{1/2})]}{2(\pi x)^{1/2}}\,\frac{1+O(|1-z|)}{(1-z)^{\delta+3/2}}.$$

Pour trouver une formule approchée du coefficient $\mu_n^{(\delta)}$ de la fonction $M_\delta(z)$ il suffit de développer la fonction auxiliaire

$$\frac{1}{2(\pi x)^{1/2}(1-z)^{\delta+3/2}} = \frac{1}{2(\pi x)^{1/2}}\sum_0^\infty A_n^{(\delta+1/2)}z^n,$$

d'où enfin

$$\lambda_n^{(\delta)} = \frac{\mu_n^{(\delta)}}{A_n^{(\delta)}n^{1/2}} = \frac{1}{2(\pi x n)^{1/2}}\cdot\frac{A_n^{(\delta+1/2)}}{A_n^{(\delta)}}[1+o(1)]$$

et par conséquent, vu que $\Gamma(\delta+1)\,A_n^{(\delta)} = n^\delta[1+o(1)]$,

$$\lim_{n=\infty}\lambda_n^{(\delta)}(x) = \frac{\Gamma(\delta+1)}{2(\pi x)^{1/2}\Gamma\left(\delta+\frac{3}{2}\right)},$$

ce qui achève la démonstration de (11).

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DISTRIBUTION FUNCTIONS AND THE RIEMANN ZETA FUNCTION*

BY

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1. Introduction

The present paper starts with a systematic study of distribution functions in k-dimensional space and in particular of their infinite convolutions representing, in the language of the calculus of probability, the distributions arising by addition of an infinite number of independent random variables. The results are applied to almost periodic functions and in particular to the Riemann zeta function.

The proper method in dealing with distribution functions and their convolutions ("Faltungen") is the method of Fourier transforms, first applied systematically by Lévy in his book on the calculus of probability [40].† The theorems concerning Fourier transforms which we need are collected at the beginning; for proofs we refer to papers of Bochner [2] and Haviland [28, 29]. These authors use Riemann-Radon integrals; we prefer for several reasons to work with Lebesgue-Radon integrals for which the proofs are simpler. Using these results on Fourier transforms we develop a general theory of infinite convolutions and in particular their convergence theory. This theory is completed at the end of the paper, where it is shown, by means of integrals in infinitely many dimensions, that the convergence problem of infinite convolutions is identical with the convergence problem of infinite series the terms of which are independent random variables, as considered by Khintchine and Kolmogoroff [37], Kolmogoroff [38], and Lévy [41]; incidentally we obtain a new treatment of the latter problem.

The dominating feature of the convolution process is its smoothing effect, although it is hardly possible to formulate a single theorem covering the whole situation. In the cases in which we are interested an appraisal of the Fourier transforms is the natural approach to the treatment of the question. This method has recently been applied in the case of circular equidistributions by Wintner [55]; in the present paper it will be applied to the more general case of distributions on convex curves, fundamental for the treatment of the zeta function. The results thus obtained are essentially finer than those obtained through geometrical considerations by Bohr and Jessen [19].

These results are then applied to the almost periodic functions of Bohr

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† Numbers in brackets refer to the Bibliography at the end of this paper.

and their generalizations. Distribution problems for almost periodic functions have been considered from various points of view by several authors. Historically, we first mention the investigations of Bohr [4–14], Bohr and Courant [18], Bohr and Jessen [19, 20] and Jessen [31, 32] concerning Dirichlet series and in particular the Riemann zeta function, which are based on the theory of diophantine approximations (more particularly on the Kronecker-Weyl theorem). These investigations are methodically independent of the general theory of almost periodic functions which is, in the main, of a later date. Distribution functions for arbitrary real functions $x=x(t)$, almost periodic in the Bohr sense, were treated by Wintner [50–54] by transforming t-averages into x-averages and then applying the moment method of the calculus of probability. It was shown by Haviland [27] that the same method is valid also if $x(t)$ is complex-valued. In the special case of linearly independent exponents as considered by Wintner [53–55] it was essential to work not only with moments but also with Fourier transforms. It was shown by Bochner and Jessen [3] that the whole problem may be treated without applying the moment theory and considering only Fourier transforms, a method which holds in the case of generalized almost periodic functions also.

Applying this method in the present paper to the Riemann zeta function $\zeta(s)=\zeta(\sigma+it)$ we obtain with regard to the functions $\log\zeta(s)$ and $\zeta(s)$ results which are in their kind essentially more precise than those obtained so far. The distribution function of $x(t)=\log\zeta(\sigma+it)$ for a fixed $\sigma>\frac{1}{2}$ is obtained in the form of an infinite convolution. This expression for the distribution function of $\log\zeta(\sigma+it)$ occurs in a geometrical form in the papers of Bohr and Jessen [19, 20], who have proved by elementary considerations that it possesses a continuous density. In the present paper it will be shown without recourse to this result that there exists a density possessing continuous partial derivatives of arbitrarily high order. In the case $\frac{1}{2}<\sigma<1$ it will even be shown that the density is a regular analytic function of the coordinates (which cannot be the case for $\sigma>1$). The distribution function of $x(t)=\zeta(\sigma+it)$ itself is obtained from that of $x(t)=\log\zeta(\sigma+it)$ by an exponential mapping. The method would enable us to discuss the dependence of the densities upon σ also. In this direction we prove that the density of the distribution function of $x(t)=\log\zeta(\sigma+it)$ tends uniformly to zero as $\sigma\to\frac{1}{2}$ and we also prove an analogous result with regard to the density of the distribution function of $x(t)=\zeta(\sigma+it)$. It may be mentioned that the presentation is independent of the theory of diophantine approximations, which is replaced by a direct argument (using, of course, the same ideas as those in the proof of the Kronecker-Weyl theorem [48]).

In order to make the present paper independent of the papers mentioned

above we have included, often with simplified proofs, some of the results of these papers. There is one aspect of the distribution problem regarding almost periodic functions and the zeta function not considered in the present paper, namely the problem of the a-points, studied by Bohr, Bohr and Jessen, Favard, and Jessen. We mention only that the methods of the present paper admit of applications to this problem also. The Bibliography at the end of the paper is intended to be complete as far as distribution problems for almost periodic functions are concerned.

2. Distribution functions

Let R_x be a k-dimensional euclidean space with $x=(\xi_1, \cdots, \xi_k)$ as variable point. A completely additive, non-negative set function $\phi(E)$ defined for all Borel sets E in R_x and having the value 1 for $E=R_x$ will be called a *distribution function in* R_x. The notation for an integral with respect to ϕ will be

$$\int_E f(x)\phi(dR_x).$$

All integrals are understood in the Lebesgue-Radon sense (Radon [45], pp. 1324–1329). Our notation for ordinary Lebesgue integrals will be

$$\int_E f(x)m(dR_x).$$

A set E is called a *continuity set of* ϕ if $\phi(E')=\phi(E'')$ where E' denotes the set formed by all interior points of E and E'' is the closure of E. There exists an at most enumerable set of real numbers such that at least those intervals $\alpha_j<\xi_j<\beta_j$ $(j=1, \cdots, k)$ for which the numbers α_j, β_j do not belong to this set are continuity sets of ϕ.

A sequence of distribution functions ϕ_n is said to be *convergent* if there exists a distribution function ϕ such that $\phi_n(E)\to\phi(E)$ for all continuity sets E of the limit function ϕ, which is then unique. The symbol $\phi_n\to\phi$ will be used only in this sense.

We have $\phi_n\to\phi$ if and only if the relation

$$\int_{R_x} f(x)\phi_n(dR_x) \to \int_{R_x} f(x)\phi(dR_x)$$

holds for all bounded continuous functions $f(x)$ in R_x. Furthermore, if $\phi_n\to\phi$ then

$$\int_{R_x} f(x)\phi(dR_x) \leqq \liminf \int_{R_x} f(x)\phi_n(dR_x) \tag{2.1}$$

holds for every non-negative, continuous function $f(x)$ in R_x.

If ϕ_1, ϕ_2 is a pair of distribution functions, a new distribution function ϕ is defined by

$$\phi(E) = \int_{R_x} \phi_1(E - x)\phi_2(dR_x).$$

Here $E-x$ denotes the set obtained from E by the translation $-x$; the function $\phi_1(E-x)$ is a bounded Baire function so that the integral exists.* We call ϕ the *convolution* (Faltung) *of* ϕ_1 *and* ϕ_2 and denote it by $\phi=\phi_1 * \phi_2$. It may be proved directly that $\phi_1 * \phi_2 = \phi_2 * \phi_1$ and $\phi_1 * (\phi_2 * \phi_3) = (\phi_1 * \phi_2) * \phi_3$, relations which also follow from the connection with the Fourier transforms discussed in §3. For later application we notice that if $\psi_n = \phi_1 * \cdots * \phi_n$ where $\phi_1, \cdots, \phi_n$ are distribution functions in R_x, and if $h(x)$ is a non-negative Baire function in R_x then†

$$(2.2) \quad \int_{R_x} h(x)\psi_n(dR_x) = \int_{R_{x_n}} \phi_n(dR_{x_n}) \cdots \int_{R_{x_1}} h(x_1 + \cdots + x_n)\phi_1(dR_{x_1}).$$

By the *spectrum*‡ $S=S(\phi)$ of a distribution function ϕ we understand the set of those points x of R_x for which $\phi(E)>0$ holds for any set E containing x as an interior point. S is always a closed set containing at least one point. The *point spectrum* $P=P(\phi)$ is defined as the set of those points x for which $\phi(x)>0$, where x is to be understood as the Borel set consisting of the point x alone. P is at most enumerable and may be empty.

The vectorial sum $A+B$ of two point sets A and B in R_x is defined as the set of those points in R_x which may be represented in at least one way as a vector sum $a+b$ where a and b are points of A and B respectively. We agree to let $A+B$ denote the empty set if at least one of the sets A and B is empty. If A and B are both closed and bounded then so is $A+B$. We have $A+B=B+A$ and $A+(B+C)=(A+B)+C$.

* That $g(x)=\phi_1(E-x)$ is a Baire function for any Borel set E follows thus: The system of sets E for which $g(x)$ is a Baire function is a Borel field; hence it is sufficient to consider the case where E is an interval $\alpha_j<\xi_j<\beta_j$ $(j=1, \cdots, k)$. Let now $h_1(x)$, $h_2(x)$, $\cdots$ be a sequence of continuous functions such that $h_n(x)=0$ when x does not belong to E and $h_n(x)\to 1$ when x belongs to E, finally $0 \leq h_n(x) \leq 1$. Then

$$g_n(x) = \int_{R_u} h_n(u + x)\phi_1(dR_u) \to \phi_1(E - x) = g(x)$$

for all x. Since every $g_n(x)$ is continuous it follows that $g(x)$ is a Baire function.

† It is sufficient to verify (2.2) for $n=2$. In this case (2.2) is trivial if $h(x)$ is $=1$ in a Borel set E and $=0$ elsewhere. Hence (2.2) holds for any $h(x)$ taking on but an enumerable number of values which implies that (2.2) holds for any Baire function $h(x) \geq 0$. The relation (2.2) is understood in the sense that the finiteness of either side implies that of the other.

‡ The above terminology, proposed by Wintner [57], is in accordance with that of Wirtinger and Hilbert and is therefore not identical with that of Wiener.

Using this notation, we have for the spectrum and the point spectrum of a convolution the *addition rules* $S(\phi_1 * \phi_2) = S(\phi_1) + S(\phi_2)$ and $P(\phi_1 * \phi_2) = P(\phi_1) + P(\phi_2)$.

By the limit $\lim B_n$ of a sequence of point sets B_n in R_x we understand the set of those points in R_x which may be represented in at least one way as the limit of a sequence of points x_n, where x_n belongs to B_n. This limit (which may be empty) always exists and is a closed set. By the vectorial sum $A_1 + A_2 + \cdots$ of an infinite sequence of sets $A_1, A_2, \cdots$ (in this order) we mean the set $\lim (A_1 + \cdots + A_n)$.

If $\phi_n \to \phi$ the spectrum $S(\phi)$ is contained in $\lim S(\phi_n)$ but the two sets need not coincide. Between $P(\phi)$ and $\lim P(\phi_n)$ there is no connection.

A distribution function ϕ will be called continuous or discontinuous according as $P(\phi)$ is or is not empty.† We shall say that ϕ is *purely discontinuous* if $\phi(P(\phi)) = 1$. A distribution function ϕ will be called *singular* if it is continuous and there exists a Borel set E of measure zero for which $\phi(E) = 1$. Finally, a distribution function ϕ is called *absolutely continuous* if $\phi(E) = 0$ for every Borel set E of measure zero; this is the case if and only if there exists in R_x a Lebesgue integrable point function $D(x)$ such that

$$\phi(E) = \int_E D(x) m(dR_x)$$

for any Borel set E; we call $D(x)$ the density of ϕ.

Any distribution function ϕ may be written (Radon [45], pp. 1321–1322) in the form $\phi(E) = \alpha_1\phi_1(E) + \alpha_2\phi_2(E) + \alpha_3\phi_3(E)$ where $\alpha_1, \alpha_2, \alpha_3$ are non-negative numbers having the sum 1 and ϕ_1, ϕ_2, ϕ_3 are distribution functions such that ϕ_1 is purely discontinuous, ϕ_2 is singular, and ϕ_3 is absolutely continuous; the three components $\alpha_1\phi_1, \alpha_2\phi_2, \alpha_3\phi_3$ are uniquely determined by ϕ.

3. Fourier transforms

Let R_y be the k-dimensional space with $y = (\eta_1, \cdots, \eta_k)$ as variable point and let xy denote the scalar product of the vectors $x = (\xi_1, \cdots, \xi_k)$ and $y = (\eta_1, \cdots, \eta_k)$. If ϕ is a distribution function in R_x then the integral

$$\Lambda(y; \phi) = \int_{R_x} e^{ixy} \phi(dR_x)$$

defines in R_y a function $\Lambda(y; \phi)$ which is uniformly continuous and bounded, the maximum of its absolute value being $\Lambda(0; \phi) = 1$. We call $\Lambda(y; \phi)$ the *Fourier transform of* ϕ. If $\Lambda(y; \phi) \equiv \Lambda(y; \psi)$ then $\phi = \psi$, that is, the correspond-

† We notice that there exist continuous distribution functions ϕ such that not every interval $\alpha_j < \xi_j < \beta_j$ $(j = 1, \cdots, k)$ is a continuity set of ϕ in the sense defined above.

ence between the class of all distribution functions and the class of their Fourier transforms is a *one-one correspondence*.

If $\phi_n \to \phi$ then $\Lambda(y; \phi_n) \to \Lambda(y; \phi)$ holds uniformly in every sphere $|y| \leqq a$; conversely, if a sequence of Fourier transforms $\Lambda(y; \phi_n)$ is uniformly convergent in every sphere $|y| \leqq a$ then the limit function also is the Fourier transform $\Lambda(y; \phi)$ of a distribution function ϕ and $\phi_n \to \phi$. We may formulate this fact by saying that the one-one correspondence between the class of all distribution functions and their Fourier transforms is a *continuous correspondence*.

Finally we have for the Fourier transform of a convolution the *multiplication rule* $\Lambda(y; \phi_1 * \phi_2) = \Lambda(y; \phi_1)\Lambda(y; \phi_2)$. This, together with the uniqueness of the correspondence, implies immediately the relations $\phi_1 * \phi_2 = \phi_2 * \phi_1$ and $\phi_1 * (\phi_2 * \phi_3) = (\phi_1 * \phi_2) * \phi_3$ mentioned in §2.

Let χ_c denote the distribution function whose spectrum consists of the single point c in R_x so that $\chi_c(E) = 1$ or 0 according as c is or is not contained in E; we have $\Lambda(y; \chi_c) = e^{icy}$ and in particular $\Lambda(y; \chi_0) \equiv 1$. If ψ is the distribution function $\psi(E) = \phi(E - c)$ then $\psi = \phi * \chi_c$; in particular, $\phi = \phi * \chi_0$ for every ϕ. For later application we notice that if $\phi_n \to \phi$ and $\phi_n * \psi_n \to \phi$ then† $\psi_n \to \chi_0$.

If the integral

$$\int_{R_y} |\, y\,|^p \,|\, \Lambda(y; \phi)\,|\, m(dR_y)$$

is finite for an integer $p \geqq 0$ then ϕ is absolutely continuous and its density $D(x) = D(\xi_1, \cdots, \xi_k)$, determined by the inversion formula

$$D(x) = (2\pi)^{-k} \int_{R_y} e^{-ixy} \Lambda(y; \phi) m(dR_y),$$

is continuous, approaches zero when $|x| \to \infty$, and possesses in the case $p > 0$ continuous partial derivatives of order $\leqq p$ which may be obtained by differentiation under the integral sign and approach zero when $|x| \to \infty$. This is in particular the case if for some $\epsilon > 0$

$$\Lambda(y; \phi) = O(|\, y\,|^{-(k+p+\epsilon)}) \text{ as } |\, y\,| \to \infty. \tag{3.1}$$

† This is proved in the following way: We know that $\Lambda(y; \phi_n) \to \Lambda(y; \phi)$ and $\Lambda(y; \phi_n)\Lambda(y; \psi_n) \to \Lambda(y; \phi)$ hold uniformly in every sphere $|y| \leqq a$ and wish to prove that $\Lambda(y; \psi_n) \to \Lambda(y; \chi_0) \equiv 1$ holds uniformly in every sphere $|y| \leqq a$. This is obvious if $\Lambda(y; \phi) \neq 0$ for all y. Otherwise, since $\Lambda(0; \phi) = 1$, we know at least that $\Lambda(y; \psi_n) \to 1$ holds uniformly in a sufficiently small sphere $|y| \leqq a$. Now it is clear from the definition of the Fourier transform that if for an arbitrary distribution function ψ and for some y the value $\Lambda(y; \psi)$ is near to 1 then the value $\Lambda(2y; \psi)$ is also near to 1; for that $\Lambda(y; \psi)$ is near to 1 means that e^{ixy} is near to 1 in a set E in R_x for which $\psi(E)$ is near to 1 and in this set e^{2ixy} is then also near to 1. Since $\Lambda(y; \psi_n) \to 1$ holds uniformly in $|y| \leqq a$ it follows that it holds uniformly in $|y| \leqq 2a$ and by repetition that it holds uniformly in any sphere $|y| \leqq 2^m a$ which proves the theorem.

A necessary condition for the absolute continuity of ϕ is that $\Lambda(y;\phi)\to 0$ as $|y|\to\infty$ (Riemann-Lebesgue lemma).

If the estimate

$$(3.2)\qquad \Lambda(y;\phi) = O(e^{-A|y|}) \text{ as } |y|\to\infty$$

holds for some $A>0$, it follows from the inversion formula by calculating the partial derivatives of $D(x)=D(\xi_1,\cdots,\xi_k)$ at an arbitrary point $x^0=(\xi_1^0,\cdots,\xi_k^0)$ of R_x that $D(x)$ may be developed according to the powers of $\xi_1-\xi_1^0,\cdots,\xi_k-\xi_k^0$ into a power series convergent if $|\xi_1-\xi_1^0|<A/k^{1/2},\cdots,$ $|\xi_k-\xi_k^0|<A/k^{1/2}$ so that $D(x)$ is regular analytic in the whole *real* space R_x. If in particular (3.2) holds for arbitrarily large A then $D(x)$ is an entire function of the k variables $\xi_1,\cdots,\xi_k$. Cf. Wintner [59].

If ϕ is such that for an integer $p>0$ the integral

$$\mathrm{M}_p(\phi) = \int_{R_x} |x|^p\phi(dR_x)$$

is finite, then $\Lambda(y;\phi)=\Lambda(\eta_1,\cdots,\eta_k;\phi)$ possesses in R_y continuous partial derivatives of order $\leqq p$ which may be obtained from the formula defining $\Lambda(y;\phi)$ by differentiation under the integral sign. These derivatives become at $y=0$ the *moments*

$$\mu_{q_1,\cdots,q_k}(\phi) = \int_{R_x} \xi_1^{q_1}\cdots\xi_k^{q_k}\phi(dR_x)$$

of ϕ of order $q=q_1+\cdots+q_k\leqq p$, multiplied by the factor i^q. For $q=1$ and $q=2$ we shall use the shorter notations

$$\mu_h(\phi)=\int_{R_x}\xi_h\phi(dR_x) \text{ and } \mu_{h,j}(\phi)=\int_{R_x}\xi_h\xi_j\phi(dR_x).$$

The existence of $\mathrm{M}_p(\phi)$ for one value of p implies of course its existence for all smaller values of p. If $\mathrm{M}_p(\phi_1)$ and $\mathrm{M}_p(\phi_2)$ are finite and $\phi=\phi_1*\phi_2$, then $\mathrm{M}_p(\phi)$ also is finite for the inequality $|x_1+x_2|^p\leqq 2^p(|x_1|^p+|x_2|^p)$ implies by (2.2)

$$\begin{aligned}\mathrm{M}_p(\phi) = \int_{R_x}|x|^p\phi(dR_x) &= \int_{R_{x_2}}\phi_2(dR_{x_2})\int_{R_{x_1}}|x_1+x_2|^p\phi_1(dR_{x_1})\\ &\leqq 2^p(\mathrm{M}_p(\phi_1)+\mathrm{M}_p(\phi_2)).\end{aligned}$$

If $\mathrm{M}_1(\phi)$ is finite we denote by $c=c(\phi)$ the point in R_x having the moments of the first order $\mu_1(\phi),\cdots,\mu_k(\phi)$ as its coordinates so that $c(\phi)$ is the center of gravity of the mass distribution determined by ϕ. We denote by $\tilde{\phi}$ the distribution function $\tilde{\phi}(E)=\phi(E+c(\phi))$ so that $c(\tilde{\phi})=0$ and $\Lambda(y;\phi)$

$=e^{ic(\phi)y}\Lambda(y;\tilde{\phi})$; the spectrum of $\tilde{\phi}$ is $S(\tilde{\phi})=S(\phi)-c(\phi)$. If $M_1(\phi_1)$ and $M_1(\phi_2)$ are finite and $\phi=\phi_1*\phi_2$, then $c(\phi)=c(\phi_1)+c(\phi_2)$ and $\tilde{\phi}=\tilde{\phi}_1*\tilde{\phi}_2$.

If $M_2(\phi)$ is finite we have $M_2(\phi)=\mu_{1,1}(\phi)+\cdots+\mu_{k,k}(\phi)$ and $M_2(\phi)=M_2(\tilde{\phi})+|c(\phi)|^2$, hence $M_2(\tilde{\phi})\leqq M_2(\phi)$. If $M_2(\phi_1)$ and $M_2(\phi_2)$ are finite and $\phi=\phi_1*\phi_2$ then $M_2(\tilde{\phi})=M_2(\tilde{\phi}_1)+M_2(\tilde{\phi}_2)$.

A point set, a set function, or a point function in R_x or R_y is said to be of *radial symmetry* if it is invariant under all rotations about the origin. A distribution function ϕ is of radial symmetry if and only if $\Lambda(y;\phi)$ is of radial symmetry. Furthermore, if R_ξ denotes a line through $x=0$ and ω the distribution function in R_ξ defined by $\omega(F)=\phi(E)$, where E consists of those x of R_x the projection of which on R_ξ belongs to the given Borel set F in R_ξ, then ω is of radial symmetry and $\Lambda(y;\phi)=\Lambda(\eta,\omega)$ where $\eta=\pm|y|$. If the distribution function ϕ of radial symmetry is absolutely continuous then its density $D(x)$ also is of radial symmetry so that $D(x)=\delta(r)$ where $r=|x|$. Furthermore,

$$\Lambda(y;\phi) = 2\pi(2\pi/|y|)^{k/2-1}\int_0^\infty \delta(r)r^{k/2}J_{k/2-1}(r|y|)dr$$

(Cauchy-Poisson). As to the explicit connection between ϕ and ω in the case $k=2$, cf. Wintner [55].

4. Convergence criteria for infinite convolutions

If $\phi_1,\phi_2,\cdots$ is a sequence of distribution functions we say that the infinite convolution $\phi_1*\phi_2*\cdots$ is *convergent* if on placing $\psi_n=\phi_1*\cdots*\phi_n$ there exists a distribution function ψ such that $\psi_n\to\psi$ as $n\to\infty$; we write then $\psi=\phi_1*\phi_2*\cdots$. Necessary and sufficient for the existence of the infinite convolution is that the infinite product $\Lambda(y;\phi_1)\Lambda(y;\phi_2)\cdots$ converges† uniformly in every sphere $|y|\leqq a$; we have then $\Lambda(y;\phi_1*\phi_2*\cdots)=\Lambda(y;\phi_1)\cdot\Lambda(y;\phi_2)\cdots$. Using the distribution function χ_0 defined in §3 we have

THEOREM 1. *A necessary and sufficient condition for the convergence of the infinite convolution $\phi_1*\phi_2*\cdots$ is that $\rho_{n,p}\to\chi_0$ as $n\to\infty$ where $\rho_{n,p}=\phi_{n+1}*\cdots*\phi_{n+p}$ and p depends on n in an arbitrary way.*

On placing $\psi_n=\phi_1*\cdots*\phi_n$ we have $\psi_{n+p}=\psi_n*\rho_{n,p}$. Consequently if $\psi_n\to\psi$ we have also $\psi_n*\rho_{n,p}\to\psi$, hence $\rho_{n,p}\to\chi_0$ by a result of §3. Conversely, if $\rho_{n,p}\to\chi_0$ then $\Lambda(y;\rho_{n,p})=\Lambda(y;\phi_{n+1})\cdots\Lambda(y;\phi_{n+p})\to 1$ uniformly in every sphere $|y|\leqq a$, hence $\Lambda(y;\phi_1)\Lambda(y;\phi_2)\cdots$ converges uniformly in every sphere $|y|\leqq a$.

† Throughout the paper convergence of an infinite product is meant in the sense that the product of the n first factors approaches a limit so that the vanishing of the limit is not excluded.

THEOREM 2. *If* $\psi=\phi_1*\phi_2*\cdots$ *is convergent, then so is* $\rho_n=\phi_{n+1}*\phi_{n+2}*\cdots$ *for every* n *and* $\rho_n\to\chi_0$ *as* $n\to\infty$.

The convergence of $\phi_{n+1}*\phi_{n+2}*\cdots$ follows from Theorem 1. Also, $\psi_n*\rho_n=\psi$ and $\psi_n\to\psi$, hence $\rho_n\to\chi_0$.

These theorems enable us to prove that the addition rule for spectra holds for infinite convolutions also:

THEOREM 3. *If* $\psi=\phi_1*\phi_2*\cdots$ *is convergent then* $S(\psi)=S(\phi_1)+S(\phi_2)+\cdots$.

The theorem states that $S(\psi)=\lim S(\psi_n)$ where $\psi_n=\phi_1*\cdots*\phi_n$. We know from §2 that $S(\psi)$ is contained in $\lim S(\psi_n)$ and shall prove that $\lim S(\psi_n)$ is contained in $S(\psi)$. Let x_0 be a point of $\lim S(\psi_n)$ and let C_ϵ denote the sphere $|x|<\epsilon$ where $\epsilon>0$ is arbitrary; then $\psi_n(x_0+C_\epsilon)>0$ for n sufficiently large. Now $\psi=\psi_n*\rho_n$, hence

$$\psi(x_0+C_{2\epsilon}) = \int_{R_x}\psi_n(x_0+C_{2\epsilon}-x)\rho_n(dR_x) \geqq \int_{C_\epsilon}\psi_n(x_0+C_{2\epsilon}-x)\rho_n(dR_x)$$

$$\geqq \psi_n(x_0+C_\epsilon)\int_{C_\epsilon}\rho_n(dR_x) = \psi_n(x_0+C_\epsilon)\rho_n(C_\epsilon).$$

Since $\rho_n\to\chi_0$ we have $\rho_n(C_\epsilon)\to 1$ as $n\to\infty$ for any fixed ϵ. Hence $\psi(x_0+C_{2\epsilon})>0$ for any ϵ which means that x_0 belongs to $S(\psi)$.

The above proof is a generalization of a proof in Bohr and Jessen [19].

On imposing conditions on the distribution functions we obtain convergence criteria which will be used later on.

THEOREM 4. *If* $\mathrm{M}_2(\phi_n)$ *is finite for every* n *then the convergence of the two series*

$$c(\phi_1)+c(\phi_2)+\cdots \quad \textit{and} \quad \mathrm{M}_2(\tilde\phi_1)+\mathrm{M}_2(\tilde\phi_2)+\cdots$$

implies the convergence of $\psi=\phi_1*\phi_2*\cdots$. *Furthermore,* $\mathrm{M}_2(\psi)$ *is finite; finally,*

$$c(\psi)=c(\phi_1)+c(\phi_2)+\cdots \quad \textit{and} \quad \mathrm{M}_2(\tilde\psi)=\mathrm{M}_2(\tilde\phi_1)+\mathrm{M}_2(\tilde\phi_2)+\cdots.$$

Since $\mathrm{M}_2(\tilde\phi_n)\leqq \mathrm{M}_2(\phi_n)$ is finite, $\Lambda(y;\tilde\phi_n)$ possesses continuous partial derivatives of order $\leqq 2$ and those of the first order vanish at $y=0$ in virtue of $c(\tilde\phi_n)=0$. Hence

$$\Lambda(y;\tilde\phi_n) = 1+\sum_{h=1}^{k}\sum_{j=1}^{k}\left[\Re\Lambda_{\eta_h,\eta_j}(\theta' y;\tilde\phi_n)+i\Im\Lambda_{\eta_h,\eta_j}(\theta'' y;\tilde\phi_n)\right]\eta_h\eta_j$$

where the subscripts denote partial differentiations and $0<\theta'<1$, $0<\theta''<1$.

Since every partial derivative of second order of $\Lambda(y;\tilde{\phi}_n)$ has an absolute value $\leqq \mathrm{M}_2(\tilde{\phi}_n)$ we have

$$(4.1)\qquad |\Lambda(y;\tilde{\phi}_n)-1| \leqq 2k^2\mathrm{M}_2(\tilde{\phi}_n)|y|^2$$

which proves the convergence of $\omega=\tilde{\phi}_1*\tilde{\phi}_2*\cdots$. The relation $\mathrm{M}_2(\tilde{\phi}_1*\cdots*\tilde{\phi}_n)=\mathrm{M}_2(\tilde{\phi}_1)+\cdots+\mathrm{M}_2(\tilde{\phi}_n)$ implies by (2.1) that $\mathrm{M}_2(\omega)$ is finite, namely $\leqq \mathrm{M}_2(\tilde{\phi}_1)+\mathrm{M}_2(\tilde{\phi}_2)+\cdots$. Now it follows from (4.1) that the derivatives of the first and second order of $\Lambda(y;\omega)$ at $y=0$ are the limits of the corresponding derivatives of $\Lambda(y;\tilde{\phi}_1*\cdots*\tilde{\phi}_n)$ as $n\to\infty$, so that $c(\omega)=0$, i.e., $\omega=\tilde{\omega}$, and $\mathrm{M}_2(\omega)=\mathrm{M}_2(\tilde{\phi}_1)+\mathrm{M}_2(\tilde{\phi}_2)+\cdots$. Finally, the relation $\Lambda(y;\phi_n)=e^{ic(\phi_n)y}\Lambda(y;\tilde{\phi}_n)$ and the convergence of $c(\phi_1)+c(\phi_2)+\cdots$ implies the convergence of $\psi=\phi_1*\phi_2*\cdots$ and also gives $c(\psi)=c(\phi_1)+c(\phi_2)+\cdots$ and $\tilde{\psi}=\omega$.

Examples show that the converse of Theorem 4 is false. We have, however, the following theorem:

THEOREM 5. *If all spectra $S(\phi_n)$ are contained in a fixed sphere $|x|\leqq K$ then the convergence of the two series*

$$c(\phi_1)+c(\phi_2)+\cdots \quad \textit{and} \quad \mathrm{M}_2(\tilde{\phi}_1)+\mathrm{M}_2(\tilde{\phi}_2)+\cdots$$

is necessary and sufficient for the convergence of $\psi=\phi_1*\phi_2*\cdots$.

The sufficiency is implied by Theorem 4. In order to prove the necessity we first show that the convergence of $\psi=\phi_1*\phi_2*\cdots$ implies the convergence of $\mathrm{M}_2(\tilde{\phi}_1)+\mathrm{M}_2(\tilde{\phi}_2)+\cdots$. For suppose that $\mathrm{M}_2(\tilde{\phi}_1)+\mathrm{M}_2(\tilde{\phi}_2)+\cdots$ is divergent; then it follows from $\mathrm{M}_2(\tilde{\phi}_n)=\mu_{1,1}(\tilde{\phi}_n)+\cdots+\mu_{k,k}(\tilde{\phi}_n)$ that for some value of j the series $\mu_{j,j}(\tilde{\phi}_1)+\mu_{j,j}(\tilde{\phi}_2)+\cdots$ is divergent. We choose $y=(0,\cdots,0,\eta_j,0,\cdots,0)$ so that

$$\Lambda(y;\tilde{\phi}_n)=\int_{R_x} e^{i\xi_j\eta_j}\tilde{\phi}_n(dR_x).$$

Instead of integrating over R_x it is sufficient to integrate over the sphere $|x|\leqq 2K$ which contains $S(\tilde{\phi}_n)=S(\phi_n)-c(\phi_n)$. Choose now an $\epsilon>0$ so small that for $|t|<2K\epsilon$

$$0\leqq \cos t\leqq 1-t^2/4 \text{ and } \sin t=t+h(t) \text{ where } |h(t)|\leqq t^2/8.$$

Then $|\xi_j\eta_j|<2K\epsilon$ in $|x|\leqq 2K$ if $|\eta_j|<\epsilon$, hence

$$0\leqq \Re\Lambda(y;\tilde{\phi}_n)\leqq 1-\eta_j^2\mu_{j,j}(\tilde{\phi}_n)/4 \text{ and } |\Im\Lambda(y;\tilde{\phi}_n)|\leqq \eta_j^2\mu_{j,j}(\tilde{\phi}_n)/8$$

so that

$$|\Lambda(y;\phi_n)|=|\Lambda(y;\tilde{\phi}_n)|\leqq 1-\eta_j^2\mu_{j,j}(\tilde{\phi}_n)/8.$$

Hence the divergence of $\mu_{j,j}(\tilde{\phi}_1)+\mu_{j,j}(\tilde{\phi}_2)+\cdots$ implies that $\Lambda(y;\psi)=\Lambda(y;\phi_1)\Lambda(y;\phi_2)\cdots=0$ for points y arbitrarily near to $y=0$ and thus leads to a contradiction. Consequently $\mathrm{M}_2(\tilde{\phi}_1)+\mathrm{M}_2(\tilde{\phi}_2)+\cdots$ is convergent. In order to prove that $c(\phi_1)+c(\phi_2)+\cdots$ also converges we observe that by the proof of Theorem 4 the convergence of $\mathrm{M}_2(\tilde{\phi}_1)+\mathrm{M}_2(\tilde{\phi}_2)+\cdots$ implies the convergence of $\tilde{\phi}_1*\tilde{\phi}_2*\cdots$. Since $\Lambda(y;\phi_1*\cdots*\phi_n)=e^{ic_ny}\Lambda(y;\tilde{\phi}_1*\cdots*\tilde{\phi}_n)$ where $c_n=c(\phi_1)+\cdots+c(\phi_n)$, we conclude that e^{ic_ny} tends in a sufficiently small sphere $|y|\leqq a$ to a limit as $n\to\infty$. This is only possible if $c(\phi_1)+c(\phi_2)+\cdots$ converges.

The infinite convolution $\psi=\phi_1*\phi_2*\cdots$ will be called *absolutely convergent* if it is convergent in any order of the terms; the connection with the Fourier transforms shows that ψ is then also independent of the order of the terms. From Theorems 4 and 5 follows immediately

THEOREM 6. *If* $\mathrm{M}_2(\phi_n)$ *is finite for every* n, *then the convergence of the two series*

$$|c(\phi_1)|+|c(\phi_2)|+\cdots \quad \textit{and} \quad \mathrm{M}_2(\tilde{\phi}_1)+\mathrm{M}_2(\tilde{\phi}_2)+\cdots$$

implies the absolute convergence of $\phi_1*\phi_2*\cdots$. *If all spectra* $S(\phi_n)$ *are contained in a fixed sphere* $|x|\leqq K$ *then the converse is also true.*

We notice that, since $\mathrm{M}_2(\phi_n)=\mathrm{M}_2(\tilde{\phi}_n)+|c(\phi_n)|^2$, the convergence of the two series is equivalent to the convergence of the series

$$|c(\phi_1)|+|c(\phi_2)|+\cdots \quad \text{and} \quad \mathrm{M}_2(\phi_1)+\mathrm{M}_2(\phi_2)+\cdots.$$

Further theorems on infinite convolutions will be given in §16.

5. CONVOLUTIONS OF SPHERICAL EQUIDISTRIBUTIONS

Let the dimension number k be >1 and let S denote the sphere $|x|=r$ where $r>0$. As *equidistribution on* S we denote the distribution function $\phi(E)$ which for a given Borel set E is the $(k-1)$-dimensional measure of ES, divided by that of S itself. We have $S(\phi)=S$, $c(\phi)=0$ and $\mathrm{M}_2(\phi)=r^2$. Let $S_1, S_2, \cdots$ denote the spheres $|x|=r_1$, $|x|=r_2, \cdots$, and $\phi_1, \phi_2, \cdots$ the corresponding equidistributions. If $\phi_1*\phi_2*\cdots$ is convergent then $\phi_n\to\chi_0$ in virtue of Theorem 1, hence $r_n\to 0$. From Theorems 5 and 6 follows therefore

THEOREM 7. *The convergence of the series* $r_1^2+r_2^2+\cdots$ *is necessary and sufficient both for the convergence and for the absolute convergence of the infinite convolution* $\phi_1*\phi_2*\cdots$.

The Fourier transform $\Lambda(y;\phi)$ of the spherical equidistribution ϕ is the mean value of e^{ixy} on S. If y is fixed and x varies on S we may write $xy=r|y|\cos\theta$, $0\leqq\theta\leqq\pi$; then $\phi(dS)=A_k\sin^{k-2}\theta\, d\theta$, where dS is the portion

of S corresponding to the interval $(\theta, \theta+d\theta)$ and $A_k=\pi^{-1/2}\Gamma(\frac{1}{2}k)/\Gamma(\frac{1}{2}k-\frac{1}{2})$. Hence

$$\Lambda(y;\phi) = A_k\int_0^\pi e^{ir|y|\cos\theta}\sin^{k-2}\theta\,d\theta = B_kJ_{k/2-1}(r|y|)/(r|y|)^{k/2-1}$$

where $B_k=2^{k/2-1}\Gamma(\frac{1}{2}k)$. Since $J_\nu(u)=O(u^{-1/2})$ as $u\to\infty$ we have

$$\Lambda(y;\phi) = O(|y|^{1/2-k/2}) \text{ as } |y|\to\infty. \tag{5.1}$$

Let now $\phi_1, \phi_2, \cdots$ be the spherical equidistributions considered above. It follows from (5.1) that $\Lambda(y;\phi_1*\cdots*\phi_n)$ satisfies (3.1) if $n>2(k+p)/(k-1)$ and that in the case of convergence $\Lambda(y;\phi_1*\phi_2*\cdots)$ satisfies (3.1) for every p in virtue of $|\Lambda(y;\phi_n)|\leqq 1$. Hence we have

THEOREM 8. *The convolution $\psi_n=\phi_1*\cdots*\phi_n$ of n spherical equidistributions is absolutely continuous with a continuous density $D_n(x)$ whenever $n>2k/(k-1)$, and $D_n(x)=D_n(\xi_1,\cdots,\xi_k)$ possesses continuous partial derivatives of order $\leqq p$ whenever $n>2(k+p)/(k-1)$. If $\psi=\phi_1*\phi_2*\cdots$ is convergent, it is absolutely continuous and its density $D(x)=D(\xi_1,\cdots,\xi_k)$ is continuous and possesses continuous partial derivatives of all orders.*

The above argument is that applied by Wintner [55] in the case $k=2$.

Remark. In view of the fact that the functions $\Lambda(y;\phi_1*\cdots*\phi_n)$ are estimated by (3.1) uniformly for all $n>2(k+p)/(k-1)$, it follows from the uniform convergence of $\Lambda(y;\phi_1*\cdots*\phi_n)$ to $\Lambda(y;\phi_1*\phi_2*\cdots)$ in any sphere $|y|\leqq a$ in the case where $\phi_1*\phi_2*\cdots$ converges that

$$\int_{R_y}|y|^p|\Lambda(y;\phi_1*\cdots*\phi_n)-\Lambda(y;\phi_1*\phi_2*\cdots)|\,m(dR_y)\to 0 \text{ as } n\to\infty.$$

This implies that $D_n(x)$ *and its partial derivatives converge uniformly to* $D(x)$ *and its partial derivatives* when $n\to\infty$. In the case where the series $r_1^2+r_2^2+\cdots$ is divergent it may easily be shown that $\Lambda(y;\phi_1*\cdots*\phi_n)\to 0$ uniformly for $|y|\geqq\epsilon$ where $\epsilon>0$ is arbitrarily small. This implies that

$$\int_{R_y}|y|^p|\Lambda(y;\phi_1*\cdots*\phi_n)|\,m(dR_y)\to 0 \text{ as } n\to\infty,$$

so that in this case $D_n(x)$ *and its partial derivatives tend uniformly to zero* when $n\to\infty$. This is a more precise statement than merely saying that the infinite convolution $\phi_1*\phi_2*\cdots$ does not converge.

Since all ϕ_n are of *radial symmetry*, the same property holds for their convolutions and hence for the densities $D_n(x)$ and $D(x)$. For the spectra $S(\phi_1*\cdots*\phi_n)=S_1+\cdots+S_n$ and $S(\phi_1*\phi_2*\cdots)=S_1+S_2+\cdots$ one easily finds (cf. Bohr [6])

THEOREM 9. *$S(\phi_1 * \cdots * \phi_n)$ is the set $a_n \leqq |x| \leqq b_n$ where $b_n = r_1 + \cdots + r_n$ and $a_n = 0$ unless one of the r's in $r_1 + \cdots + r_n$, say r_1, is larger than the sum of the other r's, in which case $a_n = r_1 - (r_2 + \cdots + r_n)$. If $r_1 + r_2 + \cdots$ is convergent then $S(\phi_1 * \phi_2 * \cdots)$ is the set $a \leqq |x| \leqq b$ where $b = r_1 + r_2 + \cdots$ and $a = 0$ unless one of the r's, say r_1, is larger than the sum of all other r's, in which case $a = r_1 - (r_2 + r_3 + \cdots)$. If $r_1 + r_2 + \cdots$ is divergent (but $r_1^2 + r_2^2 + \cdots$ is convergent) then $S(\phi_1 * \phi_2 * \cdots)$ is the whole R_x.*

Remark. It is not difficult to determine the points x for which $D_n(x)$ or $D(x)$ is positive. We prove in this direction only that in the case where $r_1 + r_2 + \cdots$ is divergent *$D(x)$ is positive for all x* (from $S(\phi_1 * \phi_2 * \cdots) = R_x$ follows only that $D(x)$ is not identically zero in any sphere). On denoting by $D'(x)$ and $D''(x)$ the densities of $\phi_1 * \phi_3 * \cdots$ and $\phi_2 * \phi_4 * \cdots$ we have

$$D(x) = \int_{R_u} D'(x - u) D''(u) m(dR_u).$$

At least one of the two functions $D'(x)$ and $D''(x)$ is not identically zero in any sphere; since the other function is positive in some sphere, we have $D(x) > 0$ for every x.

For later reference we collect the main results for the case $k = 2$ as a particular theorem.

THEOREM 10. *If ϕ_n is the circular equidistribution on $|x| = r_n$ then $\psi_n = \phi_1 * \cdots * \phi_n$ is absolutely continuous with a continuous density $D_n(x)$ if $n > 4$ and $D_n(x) = D_n(\xi_1, \xi_2)$ possesses continuous partial derivatives of order $\leqq p$ if $n > 4 + 2p$. The convergence of $r_1^2 + r_2^2 + \cdots$ is necessary and sufficient both for the convergence and for the absolute convergence of the infinite convolution $\phi_1 * \phi_2 * \cdots$ and $\psi = \phi_1 * \phi_2 * \cdots$ is then absolutely continuous and its density $D(x) = D(\xi_1, \xi_2)$ is continuous and possesses continuous partial derivatives of arbitrarily high order. Finally, $D(x) > 0$ for all x if $r_1 + r_2 + \cdots$ diverges.*

Finer results regarding $D(x)$ are contained in Theorems 15 and 16.

The results of this section could also be stated in terms of the so-called random walk problem; cf. Lord Rayleigh [46], Lüneburg [42].

6. Infinite convolutions of symmetric Bernoulli distributions

Let the dimension number k be $= 1$. Let S denote the set consisting of the two points $x = \pm r$ where $r > 0$ and let $\phi(E)$ be the distribution function which is 0, $\frac{1}{2}$ or 1 according as E contains neither, one or both of these points so that $\Lambda(y; \phi) = \frac{1}{2}(e^{-iry} + e^{iry}) = \cos ry$ and $S(\phi) = P(\phi) = S$. This ϕ is the analogue for $k = 1$ of the spherical equidistribution. We have $c(\phi) = 0$ and $M_2(\phi) = r^2$.

Hence if $S_1, S_2, \cdots$ denote the sets $x=\pm r_1$, $x=\pm r_2, \cdots$ and $\phi_1, \phi_2, \cdots$ the corresponding distribution functions, Theorem 7 holds again. On placing $\psi_n=\phi_1 * \cdots * \phi_n$ we have $\psi_n(E)=h/2^n$ if E contains h of the 2^n points $\pm r_1 \pm \cdots \pm r_n$ which form the spectrum $S(\psi_n)=P(\psi_n)=S_1+\cdots+S_n$. It follows therefore from Theorem 3 that if $r_1^2+r_2^2+\cdots$ is convergent, so that $\psi=\phi_1 * \phi_2 * \cdots$ exists, then $S(\psi)$ is either a bounded set or the whole R_x according as $r_1+r_2+\cdots$ is convergent or divergent. Furthermore, $P(\psi)$ is always empty. For suppose that there exists a point x so that $\psi(x)>0$. Let us write $\psi=\sigma_n * \phi_n$ where $\sigma_n=\phi_1 * \cdots * \phi_{n-1} * \phi_{n+1} * \cdots$; then $\psi(x) =\frac{1}{2}(\sigma_n(x-r_n)+\sigma_n(x+r_n))$ and similarly $\psi(x-2r_n)=\frac{1}{2}(\sigma_n(x-3r_n)+\sigma_n(x-r_n))$, $\psi(x+2r_n)=\frac{1}{2}(\sigma_n(x+r_n)+\sigma_n(x+3r_n))$ so that $\psi(x-2r_n)+\psi(x+2r_n)\geqq\psi(x)$. We choose a positive integer p such that $\psi(x)>1/p$ and determine then p numbers $n_1, \cdots, n_p$ such that $r_{n_1}>\cdots>r_{n_p}$ which is possible in virtue of $r_n \to 0$. Then the numbers $x \pm r_{n_1}, \cdots, x\pm r_{n_p}$ are all distinct and the sum of the corresponding values $\psi(x-2r_{n_1})+\psi(x+2r_{n_1}), \cdots, \psi(x-2r_{n_p})+\psi(x+2r_{n_p})$ is $\geqq p\psi(x)>1$ which is impossible. This proves that $P(\psi)$ is empty, i.e., that ψ is continuous. From Theorem 35, to be proved later, we conclude therefore that ψ is either singular or absolutely continuous. On collecting our results we have

THEOREM 11. *The infinite convolution $\phi_1 * \phi_2 * \cdots$, where $\phi_n(E)=0, \frac{1}{2}$, or 1 according as E contains neither, one, or both of the points $x=\pm r_n$, is convergent, and then also absolutely convergent, if and only if the series $r_1^2+r_2^2+\cdots$ is convergent. The Fourier transform of $\psi=\phi_1 * \phi_2 * \cdots$ is then $\Lambda(y;\psi)=\cos(r_1 y)\cdot\cos(r_2 y)\cdots$. The spectrum $S(\psi)$ is a bounded set or the whole R_x according as the series $r_1+r_2+\cdots$ converges or diverges; the point spectrum $P(\psi)$ is empty and ψ is either singular or absolutely continuous.*

A necessary condition for the absolute continuity of ψ is $\Lambda(y;\psi)\to 0$ as $|y|\to\infty$. It is clear that ψ is always of radial symmetry. We illustrate Theorem 11 by the following examples:

Example 1. $r_n=3^{-n}$. Here $S(\psi)$ is the Cantor null-set in $|x|\leqq\frac{1}{2}$ obtained in the usual way by successive trisections and ψ is the singular function discussed by Lebesgue. We have $\Lambda(y;\psi)=\cos(y/3)\cos(y/3^2)\cdots$, hence $\Lambda(2\pi 3^n;\psi)=\cos(2\pi/3)\cos(2\pi/3^2)\cdots\neq 0$ for all n. This is, perhaps, the simplest example of a continuous ψ for which $\Lambda(y;\psi)$ does not approach zero. With regard to this example cf. Carleman [22], pp. 223–226, Hille and Tamarkin [30].

Example 2. $r_{2n-1}=r_{2n}=3^{-n}$. Here ψ is the convolution of the previous infinite convolution with itself. This implies that $\Lambda(y;\psi)$ does not approach zero, hence ψ is singular. The spectrum $S(\psi)$ is the interval $|x|\leqq 1$. This is,

perhaps, the simplest example of a singular ψ having an interval as spectrum.

Example 3. $r_n = 2^{-n}$. Here $S(\psi)$ is the interval $|x| \leqq 1$ and ψ is absolutely continuous, its density $D(x)$ being $=\frac{1}{2}$ for $|x| < 1$ and $=0$ for $|x| > 1$ so that $D(x)$ is not continuous. Placing $s(y) = (\sin y)/y$ we have

$$\Lambda(y; \psi) = \cos (y/2) \cos (y/2^2) \cdots = s(y).$$

Example 4. $r_{2n-1} = r_{2n} = 2^{-n}$. Here ψ is the convolution of the previous infinite colvolution with itself. Hence $S(\psi)$ is the interval $|x| \leqq 2$ and ψ is absolutely continuous, its density $D(x)$ being $=\frac{1}{2} - \frac{1}{4}|x|$ for $|x| \leqq 2$ and $=0$ for $|x| > 2$ so that $D(x)$ is continuous.

Example 5. $r_1, r_2, \cdots$ is a rearrangement of the double sequence $2^{-(l+m)}$ where $l, m = 1, 2, \cdots$. Here $S(\psi)$ is the interval $|x| \leqq 1$ and $\Lambda(y; \psi) = s(y/2)s(y/2^2) \cdots$ by Example 3. Hence $\Lambda(y; \psi)$ satisfies (3.1) for every p so that ψ is absolutely continuous with a continuous density $D(x)$ possessing continuous derivatives of arbitrarily high order.

Example 6. $r_1, r_2, \cdots$ consists of the numbers $2^{-m!}$ where $m = 1, 2, \cdots$, and contains $2^{-m!}$ exactly $2^{m!}$ times. Here $r_1^2 + r_2^2 + \cdots$ is convergent and $r_1 + r_2 + \cdots$ is divergent so that ψ exists and $S(\psi) = R_x$. It is easy to see from

$$\Lambda(2\pi 2^{m!}; \psi) = \prod_{n>m} (\cos (2\pi 2^{m!-n!}))^{2^{n!}}$$

that $\Lambda(y; \psi)$ does not approach zero; hence ψ is singular.

Example 7. $r_n = 1/n$. This is a rearrangement of the double sequence $2^{-l}/(2m+1)$ where $l, m = 0, 1, 2, \cdots$. Hence $\Lambda(y; \psi) = s(y)s(y/3) \cdots$ by Example 3 (cf. Lévy [41], p. 154). Thus we have the same situation as in Example 5 except that $S(\psi)$ is now the whole R_x. Furthermore, the density $D(x)$ is regular analytic in R_x (which cannot be the case in Example 5); in fact, $|s(y)| \leqq \min (1, |y|^{-1})$, hence on writing $t = \frac{1}{2}|y|$ we have

$$|\Lambda(y; \psi)| \leqq \prod_{m=1}^{\infty} \min (1, (2m-1)|y|^{-1}) \leqq \prod_{m=1}^{\infty} \min (1, 2m|y|^{-1}) = \prod_{m \leqq t} m t^{-1}$$

so that $\Lambda(y; \psi) = O(t^{1/2}e^{-t})$, by Stirling's formula. It follows therefore from $t = \frac{1}{2}|y|$ that $\Lambda(y; \psi)$ satisfies (3.2) with every $A < \frac{1}{2}$.

Example 8. $r_n = 2^{-n}/n - 2^{-(n+1)}/(n+1)$. Here $S(\psi)$ is the null-set obtained from the interval $|x| \leqq \frac{1}{2}$ by a construction identical with that leading to the Cantor null-set except that in the nth step the length of each of the omitted intervals is $1/(n+1)$ times the length of each of the 2^{n-1} intervals obtained in the $(n-1)$th step. The interest of this example is due to the fact that $\Lambda(y; \psi)$ approaches zero although ψ is singular. Cf. Menchoff [43].

For further results cf. Wintner [60].

The results of this section could also be stated in terms of the Rademacher functions.

7. Fourier transforms of distributions on convex curves

Let the dimension number k be 2 and let S be a closed curve in R_x given by a parametric representation $x=x(\theta)=(\xi_1(\theta), \xi_2(\theta))$ such that $x(\theta)$ has the primitive period 1. This parametric representation of S determines in R_x a distribution function ϕ where $\phi(E)$ for a given Borel set E is the θ-measure of ES; we have $S(\phi)=S$. We call ϕ the *distribution function determined by the parametric representation $x=x(\theta)$ of S.* If S is the circle $|x|=r$ given in the parametric representation $x=x(\theta)=(r\cos 2\pi\theta, r\sin 2\pi\theta)$ then ϕ is the circular equidistribution on S. In this case we know from §5 that $\Lambda(y;\phi)=O(|y|^{-1/2})$ as $|y|\to\infty$. We shall now prove that this appraisal holds for a general class of convex curves. In the case of a circular equidistribution, $\Lambda(y;\phi)$ is a function of $|y|$ only; in the present case we have

$$\Lambda(y;\phi) = \int_S e^{ig_\tau(\theta)|y|}d\theta \tag{7.1}$$

where $y=(|y|\cos\tau, |y|\sin\tau)$ and $g_\tau(\theta)=\xi_1(\theta)\cos\tau+\xi_2(\theta)\sin\tau$.

Theorem 12. *Let $x=x(\theta)=(\xi_1(\theta), \xi_2(\theta))$ be a parametric representation of a convex curve S, such that*

(i) *$\xi_1(\theta)$ and $\xi_2(\theta)$ possess continuous second derivatives $\xi_1''(\theta)$ and $\xi_2''(\theta)$;*

(ii) *the second derivative $g_\tau''(\theta)$ with respect to θ of the function $g_\tau(\theta)=\xi_1(\theta)\cos\tau+\xi_2(\theta)\sin\tau$ has for every fixed value of τ exactly two zeros θ on S.*

Then for the Fourier transform of the distribution function ϕ determined by the parametric representation $x=x(\theta)$ of S we have, uniformly in τ,

$$\Lambda(y;\phi) = O(|y|^{-1/2}) \text{ as } |y|\to\infty.$$

The geometrical meaning of $g_\tau(\theta)$ implies that $g_\tau'(\theta)$ has for every fixed value of τ at least two zeros on S; since between any two zeros of $g_\tau'(\theta)$ we find at least one zero of $g_\tau''(\theta)$ it follows that $g_\tau'(\theta)$ also has exactly two zeros which separate those of $g_\tau''(\theta)$ for every fixed τ.

The zeros of $g_\tau'(\theta)$ and $g_\tau''(\theta)$ depend continuously on τ. For if $\tau_n\to\tau$, every limit point of zeros of $g_{\tau_n}'(\theta)$ or $g_{\tau_n}''(\theta)$ will, by the continuity of $g_\tau'(\theta)$ and $g_\tau''(\theta)$ as functions of θ and τ together, be a zero of $g_\tau'(\theta)$ or $g_\tau''(\theta)$. Furthermore, the two zeros of $g_{\tau_n}'(\theta)$ cannot tend to the same point on S as $n\to\infty$, for then this point would also be a limit point of zeros of $g_{\tau_n}''(\theta)$, hence $g_\tau'(\theta)$ and $g_\tau''(\theta)$ would have a common zero, which is, as we saw, impossible. The same argument shows that the two zeros of $g_{\tau_n}''(\theta)$ cannot tend to the same point on S as $n\to\infty$.

We now consider, for every fixed τ, the mid-points of the four arcs on S determined by the zeros of $g_\tau'(\theta)$ and $g_\tau''(\theta)$; these mid-points also depend continuously on τ and divide S into four arcs A_τ, B_τ, C_τ, D_τ such that A_τ and C_τ each contain one of the zeros of $g_\tau'(\theta)$ and B_τ and D_τ each contain one of the zeros of $g_\tau''(\theta)$. Since the end points of A_τ, B_τ, C_τ, D_τ depend continuously on τ there exists a constant $\alpha>0$ such that

$$(7.2)\qquad |g_\tau''(\theta)| \geqq \alpha \text{ on } A_\tau \text{ and } C_\tau \text{ and } |g_\tau'(\theta)| \geqq \alpha \text{ on } B_\tau \text{ and } D_\tau.$$

Now if $f(\theta)$, $\gamma\leqq\theta\leqq\delta$, is a real-valued function possessing a continuous monotone derivative $f'(\theta)$ which is nowhere zero, then

$$(7.3)\qquad \left|\int_\gamma^\delta e^{if(\theta)}d\theta\right| \leqq 4/\min |f'(\theta)|.$$

Furthermore, if $f(\theta)$, $\gamma\leqq\theta\leqq\delta$, is a real-valued function possessing a continuous second derivative $f''(\theta)$ which is nowhere zero, then

$$(7.4)\qquad \left|\int_\gamma^\delta e^{if(\theta)}d\theta\right| \leqq 8/\min |f''(\theta)|^{1/2}.$$

The inequality (7.3) follows from the identity

$$\left|\int_\gamma^\delta e^{if(\theta)}d\theta\right| = \left|\int_\gamma^\delta \frac{(e^{if(\theta)})'}{f'(\theta)}d\theta\right|$$

in virtue of the second mean-value theorem and (7.4) is, according to van der Corput and Landau, a consequence of (7.3); cf. Landau [39], p. 60.

On applying (7.3) and (7.4) to the four integration domains A_τ, B_τ, C_τ, D_τ and to the function $f(\theta)=g_\tau(\theta)|y|$, it follows from (7.1) and (7.2) that

$$|\Lambda(y;\phi)| \leqq 16\alpha^{-1/2}|y|^{-1/2} + 16\alpha^{-1}|y|^{-1},$$

which proves Theorem 12. Since $\alpha^{-1}|y|^{-1}\leqq\alpha^{-1/2}|y|^{-1/2}$ when $\alpha|y|\geqq 1$, and $|\Lambda(y;\phi)|\leqq 1$, we find also

$$(7.5)\qquad |\Lambda(y;\phi)| \leqq 32\alpha^{-1/2}|y|^{-1/2}.$$

Let $F(z)=a_1z+a_2z^2+\cdots$ be a power series convergent in a circle $|z|<\rho$ $(\leqq\infty)$ and such that $a_1\neq 0$. Let S denote the curve $x=x(\theta)=\xi_1(\theta)+i\xi_2(\theta)$ $=F(r\,e^{2\pi i\theta})$ where $0<r<\rho$ and let ϕ be the distribution function in R_x determined by this parametric representation of S. Then $c(\phi)=0$, i.e., $\phi=\tilde{\phi}$, and $M_2(\phi)=|a_1|^2r^2+|a_2|^2r^4+\cdots$. It is known that S is a convex curve if r is sufficiently small; condition (i) of Theorem 12 is satisfied for all r; finally if $a_n=|a_n|e^{2\pi i\gamma_n}$ we have

$$(7.6)\quad g_r(\theta) = |a_1| r\cos 2\pi(\theta+\gamma_1-\tau) + |a_2| r^2 \cos 2\pi(2\theta+\gamma_2-\tau) + \cdots,$$

showing that if r is sufficiently small then condition (ii) of Theorem 12 also is satisfied. Hence there exists a positive $\rho_0(\leqq\rho)$ such that all conditions of Theorem 12 are satisfied for $0<r<\rho_0$. Finally, there exists for any given $\rho_1<\rho_0$ a constant $c>0$ such that the number $\alpha>0$ defined by (7.2) and occurring in (7.5) may be chosen $>cr$ if $0<r\leqq\rho_1$. This is clear from (7.6) if ρ_1 is sufficiently small; on the other hand if $\rho_1<\rho_0$ and $\epsilon>0$ are arbitrary then the same argument which was applied in the proof of Theorem 12 shows that the zeros of $g_r'(\theta)$ and $g_r''(\theta)$ depend continuously on τ and r together if $\epsilon\leqq r\leqq\rho_1$. On collecting the results we have

THEOREM 13. *Let $F(z)=a_1z+a_2z^2+\cdots$, where $a_1\neq 0$, be convergent for $|z|<\rho\ (\leqq\infty)$. Let S denote the curve $x=x(\theta)=F(re^{2\pi i\theta})$ where $0<r<\rho$, and let ϕ be the distribution function in R_x determined by this parametric representation of S. Then there exists a positive $\rho_0\ (\leqq\rho)$ such that S is convex and $\Lambda(y;\phi)=O(|y|^{-1/2})$ if $0<r<\rho_0$. Furthermore, if $0<\rho_1<\rho_0$, there exists a constant β such that*

$$\Lambda(y;\phi)|\leqq \beta r^{-1/2}|y|^{-1/2}$$

if $0<r\leqq\rho_1$.

8. A TYPE OF INFINITE CONVOLUTION

As an application of previous results we prove

THEOREM 14. *Let $F(z)=a_1z+a_2z^2+\cdots$, where $a_1\neq 0$, be convergent for $|z|<\rho\ (\leqq\infty)$. Let $r<\rho$ be given and let $r_1, r_2, \cdots$ be a sequence of positive numbers such that $r_n\leqq r$ for all n. Let S_n denote the curve $x=x_n(\theta)=F(r_ne^{2\pi i\theta})$ and ϕ_n the distribution function in R_x determined by this parametric representation of S_n. Then the convergence of $r_1^2+r_2^2+\cdots$ is necessary and sufficient for both the convergence and the absolute convergence of $\phi_1*\phi_2*\cdots$. The spectrum $S(\psi)$ of $\psi=\phi_1*\phi_2*\cdots$ is then either a bounded set or the whole R_x according as $r_1+r_2+\cdots$ is convergent or divergent. The distribution function ψ is always absolutely continuous with a continuous density $D(x)=D(\xi_1,\xi_2)$ possessing continuous partial derivatives of arbitrarily high order. Finally $D(x)>0$ for all x if $r_1+r_2+\cdots$ diverges.*

Since $r_n\leqq r<\rho$, all $S(\phi_n)$ are contained in a sufficiently large circle $|x|\leqq K$. Furthermore, $c(\phi_n)=0$ so that Theorems 5 and 6 give as a necessary and sufficient condition for both the convergence and the absolute convergence of $\phi_1*\phi_2*\cdots$ the convergence of the series

$$\sum_{n=1}^{\infty} \mathrm{M}_2(\phi_n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |a_m|^2 r_n^{2m}.$$

This proves the first part of the theorem. Since $r<\rho$, there exists a constant C such that $|F(z)|\leqq C|z|$ and $|F(z)-a_1z|\leqq C|z|^2$ for $|z|\leqq r$. Hence if $r_1+r_2+\cdots$ converges, $S(\psi)$ is a bounded set. Suppose now that $r_1+r_2+\cdots$ diverges and let x_0 be any point of R_x. For a given $\epsilon>0$, let p be so large that $r_{p+1}^2+r_{p+2}^2+\cdots<\epsilon/C$. Let x_1 be an arbitrary point of the form $F(r_1e^{2\pi i\theta_1})+\cdots+F(r_pe^{2\pi i\theta_p})$; then if $n>p$ is large enough we have $|a_1|(r_{p+1}+\cdots+r_n)>|x_0-x_1|$ and none of the numbers $r_{p+1},\cdots,r_n$ is larger than the sum of the $n-p-1$ others. As in Theorem 9 we may therefore choose $\theta_{p+1},\cdots,\theta_n$ such that $a_1(r_{p+1}e^{2\pi i\theta_{p+1}}+\cdots+r_ne^{2\pi i\theta_n})=x_0-x_1$, implying that $|F(r_1e^{2\pi i\theta_1})+\cdots+F(r_ne^{2\pi i\theta_n})-x_0|<\epsilon$. Hence the circle $|x-x_0|<\epsilon$ contains for sufficiently large n points of $S_1+\cdots+S_n$, which means by Theorem 3 that $S(\psi)$ contains the arbitrary point x_0. Since $r_n\to 0$ it follows from Theorem 13 that $\Lambda(y;\psi)$ satisfies (3.1) for all p. For the proof of the last statement of Theorem 14, cf. the Remark following Theorem 9.

Remark. An analogous reasoning shows that if $r_n\to 0$ then $\psi_n=\phi_1*\cdots*\phi_n$ is absolutely continuous with a continuous density $D_n(x)$ if $n>n_0$ and $D_n(x)=D_n(\xi_1,\xi_2)$ possesses continuous partial derivatives of order $\leqq p$ if $n>n_p$. If $r_1^2+r_2^2+\cdots$ converges then $D_n(x)$ and its partial derivatives converge uniformly to $D(x)$ and its partial derivatives, whereas they tend uniformly to zero as $n\to\infty$ if $r_1^2+r_2^2+\cdots$ diverges; for a proof cf. the Remark following Theorem 8.

If $r_1+r_2+\cdots$ is convergent then the density $D(x)=D(\xi_1,\xi_2)$ cannot be regular analytic in every point of the real plane R_x since $D(x)\equiv 0$ outside of the bounded set $S(\psi)$. We have, however, the following theorem:

THEOREM 15. *If $r_n^{-1}=O(n)$, then the density $D(x)=D(\xi_1,\xi_2)$ defined in Theorem 14 is regular analytic in every point of the real plane R_x. If $r_n^{-1}=o(n)$, then $D(x)$ is an entire function of the two variables ξ_1,ξ_2.*

Let $\rho_1<\rho_0$ be fixed and let β denote the corresponding constant defined in Theorem 13. Suppose first that all $r_n\leqq\rho_1$. We have then $|\Lambda(y;\phi_n)|\leqq\beta r_n^{-1/2}|y|^{-1/2}$ for every n. Since $r_n^{-1}\leqq an$ for some $a>0$ and $|\Lambda(y;\phi_n)|\leqq 1$, it follows that

$$|\Lambda(y;\phi_n)|\leqq\min(1,\beta a^{1/2}n^{1/2}|y|^{-1/2}).$$

Thus on placing $t=\beta^{-2}a^{-1}|y|$ we have

$$|\Lambda(y;\phi_n)|\leqq\min(1,n^{1/2}t^{-1/2}).$$

It follows therefore from

$$|\Lambda(y;\psi)| = \prod_{n=1}^{\infty} |\Lambda(y;\phi_n)| \leqq \prod_{n\leqq t} |\Lambda(y;\phi_n)|$$

that

$$|\Lambda(y;\psi)| \leqq \prod_{n\leqq t} \min(1, n^{1/2}t^{-1/2}) = \prod_{n\leqq t} n^{1/2}t^{-1/2}$$

for every t. Hence $\Lambda(y;\psi)=O(t^{1/4}e^{-t/2})$, by Stirling's formula. We see therefore from $t=\beta^{-2}a^{-1}|y|$ that $\Lambda(y;\psi)$ satisfies (3.2) with every $A<\frac{1}{2}\beta^{-2}a^{-1}$. Let us now drop the assumption $r_n\leqq\rho_1$ and let $b=\limsup_{n\to\infty} r_n^{-1}/n$. Then if $a>b$ there exists an n_0 such that $r_n\leqq\rho_1$ and $r_n^{-1}\leqq a(n-n_0)$ for all $n>n_0$. Hence $\Lambda(y;\psi)$ satisfies again (3.2) with every $A<\frac{1}{2}\beta^{-2}a^{-1}$, which proves the first part of the theorem. If $b=0$ we may take a arbitrarily small; hence $\Lambda(y;\psi)$ satisfies (3.2) for arbitrarily large A which proves the second part of the theorem.

Returning to the case of general r_n we shall now give an appraisal for the density $D(x)$ occurring in Theorem 14.

THEOREM 16. *For any $\lambda>0$ the density $D(x)$ defined in Theorem 14 is $=O(e^{-\lambda|x|^2})$ as $|x|\to\infty$ and each of its partial derivatives also is $=O(e^{-\lambda|x|^2})$ as $|x|\to\infty$.*

The proof is based upon an argument of Paley and Zygmund [44]. Let λ be fixed, let q be a fixed positive integer so large that

$$d = 1 - 2\lambda|a_1|^2(r_{q+1}^2 + r_{q+2}^2 + \cdots)$$

is positive, and let n be a variable integer $>q$. Placing

$$s_n(\theta_1,\cdots,\theta_n) = F(r_1e^{2\pi i\theta_1}) + \cdots + F(r_ne^{2\pi i\theta_n})$$

and

$$t_n(\theta_{q+1},\cdots,\theta_n) = a_1r_{q+1}e^{2\pi i\theta_{q+1}} + \cdots + a_1r_ne^{2\pi i\theta_n}$$

we have

$$|s_n(\theta_1,\cdots,\theta_n) - t_n(\theta_{q+1},\cdots,\theta_n)| \leqq A$$

where A is a constant independent of n in virtue of the convergence of $r_1^2+r_2^2+\cdots$. Hence from $|s_n|^2\leqq 2|s_n-t_n|^2+2|t_n|^2$

$$\int_{c_n} d\theta_n \cdots \int_{c_1} e^{\lambda|s_n(\theta_1,\cdots,\theta_n)|^2}d\theta_1 \leqq e^{2\lambda A^2}\int_{c_n} d\theta_n \cdots \int_{c_{q+1}} e^{2\lambda|t_n(\theta_{q+1},\cdots,\theta_n)|^2}d\theta_{q+1}$$

where c_ν denotes the circle of length 1 on which θ_ν is the variable point. Now it is known (cf., for a detailed proof, Jessen [35], pp. 290–291) that the integral on the right is

$$\leqq (1 - 2\lambda(|a_1|^2 r_{q+1}^2 + \cdots + |a_1|^2 r_n^2))^{-1} \leqq d^{-1}$$

so that

$$\int_{c_n} d\theta_n \cdots \int_{c_1} e^{\lambda |s_n(\theta_1, \cdots, \theta_n)|^2} d\theta_1 \leqq K$$

where $K = e^{2\lambda A^2} d^{-1}$. Placing $h(x) = e^{\lambda |x|^2}$ in (2.2) and using the relation

$$\int_{R_{x_n}} \phi_n(dR_{x_n}) \cdots \int_{R_{x_1}} e^{\lambda |x_1 + \cdots + x_n|^2} \phi_1(dR_{x_1}) = \int_{c_n} d\theta_n \cdots \int_{c_1} e^{\lambda |s_n(\theta_1, \cdots, \theta_n)|^2} d\theta_1,$$

which is clear from the definition of ϕ_ν, one obtains

$$\int_{R_x} e^{\lambda |x|^2} \psi_n(dR_x) \leqq K.$$

It follows therefore from (2.1) by letting $n \to \infty$ that

$$\int_{R_x} e^{\lambda |x|^2} \psi(dR_x) \leqq K.$$

Consequently if E is a bounded Borel set in R_x then $\psi(x - E) = O(e^{-\lambda |x|^2})$ as $|x| \to \infty$; in fact, if E lies in the circle of radius r about the origin then $e^{\lambda(|x| - r)^2} \psi(x - E) \leqq K$ if $|x| > r$. Similarly, if n is arbitrary and $\rho_n = \phi_{n+1} * \phi_{n+2} * \cdots$ then $\rho_n(x - E) = O(e^{-\lambda |x|^2})$ as $|x| \to \infty$ for any bounded Borel set E. Now ψ_n is by the Remark following the proof of Theorem 14 absolutely continuous with a continuous density $D_n(x)$ if $n > n_0$; hence from $\psi = \psi_n * \rho_n$

$$D(x) = \int_{R_u} D_n(x - u) \rho_n(dR_u), \tag{8.1}$$

so that $D(x) \leqq M_n \rho_n(x - S(\psi_n))$ where M_n is the maximum of $D_n(x)$. Since $S(\psi_n)$ is a bounded set it follows that $D(x) = O(e^{-\lambda |x|^2})$ as $|x| \to \infty$. Also, if $p > 0$ is given, $D_n(x)$ has continuous partial derivatives of order $\leqq p$ for $n > n_p$ and the corresponding partial derivatives of $D(x)$ may be obtained from (8.1) by differentiation under the integral sign. This implies that each partial derivative of $D(x)$ is $= O(e^{-\lambda |x|^2})$ as $|x| \to \infty$.

Remark. Theorem 16 implies that $\mathrm{M}_p(\psi)$ is finite for every p so that all moments $\mu_{q_1, q_2}(\psi)$ of ψ exist; furthermore, these moments belong to a *determined* moment problem (cf. Haviland [29]). This remark applies in particular to the infinite convolution ψ_σ occurring in Theorem 19.

9. Vectorial addition of convex curves

In order to determine the spectrum $S(\psi)$ of the distribution function $\psi = \phi_1 * \phi_2 * \cdots$, occurring in Theorem 14, in the case where $r_1 + r_2 + \cdots$

converges and all S_n are convex, we first prove a theorem concerning the vectorial sum of n convex curves. If C is a convex curve we shall denote by $I(C)$ and $E(C)$ the two open domains into which C divides R_z, such that $I(C)$ is the interior and $E(C)$ the exterior domain, and we shall denote by $\bar{I}(C)$ and $\bar{E}(C)$ the closures of $I(C)$ and $E(C)$.

THEOREM 17. *If $S_1, \cdots, S_n$ are n convex curves, then their vectorial sum $T_n = S_1 + \cdots + S_n$ is either the closed convex domain $\bar{I}(B_n)$ determined by a convex curve B_n or it is a ring-shaped domain, namely a closed convex domain $\bar{I}(B_n)$ minus an open convex domain $I(A_n)$.*

For $n=1$ the theorem is true with $B_1 = A_1 = S_1$. Suppose that it is true for n and let us prove it for $n+1$. The complementary set T_n' of T_n consists either of a single domain $E(B_n)$ or of two domains $E(B_n)$ and $I(A_n)$. Now in order that a point x shall belong to the complementary set T_{n+1}' of T_{n+1} it is necessary and sufficient that the curve $x - S_{n+1}$ belongs to T_n'. This may happen in three different ways:

(i) it may belong to $I(A_n)$;

(ii) it may belong to $E(B_n)$ and contain B_n in its interior;

(iii) it may belong to $E(B_n)$ without containing B_n in its interior.

(i) occurs only when A_n exists; it means that $\bar{I}(x - S_{n+1})$ is contained in $I(A_n)$; (ii) means that $\bar{E}(x - S_{n+1})$ is contained in $E(B_n)$ and (iii) means that $\bar{I}(x - S_{n+1})$ is contained in $E(B_n)$. There are always points x for which (iii) occurs whereas (i) and (ii) need not occur and are mutually exclusive. Hence it is sufficient to prove that the set of points x for which (iii) occurs is of the type $E(B_{n+1})$ where B_{n+1} is a convex curve and that the set of points x for which (i) or (ii) occurs (if such points exist) is of the type $I(A_{n+1})$ where A_{n+1} is a convex curve. This is trivial as far as (i) and (ii) are concerned; in fact if (i) is true for both $x = x_1$ and $x = x_2$ it is obviously true for any x on the segment x_1x_2 and the same holds for (ii). We therefore only need to consider the case (iii). The complementary set H to the set of points x for which (iii) occurs is the set of points for which $\bar{I}(x - S_{n+1})$ and $\bar{I}(B_n)$ have at least one point in common. Now $\bar{I}(x - S_{n+1}) = x - \bar{I}(S_{n+1})$. Hence H is simply the vectorial sum of the two closed convex domains $\bar{I}(B_n)$ and $\bar{I}(S_{n+1})$ which is known to be a closed convex domain $\bar{I}(B_{n+1})$. This completes the proof of Theorem 17.

THEOREM 18. *If $S_1, S_2, \cdots$ are convex curves surrounding the origin such that the diameter d_n of S_n tends to zero as $n \to \infty$ then the vectorial sum $T = S_1 + S_2 + \cdots$ is a bounded set if and only if the series $d_1 + d_2 + \cdots$ is convergent and T is then either a closed convex domain $\bar{I}(B)$ or a ring-shaped domain, namely a closed convex domain $\bar{I}(B)$ minus an open convex domain $I(A)$.*

With the previous notation it is clear from the proof of Theorem 17 that the curves $B_1, B_2, \cdots$ all surround the origin and that B_{n+1} surrounds B_n. Hence, if n is sufficiently large B_n surrounds S_{n+1} so that the possibility (ii) is excluded. This implies that from a certain n on the existence of A_n is necessary for the existence of A_{n+1} which is then surrounded by A_n. Since the diameters of the curves B_n remain bounded if and only if the series $d_1+d_2+\cdots$ is convergent, Theorem 18 follows from $T=\lim T_n$.

Remark. The supporting function (Stützfunktion) of B is the sum of the supporting functions of $S_1, S_2, \cdots$. There is no corresponding rule for the supporting function of A and there is not even a simple rule enabling us to decide whether A exists. If, however, one of the given curves, say S_1, surrounds all the others we have the rule that A exists if and only if there exist points x such that $x-(S_2+S_3+\cdots)$ belongs to $I(S_1)$ and these x form then the domain $I(A)$.

Theorem 18 applies in particular to the spectrum $S(\psi)$ of the infinite convolution $\psi=\phi_1*\phi_2*\cdots$, occurring in Theorem 14, in the case where $r_1+r_2+\cdots$ is convergent and every $S(\phi_n)$ is convex.

The problem considered in this section has been studied in greater detail by Bohr [7]; cf. also Bohr and Jessen [19] and Haviland [26].

10. Distribution functions occurring in the theory of the zeta function

For later application we consider the case $F(z)=z+\frac{1}{2}z^2+\cdots=-\log(1-z)$ where $|z|<1$. The curve S defined by the parametric representation $x=x(\theta)=\xi_1(\theta)+i\xi_2(\theta)=-\log(1-re^{2\pi i\theta})$, where $0<r<1$, is convex since the angle between the tangent of S at the point θ and a horizontal line through this point is, in virtue of the conformity, equal to the angle between the tangent of the circle $z=1-re^{2\pi i\theta}$ at the point θ and the line joining the origin with this point. Furthermore, if τ is arbitrary, the function $g_\tau''(\theta)$ has exactly two zeros θ on S since $g_\tau''(\theta)$ is a trigonometrical polynomial in $2\pi\theta$ of the first order multiplied by a non-vanishing factor. Thus the number ρ_0 defined in Theorem 13 is $=1$. It may be mentioned that S surrounds the origin and that its diameter d tends to zero as $r\to 0$. Also, the distribution function determined by the parametric representation of S is symmetrical with respect to the line $\xi_2=0$. Applying the results of §8 and §9 we prove the following theorem:

THEOREM 19. *Let $p_1, p_2, \cdots$ denote the prime numbers* 2, 3, $\cdots$, *and let $\sigma>0$ be fixed. Let $S_{n,\sigma}$ denote the curve $x=x_{n,\sigma}=-\log(1-p_n^{-\sigma}e^{2\pi i\theta})$ and $\phi_{n,\sigma}$ the distribution function determined by this parametric representation of $S_{n,\sigma}$. Then the infinite convolution $\phi_{1,\sigma}*\phi_{2,\sigma}*\cdots$ is convergent and also absolutely*

convergent if and only if $\sigma > \frac{1}{2}$, *and* $\psi_\sigma = \phi_{1,\sigma} * \phi_{2,\sigma} * \cdots$ *is then symmetric with respect to the line* $\xi_2 = 0$. *If* $\frac{1}{2} < \sigma \leqq 1$, *the spectrum* $S(\psi_\sigma)$ *is* $= R_x$; *if* $\sigma > 1$, *it is either a closed convex domain* $\bar{I}(B_\sigma)$ *or a ring-shaped domain, namely a closed convex domain* $\bar{I}(B_\sigma)$ *minus an open convex domain* $I(A_\sigma)$. *The distribution function* ψ_σ *is always absolutely continuous and its density* $D_\sigma(x) = D_\sigma(\xi_1, \xi_2)$ *is continuous and possesses continuous partial derivatives of all orders. If* $\frac{1}{2} < \sigma \leqq 1$, *then* $D_\sigma(x) > 0$ *for all* x. *For any fixed* $\lambda > 0$ *we have* $D_\sigma(x) = O(e^{-\lambda |x|^2})$ *as* $|x| \to \infty$ *and every partial derivative of* $D_\sigma(x)$ *also is* $= O(e^{-\lambda |x|^2})$ *as* $|x| \to \infty$. *Finally, if* $\frac{1}{2} < \sigma < 1$, *then* $D_\sigma(x)$ *is an entire function of the two variables* ξ_1, ξ_2.

The first part of the theorem follows from Theorem 14 since $\sigma > \frac{1}{2}$ is necessary and sufficient for the convergence of $p_1^{-2\sigma} + p_2^{-2\sigma} + \cdots$. The symmetry of ψ_σ with respect to the line $\xi_2 = 0$ is obvious. Since the condition $\sigma > 1$ is necessary and sufficient for the convergence of $p_1^{-\sigma} + p_2^{-\sigma} + \cdots$, it follows from Theorem 14 that $S(\psi_\sigma)$ is $= R_x$ if $\sigma \leqq 1$ and is a bounded set if $\sigma > 1$; the description of $S(\psi_\sigma)$ for $\sigma > 1$ follows from Theorem 18. The absolute continuity of ψ_σ and the properties of $D_\sigma(x)$ including the appraisals follow from Theorems 14 and 16. Finally, the last part of the theorem follows from Theorem 15 since $p_n^\sigma = o(n)$ for every $\sigma < 1$ (this is an *elementary* property of the prime numbers).

It is clear that $D_\sigma(x)$ is not regular analytic in every point of R_x if $\sigma > 1$; we do not know what is the situation if $\sigma = 1$.

Remark. By means of the Remark following the proof of Theorem 18 it is easy to obtain a more detailed description of $S(\psi_\sigma)$ for $\sigma > 1$. The set $\bar{I}(B_\sigma)$ always contains the origin; it decreases when σ increases and the limits of $\bar{I}(B_\sigma)$ for $\sigma \to 1$ and $\sigma \to \infty$ are R_x and the point $x = 0$ respectively. Furthermore, A_σ does not exist if σ is sufficiently near to 1, say $\sigma \leqq \sigma_1$, while A_σ exists if σ is sufficiently large, say $\sigma > \sigma_2$. It is not known whether or not $\sigma_1 = \sigma_2$. Finally, $S(\psi_\sigma)$ is symmetric not only with respect to the line $\xi_2 = 0$ but also with respect to the line $\xi_1 = -\frac{1}{2} \log (1 - p_1^{-2\sigma}) - \frac{1}{2} \log (1 - p_2^{-2\sigma}) - \cdots = \frac{1}{2} \log \zeta(2\sigma)$; this line is not a symmetry axis of ψ_σ.

The explicit expression for the density $D_\sigma(x)$ as given by the inversion formula enables us to discuss $D_\sigma(x)$ as a function of σ. We give only one result in this direction:

THEOREM 20. *The function* $D_\sigma(x)$ *and each of its partial derivatives tend uniformly to zero as* $\sigma \to \frac{1}{2}$.

The Fourier transforms $\Lambda(y; \psi_\sigma)$ satisfy (3.1) uniformly for $\frac{1}{2} < \sigma < \sigma_0$ where $\sigma_0 > \frac{1}{2}$ is arbitrary but fixed. It is therefore sufficient to prove that $\Lambda(y; \psi_\sigma) \to 0$ as $\sigma \to \frac{1}{2}$ uniformly for $|y| \geqq \epsilon$ where $\epsilon > 0$ is arbitrary and this is a

simple consequence of the divergence of $p_1^{-1}+p_2^{-1}+\cdots$. Cf. the Remark following Theorem 8.

Remark. It may be mentioned that since $\rho_0=1$, the distribution function $\psi_{n,\sigma}=\phi_{1,\sigma}*\cdots*\phi_{n,\sigma}$ is for every $\sigma>0$ absolutely continuous with a continuous density $D_{n,\sigma}(x)$ whenever $n>4$, and $D_{n,\sigma}(x)=D_{n,\sigma}(\xi_1,\xi_2)$ possesses continuous partial derivatives of order $\leqq p$ whenever $n>4+2p$. If $\sigma>\frac{1}{2}$ then $D_{n,\sigma}(x)$ and its partial derivatives tend uniformly to $D_\sigma(x)$ and its partial derivatives, whereas they tend uniformly to zero as $n\to\infty$ if $0<\sigma\leqq\frac{1}{2}$.

Let $x=\xi_1+i\xi_2$ and let R_x be mapped on itself by the transformation e^x; every point $x\neq 0$ is then the image of the enumerable set of points $\log x$. If E is an arbitrary set in R_x we denote by e^E the set of all points e^x where x belongs to E and by $\log E$ the set of all points x such that e^x belongs to E; we notice that $\log(e^E)$ is not necessarily E. If E is a Borel set, then so are the sets e^E and $\log E$. If ϕ is a distribution function in R_x, a new distribution function $\bar{\phi}$ is defined by the relation $\bar{\phi}(E)=\phi(\log E)$; the spectrum $S(\bar{\phi})$ is the closure of $e^{S(\phi)}$. The set $\log E$ being a null-set if and only if E is a null-set, the absolute continuity of one of the distribution functions ϕ and $\bar{\phi}$ implies that of the other. If $D(x)$ and $\bar{D}(x)$ denote the densities of ϕ and $\bar{\phi}$ and if we write

$$\Delta(x)=\sum_{\mu=-\infty}^{\infty}D(x+2\pi i\mu),$$

we have $\bar{D}(x)=|x|^{-2}\Delta(\log x)$; it does not matter that in this expression $x=0$ is excluded, $D(x)$ and $\bar{D}(x)$ being determined only up to null-functions. We have now the following theorem:

THEOREM 21. *Let $\bar{\psi}_\sigma$ be the distribution function defined by $\bar{\psi}_\sigma(E)=\psi_\sigma(\log E)$, where $\sigma>\frac{1}{2}$ and ψ_σ denotes the distribution function defined in Theorem 19. Then $\bar{\psi}_\sigma$ is symmetric with respect to the line $\xi_2=0$. If $\frac{1}{2}<\sigma\leqq 1$ the spectrum $S(\bar{\psi}_\sigma)$ is $=R_x$; if $\sigma>1$ then $S(\bar{\psi}_\sigma)=e^{S(\psi_\sigma)}$ which is a closed bounded set not containing $x=0$. The distribution function $\bar{\psi}_\sigma$ is always absolutely continuous and its density $\bar{D}_\sigma(x)$ is continuous and possesses continuous partial derivatives of any order. For $\frac{1}{2}<\sigma\leqq 1$ we have $\bar{D}_\sigma(x)>0$ for all $x\neq 0$, while $\bar{D}_\sigma(0)=0$. Furthermore, if $\lambda>0$ is arbitrary then $\bar{D}_\sigma(x)=O(e^{-\lambda(\log|x|)^2})$ as $|x|\to\infty$ or $|x|\to 0$, and every partial derivative of $\bar{D}_\sigma(x)$ also is $=O(e^{-\lambda(\log|x|)^2})$ as $|x|\to\infty$ or $|x|\to 0$. Finally, if $\frac{1}{2}<\sigma<1$ then $\bar{D}_\sigma(e^x)$ is an entire function of the two variables ξ_1,ξ_2.*

The statements concerning $S(\bar{\psi}_\sigma)$ and the absolute continuity of $\bar{\psi}_\sigma$ are obvious consequences of Theorem 19. For the density $\bar{D}_\sigma(x)$ we find $\bar{D}_\sigma(x)=|x|^{-2}\Delta_\sigma(\log x)$ where

$$\Delta_\sigma(x)=\sum_{\mu=-\infty}^{\infty}D_\sigma(x+2\pi i\mu).$$

From Theorem 19 we conclude that $\Delta_\sigma(x)$ is a continuous function of x possessing continuous partial derivatives of arbitrarily high order. Also, if $\lambda>0$ then $\Delta_\sigma(x)=O(e^{-\lambda\xi_1^2})$ as $|\xi_1|\to\infty$ and every partial derivative of $\Delta_\sigma(x)$ also is $=O(e^{-\lambda\xi_1^2})$ as $|\xi_1|\to\infty$. This proves the continuity of $\overline{D}_\sigma(x)$ and the existence and continuity of its partial derivatives for $x\neq 0$. Furthermore,

$$\overline{D}_\sigma(x) = O(|x|^{-2}e^{-\lambda(\log|x|)^2})$$

as $|x|\to\infty$ or $|x|\to 0$, while every partial derivative of $\overline{D}_\sigma(x)$ of order p is $=O(|x|^{-2-p}e^{-\lambda(\log|x|)^2})$ as $|x|\to\infty$ or $|x|\to 0$. Since $\lambda>0$ is arbitrary the factors $|x|^{-2}$ and $|x|^{-2-p}$ may be omitted. These appraisals imply the continuity of $\overline{D}_\sigma(x)$ and its partial derivatives at $x=0$ also, at which point all these functions vanish. That $\overline{D}_\sigma(x)>0$ for $x\neq 0$ if $\frac{1}{2}<\sigma\leqq 1$ is clear from Theorem 19.

Placing $\psi_\sigma' = \phi_{1,\sigma} * \phi_{3,\sigma} * \cdots$ and $\psi_\sigma'' = \phi_{2,\sigma} * \phi_{4,\sigma} * \cdots$ we have $\Lambda(y;\psi_\sigma) = \Lambda(y;\psi_\sigma')\Lambda(y;\psi_\sigma'')$, hence

$$\frac{\partial}{\partial\eta_2}\Lambda(y;\psi_\sigma) = \Lambda(y;\psi_\sigma')\frac{\partial}{\partial\eta_2}\Lambda(y;\psi_\sigma'') + \Lambda(y;\psi_\sigma'')\frac{\partial}{\partial\eta_2}\Lambda(y;\psi_\sigma'),$$

the existence and continuity of the partial derivatives being implied by the finiteness of $\mathrm{M}_1(\psi_\sigma)$, $\mathrm{M}_1(\psi_\sigma')$, $\mathrm{M}_1(\psi_\sigma'')$, which is assured by Theorem 16. The finiteness of these numbers also implies the boundedness of the partial derivatives. Hence

$$\frac{\partial}{\partial\eta_2}\Lambda(y;\psi_\sigma) = O(|y|^{-q})$$

for every q. Since by the inversion formula we have for every n

$$\sum_{\mu=-n}^{n} D_\sigma(x+2\pi i\mu) = (2\pi)^{-2}\int_{R_y} e^{-ixy}\Lambda(y;\psi_\sigma)\,\frac{\sin(2n+1)\pi\eta_2}{\sin\pi\eta_2}\,m(dR_y),$$

where $y=\eta_1+i\eta_2$, it is clear that

$$\Delta_\sigma(x) = (2\pi)^{-2}\int_{-\infty}^{\infty}\sum_{\nu=-\infty}^{\infty} e^{-ix(\eta_1+i\nu)}\Lambda(\eta_1+i\nu;\psi_\sigma)d\eta_1. \tag{10.1}$$

From this representation of $\Delta_\sigma(x)$ as a combined Fourier series and Fourier integral follows, in particular, by an argument exactly like the one applied in §3, that $\Delta_\sigma(x)$ is an entire function of the two variables ξ_1, ξ_2 if $\frac{1}{2}<\sigma<1$. This completes the proof of Theorem 21.

The last statement of Theorem 21 implies that if $\frac{1}{2}<\sigma<1$, then $\overline{D}_\sigma(x)$ is regular analytic at every point $x\neq 0$ of the real plane R_x; the point $x=0$ is

actually an exception since at this point $\overline{D}_\sigma(x)$ and all its partial derivatives vanish.

By means of (10.1) it is easy to discuss $\overline{D}_\sigma(x)$ as a function of σ. We give only one result in this direction, which corresponds to Theorem 20:

THEOREM 22. *The function $\overline{D}_\sigma(x)$ multiplied by $|x|^2$ and each of its partial derivatives of order p multiplied by $|x|^{2+p}$ tend uniformly to zero as $\sigma \to \frac{1}{2}$.*

The theorem is equivalent to the statement that $\Delta_\sigma(x)$ and each of its partial derivatives tend uniformly to zero as $\sigma \to \frac{1}{2}$, which follows from (10.1) by the argument used in the proof of Theorem 20.

We do not know whether Theorem 22 holds if the factors $|x|^2$ and $|x|^{2+p}$ are omitted; we do not even know whether $\overline{\psi}_\sigma(E) \to 0$ as $\sigma \to \frac{1}{2}$ for any bounded set E.

11. Asymptotic distribution functions

Let G be an abstract space with t as variable point and in G let there be defined a measure m such that the system of sets A for which $m(A)$ is defined is a Borel field and m is non-negative and completely additive; we suppose that G belongs to the system and that $m(G) = \infty$. The sets A for which $m(A)$ is defined are called measurable sets. Lebesgue integrals with respect to m will be denoted by

$$\int_A f(t)m(dG).$$

We suppose that certain sequences $A_1, A_2, \cdots$ consisting of measurable sets of positive finite measure such that $m(A_n) \to \infty$ have been selected and call these sequences *admissible sequences*. A real or complex function $f(t)$ which is measurable in G is said to have the *mean value* $M(f) = M(f(t))$ if

$$M(f) = \lim_{n\to\infty} \frac{1}{m(A_n)} \int_{A_n} f(t)m(dG)$$

for each admissible sequence. A measurable set A in G is said to have the *relative measure* $\rho(A)$ if

$$\rho(A) = \lim_{n\to\infty} \frac{m(A\,A_n)}{m(A_n)}$$

for each admissible sequence. If $f(t) \geqq 0$ is measurable in G its *upper mean value* $\overline{M}(f) = \overline{M}(f(t))$ is the least upper bound of

$$\limsup_{n\to\infty} \frac{1}{m(A_n)} \int_{A_n} f(t)m(dG)$$

for all admissible sequences. Similarly, if A is a measurable set in G its *upper relative measure* $\bar{\rho}(A)$ is the least upper bound of

$$\limsup_{n\to\infty} \frac{m(A\,A_n)}{m(A_n)}$$

for all admissible sequences. These notions depend, of course, on the definition of admissible sequences, which is supposed to be fixed once for all. It is clear when for a class of functions or sets the mean values or relative measures shall be said to exist uniformly for all functions or sets of the class.

Let $x(t)$ be a measurable vector function with k components defined in G. Then if E is a Borel set in R_x, the set A_E of those points t in G for which $x(t)$ belongs to E is measurable. We say that $x(t)$ has an *asymptotic distribution function* if there exists in R_x a distribution function ϕ such that for each continuity set E of ϕ the relative measure $\rho(A_E)$ exists and is $=\phi(E)$. It is clear that there exists at most one such distribution function ϕ. The restriction imposed on E that it should be a continuity set for ϕ is essential as will be seen from later examples. Another form of the definition is the following: For a measurable set A in G of positive finite measure, let ϕ_A denote the distribution function defined by

$$\phi_A(E) = \frac{m(A_E A)}{m(A)};$$

then $x(t)$ possesses an asymptotic distribution function ϕ if and only if $\phi_{A_n}\to\phi$ for any admissible sequence.

THEOREM 23. *The vector function $x(t)$ possesses an asymptotic distribution function ϕ if and only if the mean value $M(e^{ix(t)y})$ exists and*

$$M(e^{ix(t)y}) = \lim_{n\to\infty} \frac{1}{m(A_n)} \int_{A_n} e^{ix(t)y} m(dG) \tag{11.1}$$

holds uniformly in every sphere $|y|\leqq a$ in R_y for any admissible sequence, and we have then $\Lambda(y;\phi) = M(e^{ix(t)y})$.

For any measurable set A in G of positive finite measure we have

$$\Lambda(y;\phi_A) = \frac{1}{m(A)} \int_A e^{ix(t)y} m(dG);$$

the theorem follows therefore from §3.

A sequence of vector functions $x_1(t)$, $x_2(t)$, $\cdots$ is said to be *convergent in relative measure* to the limit function $x(t)$ if $\bar{\rho}(|x(t)-x_n(t)|>\epsilon)\to 0$ as $n\to\infty$ for every fixed $\epsilon>0$. From §3 and Theorem 23 follows immediately

THEOREM 24. *If $x_1(t)$, $x_2(t)$, $\cdots$ is a sequence of measurable vector functions which converges in relative measure to the vector function $x(t)$ and if every $x_n(t)$ has an asymptotic distribution function ϕ_n, then $x(t)$ also possesses an asymptotic distribution function ϕ and $\phi_n \to \phi$.*

The existence of a distribution function ϕ of $x(t)$ implies immediately the existence of the mean value $M(F(x(t)))$ for any bounded continuous function $F(x)$ in R_x and also gives the formula

$$(11.2) \qquad M(F(x(t))) = \int_{R_x} F(x)\phi(dR_x).$$

Hence we conclude that if the mean value $M(F(x(t)))$ exists for the function $F(x) = e^{ixy}$ where y is arbitrary, and if the limit relation (11.1) holds uniformly in every sphere $|y| \leqq a$ for any admissible sequence, then $M(F(x(t)))$ exists for any bounded continuous function $F(x)$ in R_x. One may start, of course, with other systems than the system of the functions $F(x) = e^{ixy}$ in order to obtain conditions for the existence of ϕ. In this direction we mention the theorem that if $|x(t)|$ is bounded then the existence of $M(F(x(t)))$ for any $F(x) = \xi_1^{q_1} \cdots \xi_k^{q_k}$ is necessary and sufficient for the existence of ϕ. This condition is equivalent to the one that every moment $\mu_{q_1,\cdots,q_k}(\phi_A)$ approaches for any admissible sequence a limit which is then $\mu_{q_1,\cdots,q_k}(\phi)$. This method still applies when $|x(t)|$ is not bounded but such that the limits of the moments belong to a *determined* moment problem (cf. Wintner [49], [54] and Fréchet and Shohat [25]). Hence the moment method applies only under restrictive conditions regarding $x(t)$ while the method of the Fourier transform applies whenever $x(t)$ has an asymptotic distribution function.

It is sometimes of interest to establish the existence of the mean value $M(F(x(t)))$ also for unbounded functions $F(x)$. In this direction we have the theorem (cf. Bohr and Jessen [21]) that if $x(t)$ has an asymptotic distribution function ϕ and if for some continuous function $H(x) \geqq 0$ the upper mean value $\overline{M}(H(x(t)))$ is finite, then $M(F(x(t)))$ exists for any continuous function $F(x)$ satisfying the condition $F(x) = o(H(x))$ as $|x| \to \infty$ and (11.2) is valid.

12. Asymptotic distribution functions of almost periodic functions

Let G be the real axis $-\infty < t < \infty$ and m the Lebesgue measure on it. Two cases will be considered; in the first case an admissible sequence is an arbitrary sequence of intervals (a_n, b_n) where $b_n - a_n \to \infty$; in the second case we allow only sequences $(0, b_n)$ where $b_n \to \infty$ and sequences $(a_n, 0)$ where $a_n \to -\infty$. We shall refer to the two cases as the *unrestricted* case and the *restricted* case, but we shall not distinguish the two cases by the use of different

notations for mean values, relative measures, etc. The existence of an asymptotic distribution function in the unrestricted case implies of course the existence in the restricted case also.

A vector function $x(t)$ is called almost periodic if each of its k components is almost periodic. Our notations will be those used by Besicovitch [1] so that in particular a *u.a.p.* function means a function almost periodic in the original sense of Bohr.

THEOREM 25. *Any u.a.p. vector function $x(t)$ possesses an asymptotic distribution function in the unrestricted case.*

The functions $e^{ix(t)y}$ form for $|y|\leqq a$ a majorisable class of *u.a.p.* functions; consequently the mean value

$$M(e^{ix(t)y}) = \lim_{n\to\infty} \frac{1}{b_n - a_n}\int_{a_n}^{b_n} e^{ix(t)y}dt$$

exists uniformly for $|y|\leqq a$ for any admissible sequence.

THEOREM 26. *The spectrum $S(\phi)$ of the asymptotic distribution function ϕ of a u.a.p. vector function $x(t)$ is the closure of the range of $x(t)$.*

Since the range of $x(t)$ is defined as the set of those points x_0 of R_x for which $x_0=x(t_0)$ holds for some t_0 it is clear that any point of $S(\phi)$ belongs to the closure of the range of $x(t)$. Conversely, if $x_0=x(t_0)$ for some t_0 then x_0 belongs to $S(\phi)$, which means that $\bar{p}(|x(t)-x(t_0)|<\epsilon)>0$ for every $\epsilon>0$. This is an easy consequence of the uniform continuity of $x(t)$ and the fact that $|x(t)-x(t_0)|<\epsilon/2$ for a relatively dense set of values t.

A measurable vector function $x(t)$ is *Wa.p.* if and only if there exists a sequence $x_1(t), x_2(t), \cdots$ of *u.a.p.* vector functions such that

$$\overline{M}(|x(t) - x_n(t)|) \to 0 \text{ as } n \to \infty \tag{12.1}$$

where the upper mean value belongs to the unrestricted case. A measurable vector function $x(t)$ is *B a.p.* if and only if the same holds with the sole difference that the upper mean value belongs to the restricted case.

THEOREM 27. *Any W a.p. vector function $x(t)$ possesses an asymptotic distribution function in the unrestricted case. Any B a.p. vector function possesses an asymptotic distribution function in the restricted case.*

Let $x(t)$ be *W a.p.*; from (12.1) and the inequality

$$\bar{p}(|x(t) - x_n(t)| > \epsilon) \leqq \epsilon^{-1}\overline{M}(|x(t) - x_n(t)|)$$

we deduce

$$\bar{p}(|x(t) - x_n(t)| > \epsilon) \to 0 \text{ as } n \to \infty$$

where the upper relative measure belongs to the unrestricted case. Hence the result follows from Theorem 24. If $x(t)$ is B *a.p.* the proof is the same with the sole difference that the upper relative measure belongs to the restricted case.

Let $x(t)$ be an almost periodic vector function (of one of the types considered) and ϕ its asymptotic distribution function; let E be a Borel set in R_x; then if A_E denotes the set of those points t for which $x(t)$ belongs to E we know that $\rho(A_E)$ exists and is $=\phi(E)$ for any continuity set E of ϕ, the relative measure belonging to the unrestricted case if $x(t)$ is W *a.p.* and to the restricted case if $x(t)$ is B *a.p.* If E is not a continuity set of ϕ the relative measure $\rho(A_E)$ need not exist and even when it exists it need not be $=\phi(E)$. A simple example (for $k=1$) of the first behavior was given by Bohr [17] who constructed a *u.a.p.* function $x(t)$ for which $\rho(A_E)$ does not exist (not even in the restricted case) for a certain interval E. An example of the second behavior (for $k=2$) is the function $x(t)=\log \zeta(\sigma+it)$ where $\sigma>1$ is fixed; this function is *u.a.p.* and its asymptotic distribution function is absolutely continuous (§14); on the other hand, the range of $x(t)$ is a null-set; hence if we take E to be the range of $x(t)$, we have $\rho(A_E)=1$ and $\phi(E)=0$.

Let now E be a continuity set of ϕ and let $f(t)$ denote the function which is 1 or 0 according as $x(t)$ does or does not belong to E. Then we know that $M(f(t))$ exists and is $=\phi(E)$. It is natural to ask if $f(t)$ is also almost periodic (in some sense). We shall prove that $f(t)$ is W *a.p.* if $x(t)$ is W *a.p.* and B *a.p.* if $x(t)$ is B *a.p.* We prove this as follows: The fact that E is a continuity set of ϕ makes it possible, corresponding to any given $\epsilon>0$, to find two continuous functions $F(x)$ and $G(x)$ in R_x such that $0\leqq F(x)\leqq G(x)\leqq 1$ for all x, $F(x)=0$ when x does not belong to E, $G(x)=1$ when x belongs to E, and finally

$$\int_{R_x} (G(x) - F(x))\phi(dR_x) < \epsilon.$$

This implies $F(x(t))\leqq f(t)\leqq G(x(t))$ and also

$$M(G(x(t)) - F(x(t))) < \epsilon$$

where the mean value belongs to the unrestricted or to the restricted case according as $x(t)$ is W *a.p.* or B *a.p.* Now $F(x(t))$ and $G(x(t))$ are W *a.p.* if $x(t)$ is W *a.p.* and B *a.p.* if $x(t)$ is B *a.p.* This leads to the desired conclusion.

If $x(t)$ is *u.a.p.* we find that $f(t)$ is W *a.p.* and we cannot say more than this; in particular we cannot say that $f(t)$ is S *a.p.* This is shown (for $k=1$) by the following example: Let $x_1(t)$ denote the periodic function with period 4 which is $=0$ for $|x|\leqq 1$ and $=|x|-1$ for $1\leqq|x|\leqq 2$ and let $x_n(t)$ denote the function $x_n(t)=2^{-n}x_1(2^{-n}t)$. Let $x(t)$ be the *u.a.p.* function defined by the

uniformly convergent series $x(t)=x_1(t)+x_2(t)+\cdots$. Then the set $x=0$ is a continuity set of the asymptotic distribution function ϕ of $x(t)$. The corresponding function $f(t)$ is $=1$ for $|x|\leqq 1$ and $=0$ elsewhere; hence $f(t)$ is not S *a.p.*

The methods of this section may be extended to functions of two or more variables; for a result in this direction cf. Wintner [58].

13. Independent moduli

By the modul of a B *a.p.* vector function is understood the smallest modul containing the Fourier exponents of each of the k components $\xi_1(t),\cdots,\xi_k(t)$ of $x(t)$. The B *a.p.* vector functions $x_1(t), x_2(t),\cdots$ are said to have independent moduli if a finite sum $\alpha_1+\cdots+\alpha_n$, where α_ν belongs to the modul of $x_\nu(t)$, is equal to zero only when all $\alpha_\nu=0$.

Theorem 28. *If $x_1(t),\cdots,x_n(t)$ are B a.p. vector functions with independent moduli then the asymptotic distribution function ϕ of $x(t)=x_1(t)+\cdots+x_n(t)$ is $\phi_1*\cdots*\phi_n$ where $\phi_1,\cdots,\phi_n$ are the asymptotic distribution functions of $x_1(t),\cdots,x_n(t)$ respectively.*

The statement is that $\Lambda(y;\phi)=\Lambda(y;\phi_1)\cdots\Lambda(y;\phi_n)$ which may be written according to Theorem 23 in the form

$$M(e^{i(x_1(t)+\cdots+x_n(t))y}) = M(e^{ix_1(t)y})\cdots M(e^{ix_n(t)y})$$

where the mean values belong to the restricted case. In virtue of the approximation theorem for B *a.p.* functions it is enough to verify the last relation in the case where all components of $x_1(t),\cdots,x_n(t)$ are exponential polynomials in which case it follows by a direct calculation.

With regard to this argument cf. Wintner [53–55], Bochner and Jessen [3]; this is the point where the explicit use of the theory of diophantine approximations is avoided as pointed out in §1.

The conditions of Theorem 28 are in particular satisfied if $x_1(t),\cdots,x_n(t)$ are periodic vector functions with periods $2\pi/\lambda_1,\cdots,2\pi/\lambda_n$ where $\lambda_1,\cdots,\lambda_n$ are linearly independent.

For $k=2$ we have as a simple application

Theorem 29. *Let $r_1, r_2,\cdots$ be positive, $\lambda_1,\lambda_2,\cdots$ linearly independent, and $\delta_1,\delta_2,\cdots$ real. Then the asymptotic distribution function of $s_n(t)=r_1e^{i(\lambda_1t+\delta_1)}+\cdots+r_ne^{i(\lambda_nt+\delta_n)}$ is the distribution function $\psi_n=\phi_1*\cdots*\phi_n$ described in Theorem 10. If $r_1e^{i(\lambda_1t+\delta_1)}+r_2e^{i(\lambda_2t+\delta_2)}+\cdots$ is the Fourier series of a B a.p. function $s(t)$, then $r_1^2+r_2^2+\cdots$ is convergent and the asymptotic distribution function of $s(t)$ is the distribution function $\psi=\phi_1*\phi_2*\cdots$ described in Theorem 10. Finally, if $s(t)$ is bounded then $r_1+r_2+\cdots$ converges.*

The first part of the theorem is an immediate consequence of Theorem 28 since the asymptotic distribution function of a pure oscillation $re^{i(\lambda t+\delta)}$ is the circular equidistribution on $|x|=r$. The second part of the theorem is clear from the relation

$$M(|s(t)-s_n(t)|)\to 0 \quad \text{as } n\to\infty$$

where the mean value belongs to the restricted case; this relation is a consequence of the approximation theorem for B *a.p.* functions .Use is made of the linear independence of the exponents. Finally, the last part of the theorem follows from Theorem 10 since the boundedness of $s(t)$ implies the boundedness of $S(\psi)$. In this last part of the theorem is contained the classical theorem of Bohr that the Fourier series of a *u.a.p.* function with linearly independent exponents is absolutely convergent; it is interesting that the original proof of Bohr [16], without using the notion of a distribution function, was built precisely on the same ideas as the present proof.

It is interesting that the smoothness of the asymptotic distribution function ψ of $s(t)$ established by Theorem 29 does not imply any smoothness for $s(t)$ itself. In fact (cf. Wintner [56]), if $r_n=a^n$, $\lambda_n=b^n$ and $\delta_n=0$ where $0<a<1$, $ab\geqq 1$ and b is a transcendental number, then $s(t)$ is *u.a.p.* with linearly independent exponents but is nowhere differentiable.

14. The Riemann zeta function

We now consider the Riemann zeta function $\zeta(s)=\zeta(\sigma+it)$. In the halfplane $\sigma>1$ we have

$$\zeta(s)=\prod_{n=1}^{\infty}(1-p_n^{-s})^{-1}$$

where $p_1, p_2, \cdots$ denote the primes $2, 3, \cdots$; in particular, $\zeta(s)\neq 0$ for $\sigma>1$. We write

$$\zeta_n(s)=\prod_{\nu=1}^{n}(1-p_\nu^{-s})^{-1};$$

$\zeta_n(s)$ is regular and $\neq 0$ for $\sigma>0$. By $\log\zeta(s)$ and $\log\zeta_n(s)$ we denote the functions

$$\log\zeta(s)=\sum_{n=1}^{\infty}-\log(1-p_n^{-s})$$

and

$$\log\zeta_n(s)=\sum_{\nu=1}^{n}-\log(1-p_\nu^{-s}),$$

where in each term on the right $-\log(1-z)=z+\frac{1}{2}z^2+\cdots$. The function $\log\zeta(s)$ is regular for $\sigma>1$ and $\log\zeta_n(s)$ for $\sigma>0$. By H we denote the domain obtained from the half-plane $\sigma>\frac{1}{2}$ by leaving out the segment $\frac{1}{2}<\sigma\leqq 1, t=0$, and all segments $\frac{1}{2}<\sigma\leqq\sigma_0$, $t=t_0$, where σ_0+it_0 denote the zeros (if any) of $\zeta(s)$ in $\sigma>\frac{1}{2}$; by $\log\zeta(s)$ for $\sigma>\frac{1}{2}$ we understand the analytic continuation of $\log\zeta(s)$ in H. For any fixed $\sigma>1$ the functions $\zeta(\sigma+it)$ and $\log\zeta(\sigma+it)$ are both *u.a.p.*; similarly, $\zeta_n(\sigma+it)$ and $\log\zeta_n(\sigma+it)$ are *u.a.p.* for any fixed $\sigma>0$ and tend uniformly to $\zeta(\sigma+it)$ and $\log\zeta(\sigma+it)$ if $\sigma>1$. For any fixed $\sigma>\frac{1}{2}$ the function $\zeta(\sigma+it)$ is B^2 *a.p.* and

$$M(|\zeta(\sigma+it)-\zeta_n(\sigma+it)|^2)\to 0 \text{ as } n\to\infty,$$

where the mean value belongs to the restricted case. This follows, e.g., from a result of Besicovitch [1], pp. 163–169, but is in the main of an older date. In the case $\sigma=1$ it is necessary in all integrations to leave out a vicinity of the pole $t=0$. Finally, it was proved by Bohr [12] that if $\sigma>\frac{1}{2}$ is fixed then

$$\bar{p}(|\log\zeta(\sigma+it)-\log\zeta_n(\sigma+it)|>\epsilon)\to 0 \text{ as } n\to\infty$$

for any $\epsilon>0$; the upper relative measure belongs to the restricted case.

Using our previous results we can now prove very easily

THEOREM 30. *The function* $\log\zeta(\sigma+it)$ *possesses an asymptotic distribution function in the unrestricted case if* $\sigma>1$ *and in the restricted case if* $\frac{1}{2}<\sigma\leqq 1$. *This asymptotic distribution function is the distribution function* $\psi_\sigma=\phi_{1,\sigma}*\phi_{2,\sigma}*\cdots$ *described in Theorems* 19 *and* 20. *The closure of the range of* $\log\zeta(\sigma+it)$ *is* $S(\psi_\sigma)$.

The first part of the theorem follows from Theorems 25 and 24. In order to prove the second part of the theorem we first observe that if S denotes the curve $x=x(\theta)=-\log(1-re^{2\pi i\theta})$ where $0<r<1$, and if ϕ is the distribution function in R_z determined by this parametric representation of S, then ϕ is also for an arbitrary $\lambda\neq 0$ the asymptotic distribution function of the function $-\log(1-re^{i\lambda t})$. Now if $\sigma>0$ then

$$\log\zeta_n(\sigma+it)=\sum_{\nu=1}^{n}-\log(1-p_\nu^{-\sigma}e^{-i\log p_\nu t});$$

since the numbers $\log p_1,\cdots,\log p_n$ are linearly independent it follows from Theorem 28 that $\log\zeta_n(\sigma+it)$ has the asymptotic distribution function $\phi_{1,\sigma}*\cdots*\phi_{n,\sigma}$. The second part of the theorem follows then by Theorem 24. Finally, the last part of the theorem is a consequence of Theorem 26 if $\sigma>1$ and is clear if $\frac{1}{2}<\sigma\leqq 1$ since $S(\psi_\sigma)$ is then the whole R_z.

For $\zeta(s)$ itself we have a corresponding result:

THEOREM 31. *The function $\zeta(\sigma+it)$ possesses an asymptotic distribution function in the unrestricted case if $\sigma>1$ and in the restricted case if $\frac{1}{2}<\sigma\leqq 1$. This asymptotic distribution function is the distribution function $\bar{\psi}_\sigma$ described in Theorems* 21 *and* 22. *The closure of the range of $\zeta(\sigma+it)$ is $S(\bar{\psi}_\sigma)$.*

The first part of the theorem follows from Theorems 25 and 27; the second part is an immediate consequence of the definition of $\bar{\psi}_\sigma$; finally, the last part of the theorem is a consequence of Theorem 26 if $\sigma>1$ and is clear if $\frac{1}{2}<\sigma\leqq 1$ since $S(\bar{\psi}_\sigma)$ is then the whole R_z.

It is clear that the relations

$$\bar{D}_\sigma(0) = 0; \quad \bar{D}_\sigma(x) > 0, \quad x \neq 0 \qquad (\tfrac{1}{2} < \sigma \leqq 1)$$

may be interpreted as an illustration to the Riemann hypothesis.

15. MEASURE AND INTEGRATION IN PRODUCT SPACES

We obtain further results concerning infinite convolutions by using the theory of measure and integration in product spaces of an infinite number of spaces. In the present section we collect some of the results of this theory. These results are proved for a special case in Jessen [35] where references to the literature are to be found. The proofs for the general case will be given by Jessen in a forthcoming paper.

Let Q be an abstract space with t as variable point and let there be defined in Q a measure m such that the system of sets A for which $m(A)$ is defined is a Borel field and m is non-negative and completely additive. We suppose that Q itself belongs to the field and that $m(Q)=1$. The sets A for which $m(A)$ is defined are called measurable sets; Lebesgue integrals with respect to m are denoted by

$$\int_A f(t)m(dQ).$$

It is well known that if q_1 and q_2 are two spaces of the type described before, with τ_1 and τ_2 as variable points and with μ_1 and μ_2 as measures, then these measures generate in the product space $Q=(q_1, q_2)$ with the variable point $t=(\tau_1, \tau_2)$ a measure $m=(\mu_1, \mu_2)$ in the following way. The system of sets in Q for which m is defined is the smallest Borel field containing all sets $A=(a_1, a_2)$ of Q where a_1 and a_2 are measurable sets in q_1 and q_2 respectively and m is characterized by the property that if $A=(a_1, a_2)$ then $m(A) =\mu_1(a_1)\mu_2(a_2)$. For integrals with respect to m we have Fubini's theorem

$$(15.1) \qquad \int_Q f(t)m(dQ) = \int_{q_2} \mu_2(dq_2) \int_{q_1} f(\tau_1, \tau_2)\mu_1(dq_1).$$

Let $q_1, q_2, \cdots$ be a finite or infinite sequence of abstract spaces of the type described above, with $\tau_1, \tau_2, \cdots$ as variable points and $\mu_1, \mu_2, \cdots$ as measures. Let Q denote the space $Q=(q_1, q_2, \cdots)$ where $t=(\tau_1, \tau_2, \cdots)$ is the variable point; then the measures $\mu_1, \mu_2, \cdots$ generate a measure $m=(\mu_1, \mu_2, \cdots)$ in Q in the following way. The system of sets in Q for which m is defined is the smallest Borel field containing all sets $A=(a_1, a_2, \cdots)$, where $a_1, a_2, \cdots$ are measurable sets in $q_1, q_2, \cdots$ respectively and m is characterized by the property that for sets of this type we have $m(A) =\mu_1(a_1)\mu_2(a_2)\cdots$.

Suppose now that the sequence $q_1, q_2, \cdots$ is infinite; then we may for every n consider the space $Q_n=(q_1, \cdots, q_n)$ with $t_n=(\tau_1, \cdots, \tau_n)$ as variable point and with $m_n=(\mu_1, \cdots, \mu_n)$ as measure and the space $Q_{n,\omega}=(q_{n+1}, q_{n+2}, \cdots)$ with $t_{n,\omega}=(\tau_{n+1}, \tau_{n+2}, \cdots)$ as variable point and $m_{n,\omega}=(\mu_{n+1}, \mu_{n+2}, \cdots)$ as measure. Then $Q=(q_1, q_2, \cdots)=(Q_n, Q_{n,\omega})$, $t=(\tau_1, \tau_2, \cdots) =(t_n, t_{n,\omega})$ and it is easily seen that $m=(\mu_1, \mu_2, \cdots)=(m_n, m_{n,\omega})$. Hence if $f(t)$ is integrable in Q we have

$$\begin{aligned}\int_Q f(t)m(dQ) &= \int_{Q_n} m_n(dQ_n)\int_{Q_{n,\omega}} f(t_n, t_{n,\omega})m_{n,\omega}(dQ_{n,\omega})\\ &= \int_{Q_{n,\omega}} m_{n,\omega}(dQ_{n,\omega})\int_{Q_n} f(t_n, t_{n,\omega})m_n(dQ_n)\end{aligned}$$

for every n. We have now the following theorems:

THEOREM A. *Let $f(t)$ be integrable in Q and let $f_n(t)$ denote the function*

$$f_n(t) = \int_{Q_{n,\omega}} f(t_n, t_{n,\omega})m_{n,\omega}(dQ_{n,\omega})$$

so that $f_n(t)$ depends only on t_n. Then $f_n(t)\to f(t)$ almost everywhere in Q as $n\to\infty$.

THEOREM B. *Let A be a measurable set in Q with the property that two points $t'=(\tau_1', \tau_2', \cdots)$ and $t''=(\tau_1'', \tau_2'', \cdots)$ such that $\tau_n'=\tau_n''$ when $n>n_0 =n_0(t', t'')$ always either both belong to A or both do not belong to A. Then $m(A)$ is either* 0 *or* 1.

THEOREM C. *Let $f(t)$ be integrable in Q and let $f_{n,\omega}(t)$ denote the function*

$$f_{n,\omega}(t) = \int_{Q_n} f(t_n, t_{n,\omega})m_n(dQ_n)$$

so that $f_{n,\omega}(t)$ depends only on $t_{n,\omega}$. Then $f_{n,\omega}(t)\to I$ almost everywhere in Q as $n\to\infty$ where I is the constant

$$I = \int_Q f(t)m(dQ).$$

16. The convergence problem of infinite convolutions

Let Q be an abstract space of the type considered at the beginning of §15 and let $x(t)$ be a measurable vector function with k components defined on Q. Then if E is a Borel set in R_x the set A_E of those points t for which $x(t)$ belongs to E is a measurable set in Q. The distribution function ϕ in R_x defined by $\phi(E)=m(A_E)$ will be called the *distribution function of* $x(t)$ *in* Q. For the Fourier transform of ϕ we have

$$\Lambda(y;\phi) = \int_Q e^{ix(t)y} m(dQ). \tag{16.1}$$

For later application we notice that *any distribution function* ϕ *in* R_x *is the distribution function of a measurable vector function* $x(t)$ *in a suitable abstract space* Q. The simplest possibility is to choose Q as R_x itself, except for the change in the notation for the variable point, and to choose ϕ as measure in Q. Then $x(t)=t$ is a measurable vector function in Q and ϕ is its distribution function.

Let $Q=(q_1, q_2)$ and let $x_1(\tau_1)$ and $x_2(\tau_2)$ be measurable vector functions with k components in q_1 and q_2 and having the distribution functions ϕ_1 and ϕ_2. Then the distribution function ϕ of $x(t)=x_1(\tau_1)+x_2(\tau_2)$ in Q is $\phi=\phi_1 * \phi_2$. This follows readily from the definitions and is obvious also from (15.1) and (16.1) which imply $\Lambda(y;\phi)=\Lambda(y;\phi_1)\Lambda(y;\phi_2)$. We shall now prove that a corresponding theorem holds for infinite convolutions also:

THEOREM 32. *A necessary and sufficient condition for the convergence of the infinite convolution* $\phi_1 * \phi_2 * \cdots$ *is that if* $q_1, q_2, \cdots$ *are abstract spaces and* $x_1(\tau_1), x_2(\tau_2), \cdots$ *measurable functions in* $q_1, q_2, \cdots$ *having the distribution functions* $\phi_1, \phi_2, \cdots$, *then the series* $x_1(\tau_1)+x_2(\tau_2)+\cdots$ *is convergent almost everywhere in* $Q=(q_1, q_2, \cdots)$. *The distribution function of* $s(t)=x_1(\tau_1)+x_2(\tau_2)+\cdots$ *is then* $\psi=\phi_1 * \phi_2 * \cdots$.

The sum $s_n(t)=x_1(\tau_1)+\cdots+x_n(\tau_n)$ is for every n a measurable vector function in Q and its distribution function is $\psi_n=\phi_1 * \cdots * \phi_n$. Similarly, the distribution function of $r_{n,p}(t)=x_{n+1}(\tau_{n+1})+\cdots+x_{n+p}(\tau_{n+p})$ is $\rho_{n,p}=\phi_{n+1} * \cdots * \phi_{n+p}$. From Theorem 1 a necessary and sufficient condition for the convergence of $\phi_1 * \phi_2 * \cdots$ is that $\rho_{n,p}\to\chi_0$ as $n\to\infty$, which means that $r_{n,p}(t)\to 0$ *in measure* (en mesure) as $n\to\infty$. Hence the convergence in measure of the series $x_1(\tau_1)+x_2(\tau_2)+\cdots$ is necessary and sufficient for the convergence of $\phi_1 * \phi_2 * \cdots$, and it is also seen from (16.1) that the sum $s(t)=x_1(\tau_1)+x_2(\tau_2)+\cdots$ has then $\psi=\phi_1 * \phi_2 * \cdots$ as distribution function. Since convergence almost everywhere implies convergence in measure, it remains only to prove that for series of the type $x_1(\tau_1)+x_2(\tau_2)+\cdots$ con-

vergence in measure implies convergence almost everywhere. If $s(t)=x_1(\tau_1)+x_2(\tau_2)+\cdots$ is convergent in measure then

$$e^{is(t)y} = e^{ix_1(\tau_1)y}e^{ix_2(\tau_2)y}\cdots$$

also holds in the sense of convergence in measure for every y in R_y. For a fixed y we write $f(t)=e^{is(t)y}$; then $f(t)$ is measurable and bounded, hence integrable, so that we may apply Theorem A. We find

$$f_n(t) = e^{ix_1(\tau_1)y}\cdots e^{ix_n(\tau_n)y}a_n$$

where the constant $a_n \to 1$ as $n\to\infty$; hence

$$e^{is(t)y} = e^{ix_1(\tau_1)y}e^{ix_2(\tau_2)y}\cdots$$

holds in the sense of convergence almost everywhere for every y, which shows that $x(t)=x_1(\tau_1)+x_2(\tau_2)+\cdots$ is convergent almost everywhere.

Since the set A of points t in Q in which a series of the form $x_1(\tau_1)+x_2(\tau_2)+\cdots$ is convergent satisfies the conditions of Theorem B, we have also

THEOREM 33. *An infinite series $x_1(\tau_1)+x_2(\tau_2)+\cdots$ whose terms are measurable vector functions is always either convergent almost everywhere or divergent almost everywhere.*

Theorems 32 and 33 together give, when translated into the language of the calculus of probability, a new solution of the convergence problem for series $x_1+x_2+\cdots$ whose terms are independent random variables; this problem (for $k=1$) was first treated by Khintchine and Kolmogoroff [37] and later by Kolmogoroff [38] and Lévy [41]. Theorem 33 states that the probability for convergence is always either 0 or 1 and Theorem 32 shows that the probability is 1 if and only if the distribution function ψ_n of $s_n=x_1+\cdots+x_n$ tends to a distribution function ψ when $n\to\infty$. Combining Theorem 5 with a remark due to Khintchine and Kolmogoroff, we find also the main result of these authors, which we formulate as a convergence criterion for infinite convolutions. Denoting by C_K the sphere $|x|\leqq K$ and by $\phi_{n,K}$ the distribution function for which $\phi_{n,K}(E)$ is $=\phi_n(EC_K)+1-\phi_n(C_K)$ or $=\phi_n(EC_K)$ according as E does or does not contain the point $x=0$, we have

THEOREM 34. *A necessary and sufficient condition for the convergence of the infinite convolution $\phi_1 * \phi_2 * \cdots$ is the convergence of the three series*

$$(1-\phi_1(C_K)) + (1-\phi_2(C_K)) + \cdots,$$

$$c(\phi_{1,K}) + c(\phi_{2,K}) + \cdots \quad \textit{and} \quad \mathrm{M}_2(\tilde{\phi}_{1,K}) + \mathrm{M}_2(\tilde{\phi}_{2,K}) + \cdots$$

for a fixed $K>0$ (or for all $K>0$).

With the notation of Theorem 32, let $x_{n,K}(\tau_n)$ be $=x_n(\tau_n)$ when $|x_n(\tau_n)| \leqq K$ and $=0$ when $|x_n(\tau_n)| > K$ so that $x_{n,K}(\tau_n)$ has $\phi_{n,K}$ as distribution function. On combining Theorems 5 and 32 we see that the convergence of the series $c(\phi_{1,K})+c(\phi_{2,K})+\cdots$ and $M_2(\tilde{\phi}_{1,K})+M_2(\tilde{\phi}_{2,K})+\cdots$ is necessary and sufficient for the convergence almost everywhere of $x_{1,K}(\tau_1)+x_{2,K}(\tau_2)+\cdots$. On the other hand, $x_1(\tau_1)+x_2(\tau_2)+\cdots$ converges almost everywhere if and only if $(1-\phi_1(C_K))+(1-\phi_2(C_K))+\cdots$ is convergent and $x_{1,K}(\tau_1)+x_{2,K}(\tau_2)+\cdots$ converges almost everywhere.

As an application of Theorem B we finally prove

THEOREM 35. *If $\psi=\phi_1 * \phi_2 * \cdots$ is a convergent infinite convolution of distribution functions ϕ_n each of which is purely discontinuous, then ψ is either purely discontinuous or singular or absolutely continuous.*

Using the notation of Theorem 32, we may suppose that each $x_n(\tau_n)$ takes on an at most enumerable set of values $x_{n,1}, x_{n,2}, \cdots$. Let M denote the smallest modul in R_x containing all points $x_{n,m}$ so that for an arbitrary set E in R_x the vectorial sum $E+M$ is at most enumerable if E is at most enumerable and $E+M$ is a null-set if E is a null-set. If E is a Borel set then the set A of those points t in Q for which $s(t)=x_1(\tau_1)+x_2(\tau_2)+\cdots$ is convergent and belongs to $E+M$ satisfies the conditions of Theorem B, so that $m(A)=1$ whenever $\bar{m}(A)>0$. This means that $\psi(E+M)=1$ whenever $\psi(E+M)>0$ and a fortiori when $\psi(E)>0$ which implies Theorem 35.

Theorem 35 has been used already in §6. Further infinite convolutions of the type considered in Theorem 35 have recently been investigated by Schoenberg [47] in connection with distribution problems for arithmetical functions.

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TRAJECTORIES AND LINES OF FORCE*

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In this paper we generalize certain theorems of Kasner† relative to the geometry of arbitrary fields of force in the plane.

Consider the motion of a particle which starts from rest in a positional field of force at a point where the force does not vanish. It begins to move along the line of force on which it is situated. However, due to the effect of inertia, it does not remain on this line of force, but travels in a somewhat straighter path. In general, the line of force and the trajectory will have the same initial direction but different initial curvatures. Kasner has shown that *the curvature of the trajectory is always one-third the curvature of the line of force.* If the initial curvature of the line of force vanishes, this result, while still valid, is not significant. In this case Kasner studies the ratio between the infinitesimal departures of the path and the line of force from their common tangent line. He proves the following theorem:

THEOREM. *If the line of force has contact of nth order with the tangent line, the trajectory produced by starting a particle from rest will also have contact of nth order; and the limiting ratio of the departure of the trajectory to the departure of the line of force from the common tangent will be* $1:(2n+1)$.

We extend this result to the more general cases in which the contact between the line of force and its tangent is of any order, finite or infinite, as well as to some cases in which no definite order of contact exists.‡ The theo-

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† For a complete report of Kasner's work, see Proceedings of the National Academy of Sciences, vol. 20 (1934), pp. 130–136. Some results also appear in these Transactions, 1906–1910; Bulletin of the American Mathematical Society, vol. 16 (1909–1910), p. 172; Princeton Colloquium Lectures, *Differential Geometric Aspects of Dynamics*, 1913, p. 9; Science, vol. 75 (1932), p. 671; Zurich Congress Proceedings, 1932, vol. 2, p. 180.

‡ A curve, $y=f(x)$, where $f(x)$ is single-valued, continuous and

$$\lim_{x\to+0} \frac{f(x)}{x} = 0,$$

has contact of finite order α with the x-axis if $\lim_{x\to+0} f(x)/x^{\alpha+1}$ is a non-zero constant. If

$$\lim_{x\to+0} \frac{f(x)}{x^{\alpha+1}} = 0$$

for all values of α, $f(x)$ has contact of infinite order. In all other cases, $f(x)$ has no definite order of contact.

rems are stated more simply in terms of the inverse of Kasner's ratio, i.e., the ratio of the departure of the line of force to the departure of the trajectory from their common tangent. For brevity, we call the limits of this ratio the *ratio set*. The trajectory produced by starting a particle from rest will be referred to simply as "the trajectory." In general the ratio set will not be a single number but will consist of a set of numbers. An easy application of a theorem of Hardy leads to the result that for certain simple types of fields the ratio set is a unique number. In the course of the work, we give an indication of the extent to which the ratio set determines the field. The theorems which Kasner obtains when friction is allowed or when the particle is projected with non-zero velocity in the direction of the force are also generalized.

We proceed to obtain a formula for the ratio set. The components of the field of force are assumed to be continuous and to possess continuous first partial derivatives. Furthermore we assume that the direction of the force at each point of some neighborhood of the initial point differs from that at the initial point. In fact, it is sufficient for this property to hold in a sufficiently small portion of a neighborhood of the initial point, containing some first part of the trajectory and the tangent in its interior, and having the initial point on its boundary. In all that follows, we choose the initial point as the origin of coordinates and the tangent to the line of force as the x-axis; we assume unit mass and we write f for the force at the origin. It is clear that this causes no loss in generality. An equation in x and y in which the variables are referred to the above set of axes will be called *normal*. The formula for the ratio set is given by

THEOREM I. *Let $y=g(x)$ and $y=h(x)$ be the normal equations of the trajectory and the line of force respectively. Then the ratio set is identical with the set of limits of the expression*

$$\frac{2x\,\dfrac{dg(x)}{dx}}{g(x)}-1$$

or of the equivalent expression

$$\frac{2h(x)}{x^{1/2}\displaystyle\int_0^x \frac{h(x)}{x^{3/2}}\,dx}$$

as x approaches zero.

In the course of the proof of this theorem, we shall also discover a sufficient condition that two different fields of force have the same ratio set at a point. For this purpose, we introduce the notion of the *direction function* of a field of force. Through a fixed point in the plane, there passes a single line

of force. The slope of the force at each point of the tangent to this line of force is a function of x. This function approaches zero with x and is the direction function of the field at the fixed point.

THEOREM II. *Two fields of force have the same ratio set at a fixed point if the quotient of the direction functions of the fields approaches a finite non-zero limit at the given point.*

Thus the ratio set at a point is completely determined merely by the limiting behavior of the direction function. The two fields need not have the same direction at the point. The proofs of these theorems follow.

By the hypothesis of Theorem I, the equation of the trajectory is

$$y = g(x). \tag{1}$$

The components of the field are $\phi(x, y)$ and $\psi(x, y)$ where

$$\phi(0, 0) = f \quad (f \neq 0), \qquad \psi(0, 0) = 0. \tag{2}$$

By the theorem of the mean*

$$\frac{\psi(x, y)}{\phi(x, y)} = \frac{\psi(x, 0)}{\phi(x, 0)} + \left[\frac{\psi(x, \theta y)}{\phi(x, \theta y)}\right]_y \cdot y, \qquad 0 < \theta < 1.$$

By hypothesis, ϕ_y and ψ_y, and hence also $[\psi/\phi]_y$, are continuous in a sufficiently small neighborhood of the origin. We may write the last equation as

$$\frac{\psi(x, y)}{\phi(x, y)} = D(x) + A(x, y) \cdot y, \tag{3}$$

where, by definition, $D(x)$ is the direction function; $D(0) = 0$ and $A(x, y)$ is bounded in a neighborhood of the origin.

Now the trajectory is a solution† of

* The only partial derivatives whose existence we require in our work are those with respect to y. Since we do not assume the existence of ϕ_x and ψ_x, the mathematics may allow more than one trajectory through a point. Of course, this theory would have physical application only when a unique trajectory existed.

† For this solution dg/dx which appears in Theorem I exists and is a continuous function for a sufficiently small neighborhood of the origin. d^2g/dx^2 which also appears in the work is equal to

$$\frac{\dfrac{d^2y}{dt^2}\cdot\dfrac{dx}{dt} - \dfrac{d^2x}{dt^2}\cdot\dfrac{dy}{dt}}{\left(\dfrac{dx}{dt}\right)^3}$$

and hence is continuous wherever $dx/dt \neq 0$. Since $(d^2x/dt^2)_0 \neq 0$, it follows from Rolle's Theorem that d^2g/dx^2 is continuous in a sufficiently small positive neighborhood of the origin.

$$\ddot{x} = \phi(x, y),^* \qquad \ddot{y} = \psi(x, y). \tag{4}$$

Since the initial velocity is zero, the parametric equations of the trajectory (1) in terms of the time are

$$x = \tfrac{1}{2}ft^2 + k(t), \qquad y = y(t),$$

where $k(t)$ and $y(t)$ and their first two derivatives vanish at the origin. If we eliminate t from these two equations we obtain (1).† Now $\dot{y} = g' \cdot \dot{x}$ and $\ddot{y} = g''\dot{x}^2 + g'\ddot{x}$. Hence

$$\frac{\ddot{y}}{\ddot{x}} = g''\frac{\dot{x}^2}{\ddot{x}} + g' = g''\left[\frac{f^2t^2 + 2ft\dot{k}(t) + \dot{k}(t)^2}{f + \ddot{k}(t)}\right] + g',$$

$$\frac{\ddot{y}}{\ddot{x}} = [2x + m(x)]g'' + g', \tag{5}$$

where $m(x)/x \to 0$ as $x \to 0$. Comparing (3), (4), and (5), we have

$$[2x + m(x)]g'' + g' = D(x) + A(x, g) \cdot g. \tag{6}$$

Thus (6) is a differential equation whose solution through the element (0, 0, 0) is the trajectory.

We now obtain a similar differential equation for the corresponding line of force. The equation of the line of force is

$$y = h(x). \tag{7}$$

By definition of a line of force, (7) must satisfy the equation

$$h' = \frac{\psi(x, h)}{\phi(x, h)},$$

or

$$h' = D(x) + A(x, h) \cdot h. \tag{8}$$

Since the ratio set consists of the limiting values of $h(x)/g(x)$ as $x \to 0$, we now proceed to compare the solutions of (6) and (8) through the element (0, 0, 0). For this purpose we prove two lemmas.

LEMMA I. *Let $h = h(x)$ be a solution of* (8) *through the element* (0, 0, 0). *Then*

$$\lim_{x \to 0} h(x) \Big/ \int_0^x D(x)dx = 1.$$

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* In all that follows, primes denote differentiation with respect to x, and dots differentiation with respect to time.

† It is an easy consequence of the theorem on implicit functions that the elimination of t gives a unique solution for $t > 0$ and a unique solution for $t < 0$. In each case $x > 0$ for small values of t. In what follows, $y = g(x)$ signifies either branch of the trajectory.

Let $A(x, h(x)) = B(x)$. Then a solution of

$$w' = D(x) + B(x)\cdot w \tag{9}$$

is $w = h(x)$. Since (9) is a linear differential equation, its solution through the origin is

$$h(x) = \exp\left[\int_0^x B(x)dx\right]\int_0^x \exp\left[-\int_0^x B(x)dx\right]D(x)dx.$$

Hence

$$h(x) = (1 + E_1(x))\int_0^x (1 + E_2(x))D(x)dx,$$

where $E_1(x)$ and $E_2(x)$ approach zero with x, since $B(x)$ is bounded as $x\to 0$. Since, by hypothesis, $D(x)\neq 0$ in a positive neighborhood of the origin, we may apply L'Hospital's Rule. Therefore

$$\lim_{x\to 0} h(x)\bigg/\int_0^x D(x)dx = 1\cdot\lim_{x\to 0}(1 + E_2(x))D(x)/D(x) = 1,$$

which proves Lemma I. As a consequence of this lemma, the line of force must be on one side of its tangent in a neighborhood of the origin.

Lemma II. *Let $g = g(x)$ be a solution of* (6) *through the element* (0, 0, 0). *Then*

$$\lim_{x\to 0}\frac{2xg'(x) - g(x)}{\int_0^x D(x)dx} = 1.$$

Let $A(x, g(x)) = C(x)$. Then

$$[2x + m(x)]z'' + z' = D(x) + C(x)\cdot z \tag{10}$$

has the solution $z = g(x)$. Consider the equation

$$w' = D(x) + C(x)\cdot w. \tag{11}$$

Let $w = w(x)$ be the solution of (11) through the origin. Subtracting (10) from (11),

$$w' - z' - [2x + m(x)]z' = C(x)\cdot[w - z]. \tag{12}$$

For the solution of (12) through the origin,

$$u' - C(x)u = [2x + m(x)]g''(x),$$

where $u(x) = w(x) - g(x)$. Since $m(x)/x\to 0$ as $x\to 0$, $[2x+m(x)]g'' = 2xg''\cdot(1+E_3(x))$. Hence, as in Lemma I,

$$\lim_{x\to 0} \frac{\int_0^x 2xg''(1+E_4(x))\,dx}{u(x)} = 1,$$

where $E_4(x)$ approaches zero with x.

We now show that $E_4(x)$ may be neglected. By hypothesis $\psi(x, y)/\phi(x, y) \neq 0$ in some initial neighborhood of the trajectory and the x-axis. From (4), this is also true for $\ddot{y}(t)$. By Rolle's Theorem the same obtains for $\dot{y}(t)$ and consequently for $g'(x)$ and $g(x)$. It is easy to show that the line of force and the trajectory are on the same side of the tangent line near the origin.

Now, from (6),

$$(2xg'' + g')(1 + E_3(x)) = \frac{\dddot{y}}{\ddot{x}} + E_3(x)\,\frac{\ddot{y}}{\ddot{x}}.$$

When this expression is zero,

$$E_3(x) = -\frac{\dot{x}\dddot{y}}{\ddot{x}\dot{y}} = -\frac{t\dddot{y}}{\dot{y}}(1 + E_5(t))$$

where $E_5(t)$ approaches zero with t. As will be shown later,

$$\lim_{t\to 0} \frac{t\dddot{y}}{\dot{y}} \neq 0.$$

Hence $2xg''+g'$ does not change sign in some deleted neighborhood of the origin. Hence, by L'Hospital's Rule,

$$\lim_{x\to 0} \frac{\int_0^x (2xg'' + g')(1 + E_4(x))dx}{\int_0^x (2xg'' + g')dx} = 1.$$

Furthermore

$$\lim_{x\to 0} \frac{\int_0^x g'(x)(1 + E_4(x))dx}{\int_0^x g'(x)dx} = 1$$

and therefore

$$\lim_{x\to 0} \frac{\int_0^x (2xg'' + g')(1 + E_4(x))dx}{u + g} = 1.$$

It follows easily that

$$\lim_{x\to 0} \frac{\int_0^x (2xg'' + g')dx}{u + g} = 1.$$

Performing the indicated integration in the numerator and using $w(x) = u(x) + g(x)$ and

$$\lim_{x\to 0} \frac{w(x)}{\int_0^x D(x)dx} = 1$$

(Lemma I), we have

$$\lim_{x\to 0} \frac{\int_0^x (2xg'' - g')dx}{\int_0^x D(x)dx} = 1.$$

A comparison of the results of Lemmas 1 and 2 shows that

$$2xg'(x) - g(x) = h(x)[1 + E_6(x)] \tag{13}$$

where $E_6(x)\to 0$ as $x\to 0$. From this it follows that the limits of $h(x)/g(x)$ and of $2xg'(x)/g(x) - 1$ as x approaches zero are identical. This proves the first part of Theorem I.

Solving (13) for $g(x)$,

$$g(x) = \frac{x^{1/2}}{2}\int_0^x \frac{h(x)[1 + E_6(x)]}{x^{3/2}}\,dx. \tag{14}$$

We now show that a suitable approximation for $g(x)$ in terms of $h(x)$ may be derived by neglecting $E_6(x)$ in (14). Now

$$\lim_{x\to 0} \frac{h(x)}{x} = 0.$$

Hence $|h(x)/x^{3/2}| < x^{-1/2}$ for x sufficiently small. Also

$$\left|\int_0^x \frac{h(x)}{x^{3/2}}\,dx\right| < \int_0^x x^{-1/2}dx = 2x^{1/2}.$$

Hence this integral approaches zero with x, as does the similar integral in (14), and we may apply L'Hospital's Rule to their quotient. We have

$$\lim_{x\to 0} \frac{\displaystyle\int_0^x \frac{h(x)}{x^{3/2}}\,dx}{\displaystyle\int_0^x \frac{h(x)[1 + E_6(x)]}{x^{3/2}}\,dx} = \lim_{x\to 0} \frac{\dfrac{h(x)}{x^{3/2}}}{\dfrac{h(x)[1 + E_6(x)]}{x^{3/2}}} = 1.$$

Hence*

$$\lim_{x\to 0} \frac{g(x)}{\dfrac{x^{1/2}}{2}\displaystyle\int_0^x \frac{h(x)}{x^{3/2}}\,dx} = 1. \tag{15}$$

* Therefore the limiting ratio of both branches of the trajectory is unity and the same ratio set is obtained by using either branch.

The second part of Theorem I is an easy consequence of (15) and the definition of the ratio set. The ratio set is therefore determined by the complete equation of the trajectory or line of force.

To establish the truth of Theorem II, we show that the ratio set is identical with the limits of yet a third expression, depending only upon the direction function at the point. By Lemma I,

$$\lim_{x\to 0}\frac{h(x)}{\int_0^x D(x)dx} = 1.$$

On the basis of this result and the second part of Theorem I, it follows by a proof analogous to that used in deriving (15) that *the ratio set is identical with the limits of the expression*

$$\frac{2\int_0^x D(x)dx}{x^{1/2}\int_0^x \frac{\int_0^x D(x)dx}{x^{3/2}}dx}$$

as x approaches zero. By means of this formula the ratio set may be calculated directly from the components of the force without integrating the equations of motion.

Now consider two different fields of force whose direction functions at a fixed point are $D(x)$ and $D_1(x)$. It is easy to show that if

$$\lim_{x\to 0}\frac{D(x)}{D_1(x)} = c,$$

where c is a non-zero constant, the ratio set for each field, computed from the above expression, will be the same. For, by the argument used to derive (15), a suitable approximation for $D(x)$ in the formula for the ratio set is $cD_1(x)$ which obviously gives the same values for the ratio set as does $D_1(x)$. This proves Theorem II.

We now apply Theorem I to the case in which the line of force has any finite contact, integral, fractional, or irrational, with its tangent. This is a first generalization of Kasner's theorem and includes it as a special case.

THEOREM III. *If the line of force has contact of order α with the tangent line, the trajectory will also have contact of order α; and the ratio set will be $2\alpha+1$.*

By the hypothesis of the theorem,

$$\lim_{x\to 0}\frac{h(x)}{x^{\alpha+1}} = c\,(\neq 0).$$

Hence, as shown in the derivation of (15), $h(x)$ may be replaced by $cx^{\alpha+1}$ in taking the limit of the second expression in Theorem I:

$$\lim_{x\to 0} \frac{2h(x)}{x^{1/2}\int_0^x \frac{h(x)}{x^{3/2}}\,dx} = \lim_{x\to 0} \frac{2cx^{\alpha+1}}{x^{1/2}\int_0^x cx^{\alpha-1/2}dx} = 2\alpha + 1.$$

Since

$$\lim_{x\to 0}\frac{g(x)}{h(x)} = \frac{1}{2\alpha+1}, \qquad \text{we have} \quad \lim_{x\to 0}\frac{g(x)}{x^{\alpha+1}} = \frac{c}{2\alpha+1},$$

which shows that the trajectory also has contact of order α with the common tangent.

If we consider the case in which the line of force has infinite contact with its tangent, the corresponding theorem is

THEOREM IV. *If the line of force has contact of infinite order with the tangent line, the trajectory will also have contact of infinite order; and the ratio set will be* $+\infty$ *or all numbers in some non-negative closed interval including* $+\infty$. *Furthermore, any given interval of this kind will be the ratio set of some field of force for which the line of force has contact of infinite order with its tangent.*

We first prove that the trajectory has contact of infinite order with its tangent line. From (14), for every $k>\frac{1}{2}$,

$$\lim_{x\to 0}\frac{g(x)}{x^k} = \lim_{x\to 0}\frac{\int_0^x \frac{h(x)[1+E_6(x)]}{2x^{3/2}}\,dx}{x^{k-1/2}} = \frac{1}{2k-1}\lim_{x\to 0}\frac{h(x)}{x^k}.$$

By the hypothesis this last limit is zero, which proves the preliminary result of Theorem IV. For the rest of the theorem, we consider the expression

$$(16) \qquad G(x) = \frac{xg'(x)}{g(x)}$$

which appears in the formula for the ratio set. The possible limits of $G(x)$ are investigated under the assumption that

$$\lim_{x\to 0}\frac{g(k)}{x^k} = 0$$

for every k.

As already shown the origin is an isolated point of the zeros of $g(x)$. Then $G(x)$ is a continuous function in a sufficiently small positive neighborhood of the origin. The limiting values of $G(x)$ for $x>0$ must be a closed interval (which may degenerate into a single point). For if a and b $(>a)$ are lower and upper limits of $G(x)$, then $G(x)$ assumes values which lie in the bands

$[a-\epsilon,\ a+\epsilon]$ and $[b-\epsilon,\ b+\epsilon]$, E arbitrarily small, an infinite number of times in every neighborhood of the origin. Since $G(x)$ is continuous, it assumes each value between these bands an infinite number of times. Therefore the closed interval $[a, b]$ is the set of limits of $G(x)$.

We now indicate which closed intervals may actually appear as limits of $G(x)$. In the following, we suppose that $g(x)\neq 0$ in the interval $0 < x \leqq 1$. Integrating (15),

$$g(x) = c \exp\left[\int_1^x (G(x)/x)dx\right]. \tag{17}$$

From (17),

$$\frac{g(x)}{cx^k} = \exp\left[\int_1^x \{(G(x) - k)/x\}dx\right].$$

Since $g(x)$ has contact of infinite order, $G(x)$ cannot remain less than $+k$ in any neighborhood of the origin. For suppose $G(x)\leqq k-\epsilon$ when $0<x\leqq\delta$:

$$\frac{g(x)}{cx^k} = d + \exp\left[\int_\delta^x \{(G(x) - k)/x\}dx\right]$$

where

$$d = \exp\left[\int_1^\delta \{(G(x) - k)/x\}dx\right].$$

Then

$$\lim_{x\to 0}\left|\frac{g(x)}{cx^k}\right| \geqq d + \lim_{x\to 0} \exp\left[\int_\delta^x (-\epsilon/x)dx\right] = +\infty$$

contrary to hypothesis. Therefore $G(x)$ assumes values as great as any fixed k an infinite number of times in every neighborhood of the origin. Since k is any positive number, lim sup $G(x)=+\infty$. Hence the only closed intervals which may occur as limits of $G(x)$ are those which include $+\infty$. No negative ratio may appear since the line of force and trajectory are both on the same side of their common tangent near the origin.

Furthermore, any interval of this kind will be the limit of $G(x)$ for some field of force. To prove this last statement, it will suffice to present a trajectory, $y=g(x)$, having contact of infinite order with its tangent, such that the limit of the associated function $G(x)$ is a given closed interval $[a, +\infty]$. For then the field of force mentioned above surely exists. For example, a field which generates the trajectory, $y=g(x)$, is

$$\phi(x, y) = 1, \qquad \psi(x, y) = 2xg''(x) + g'(x). \tag{18}$$

We list the possible limits of $G(x)$ together with the corresponding trajectories:

$$(19_1)\quad \lim_{x\to 0} G(x) = +\infty, \qquad y = \exp\,[-1/x^2],$$

$$(19_2)\quad \lim_{x\to 0} G(x) = [a, +\infty], \quad y = \exp\left[\int_1^x \{(\sin(1/x)+1)/x^2 + a)/x\}\,dx\right].$$

It is easily seen that in each case $G(x)$, calculated from (16), has the prescribed limit. It only remains to show that the line of force corresponding to each case actually has contact of infinite order. We first prove that each trajectory has contact of infinite order. This is immediate for (19_1). For (19_2), we must show that

$$\int_1^0 \frac{\dfrac{1}{x^2}\left(\sin\dfrac{1}{x}+1\right)+a-n}{x}\,dx = -\infty$$

for all values of n. The substitution $y=1/x$ makes it possible to perform the integration in finite terms and establishes the required result.

Now by Theorem I

$$\lim_{x\to 0}\frac{h(x)}{g(x)} = \lim_{x\to 0}\,(2G(x)-1)$$

or

$$(20)\qquad \lim_{x\to 0}\frac{h(x)}{x^k} = \lim_{x\to 0}\,(2G(x)-1)\cdot\frac{g(x)}{x^k}.$$

Since $g(x)$ in each case has contact of infinite order and $G(x)$ involves only powers of x, the right hand member of (20) approaches zero. Hence the corresponding line of force has contact of infinite order with its tangent.

There still remains the case in which no definite order of contact exists. We make the following definition:

A curve, $y=f(x)$, where $f(x)$ is single-valued, continuous and

$$\lim_{x\to +0}\frac{f(x)}{x} = 0,$$

has generalized contact of order α with the x-axis if α is the upper bound of all numbers k such that

$$\lim_{x\to +0}\frac{f(x)}{x^{k+1}} = 0.$$

Note that if a curve has ordinary contact of order α, it also has generalized contact of order α. For infinite contact, the two definitions coincide. To every curve there is assigned some generalized contact $\alpha \geqq 0$.

THEOREM V. *If the line of force has generalized contact of order α with the tangent line, the trajectory will have generalized contact $\geqq \alpha$; and the ratio set will be a non-negative closed interval containing at least one of the numbers $2\alpha+1$, $+\infty$. This interval may degenerate into a single point. Furthermore, any given interval of this kind will be the ratio set of some field for which the line of force has generalized contact of order α.*

The case $\alpha = +\infty$ has been treated in Theorem IV. We therefore assume α is finite. As shown in the beginning of the proof of Theorem IV, since

$$\lim_{x\to 0} \frac{h(x)}{x^{k+1}} = 0, \quad \text{we have} \quad \lim_{x\to 0} \frac{g(x)}{x^{k+1}} = 0.$$

This proves the first part of Theorem V.

We proceed to study the possible limits of $G(x)$ defined by (16). The proof is parallel with that of Theorem IV and is outlined in what follows. We note again that the limiting values of $G(x)$ for $x>0$ form a non-negative closed interval. Repeating the proof following (17), we conclude that $G(x)$ assumes values as great as any fixed $k+1<\alpha+1$ an infinite number of times in every neighborhood of the origin. Therefore $\limsup G(x) \geqq \alpha+1$.

We now show that either $\alpha+1$ or $+\infty$ is a limit of $G(x)$. For suppose $\alpha+1$ is not a limit of $G(x)$. Then $\liminf G(x) = \gamma > \beta > \alpha+1$. If, in addition, $+\infty$ is not a limit of $G(x)$, we shall prove that

$$\lim_{x\to 0} \frac{h(x)}{x^{\beta}} = 0,$$

which contradicts the hypothesis. From (17),

$$\begin{aligned} \lim_{x\to 0} \frac{g(x)}{x^{\beta}} &= \lim_{x\to 0} c \exp\left[\int_1^x \{(G(x)-\beta)/x\}\,dx\right] \\ &< \lim_{x\to 0} c \exp\left[\int_1^x \{(\gamma-\beta)/x\}\,dx\right] = 0. \end{aligned} \tag{21}$$

In (19), replace k by β. Now since $+\infty$ is not a limit of $G(x)$, $[2G(x)-1]$ remains bounded. Hence, from (20) and (21), it follows that

$$\lim_{x\to 0} \frac{h(x)}{x^{\beta}} = 0$$

which is the predicted contradiction. Therefore either $\alpha+1$ or $+\infty$ is a limit of $G(x)$.

It remains to show that any interval of this kind is the limit of $G(x)$ for some field of force. As in the proof of Theorem IV, it will suffice to present suitable trajectories.

We first introduce several auxiliary functions. Let

$$\phi_1(x) = \sin^2\frac{1}{x} \quad \text{when } \frac{1}{2^m\pi} \geqq x \geqq \frac{1}{(2^m+1)\pi} \qquad (m = 0, 1, 2, \cdots),$$

$$\phi_1(x) = 0 \qquad \text{for all other values of } x.$$

Then $\phi_1(x)$ oscillates between 0 and $+1$ and $\int_0^1[\phi_1(x)/x]dx$ converges to a negative constant. For

$$\left|\int_0^1 \frac{\phi_1(x)}{x}\,dx\right| = \left|\int_1^\infty \frac{\phi_1\left(\frac{1}{y}\right)}{y}\,dy\right| = \left|\sum_{m=0}^{\infty}\int_{2^m\pi}^{(2^m+1)\pi}\frac{\phi_1\left(\frac{1}{y}\right)}{y}\,dy\right| < \sum_{m=0}^{\infty}\frac{1}{2^m} = 2.$$

Note also that $\phi_1(x)$ is continuous and has a continuous first derivative. It is clear that similar functions having any finite number of continuous derivatives may be constructed by using sufficiently high powers of $\sin^2(1/x)$. Let

$$\phi_2(x) = \sin^2\frac{1}{x} \quad \text{when } \frac{1}{2^{2^m}\pi} \geqq x \geqq \frac{1}{(2^{2^m}+1)\pi} \qquad (m = 0, 1, 2, \cdots),$$

$$\phi_2(x) = 0 \qquad \text{for all other values of } x.$$

Obviously $\phi_2(x)$ has the same properties as $\phi_1(x)$. Similarly let $\phi_3(x)$ and $\phi_4(x)$ be continuous differentiable functions which oscillate between 0 and $+1$ in the neighborhood of the origin and such that

$$\int_1^0 x^{\alpha+1}e^{1/x}\phi_3(x)dx \quad \text{and} \quad \int_1^0 x^{\alpha+1-c}\phi_4(x)dx \qquad (c > \alpha)$$

converge to negative constants.

We now list the possible limits of $G(x)$ together with the corresponding trajectories*:

* In (22_2), if $b=+\infty$, it is replaced by $-\log x$. If $\alpha \leqq 1$, the expression $2xg''(x)+g'(x)$ in (18) will not be zero at the origin. In this case, more complicated ϕ functions must be used.

$$(22_1)\quad \lim_{x\to 0} G(x) = \alpha + 1, \qquad y = x^{\alpha+1} \text{ or } y = \frac{x^{\alpha+1}}{\log x};$$

$$(22_2)\quad \begin{aligned} &\lim_{x\to 0} G(x) = [a, b], \qquad a \leqq \alpha + 1 \leqq b, \\ &y = \exp\left[\int_1^x \{((a - \alpha - 1)\phi_1(x) + (b - a)\phi_2(x) + \alpha + 1)/x\} dx\right]; \end{aligned}$$

$$(22_3)\quad \lim_{x\to 0} G(x) = +\infty, \qquad y = \exp\left[\int_1^x \{(1/x + x^{\alpha+1}e^{1/x}\phi_3(x))/x\} dx\right];$$

$$(22_4)\quad \begin{aligned} &\lim_{x\to 0} G(x) = [c, +\infty], \qquad \alpha + 1 < c, \\ &y = \exp\left[\int_1^x \{(c + x^{\alpha+1-c}\phi_4(x))/x\} dx\right]. \end{aligned}$$

It is easy to verify that in each case $G(x)$ has the prescribed limit. It is necessary to show that the corresponding line of force has generalized contact of order α. This follows at once for the first line of force from (20) and (22_1). The trajectory (22_2) has contact of order α. Hence, from (20), the corresponding line of force has generalized contact of order α, if $\lim_{x\to 0} G(x)$ is bounded. This remains true even if $b = -\log x$, since $\log x$ is greater than any power of x in the neighborhood of the origin. The trajectory (22_3) has the same contact as $e^{-1/x}$ and (22_4) as x^c. By substituting these values in (20), we find that the corresponding lines of force have generalized contact of order α. This completes the proof.

These theorems indicate how the field of force determines the ratio set. Indeed, as proved in Theorem II, the ratio set depends only upon the limiting behavior of the direction function. The converse question arises: To what extent does the ratio set determine the field? The simplest answer seems to be in terms of the trajectory, although there are similar statements about the line of force and the direction function.

THEOREM VI. *Let the ratio set be the closed interval $[a, b]$. Let $y = g(x)$ be the normal equation of the trajectory. For a sufficiently small neighborhood of the origin, let e_1 be the upper bound of e such that $x^{-e}|g(x)|$ is an increasing function and let e_2 be the lower bound of e such that $x^{-e}|g(x)|$ is a decreasing function. Then $e_1 = (a+1)/2$ and $e_2 = (b+1)/2$.*

As shown in the proof of Lemma II, $g(x)$ is either an increasing or a decreasing function in a sufficiently small neighborhood of the origin. Hence $|g(x)|$ is an increasing function in this neighborhood. From (17),

$$p_e(x) = x^{-e} \mid g(x) \mid = \mid c \mid \exp\left[\int_1^x \{(G(x) - e)/x\} dx\right],$$
$$p_e'(x) = \mid c \mid \frac{G(x) - e}{x} p_e(x).$$

Hence $p_e(x)$ is an increasing function as long as $(G(x)-e)>0$ for small values of x. Then e_1 is the lower limit of $G(x)$. The first part of the theorem follows from (20). Since $p_e(x)$ is a decreasing function if $(G(x)-e)<0$ for small values of x, we may prove the remainder of the theorem in a similar manner.

THEOREM VII. *Let $y=g(x)$ be the normal equation of the trajectory. Let $g(x)$ be an L-function of x. Then the ratio set will be a unique number.*

Since $xg'(x)/g(x)$ is also an L-function, this follows from Theorem I and a theorem of Hardy on L-functions.*

All these theorems are derived upon the assumption that the particle encounters no resistance. For those cases in which resistance is allowed, we have

THEOREM VIII. *Let a particle start from rest in a continuous resisting medium. Let f be the intensity of the force at the initial point and let R_0 be the resistance due to zero speed. Let α be a number of the ratio set of the field of force which gives the same trajectory when the resistance is neglected. Then the ratio set will consist of the numbers $\alpha(1-R_0/f)+R_0/f$ when the motion takes place in the resisting medium.*

If $R_0=0$, as in a gas, the resisting medium may be entirely disregarded in calculating the ratio set. If the initial point is not a point of inflection of the line of force, $\alpha=3$, and we obtain Kasner's result: $3-2R_0/f$.

The proof follows. Let (1) and (7) be the equations of the trajectory and line of force respectively and (2) the components of the field of force. Then the trajectory is a solution of

$$\ddot{x} = \phi(x, y) - R(v)\cdot\cos\theta,$$
$$\ddot{y} = \psi(x, y) - R(v)\cdot\sin\theta,$$

where tan θ is the slope of the tangent of the trajectory. Hence

$$\frac{\psi(x, y)}{\phi(x, y)} = \frac{\dfrac{\ddot{y}}{\ddot{x}} + \dfrac{R_0}{f - R_0}(1 + E_1(x))\cdot g'(x)\cdot\cos\theta}{1 + \dfrac{R_0}{f - R_0}(1 + E_1(x))\cdot\cos\theta}, \tag{23}$$

* G. H. Hardy, *Orders of Infinity*, 1924. An L-function is a real one-valued function defined by a finite combination of the ordinary algebraic symbols and the function symbols log ( ) and exp ( ) operating on the variable x and on real constants. The theorem referred to above is as follows: An L-function is ultimately continuous, of constant sign, and monotonic, and tends as $x\to+\infty$ to infinity or to zero or to some other definite limit. This applies also if $x\to 0$ through positive values.

where $E_1(x)\to 0$ as $x\to 0$ since $R(v)$ is continuous and $R(0)=R_0$ (of course, $R_0<f$). Now $\cos\theta\to 1$ as $x\to 0$. Therefore, from (3), (5), and (23), we obtain

$$(24)\qquad (2x+n(x))\left(1-\frac{R_0}{f}\right)g''+g'=D(x)+A(x,g)\cdot g,$$

where $n(x)/x\to 0$ as $x\to 0$. Thus (24) is a differential equation whose solution through the element (0, 0, 0) is the trajectory. Similarly a differential equation for the line of force is (8). Proceeding as in Lemmas 1 and 2 of Theorem I, we find that

$$\lim_{x\to 0}\frac{h(x)}{g(x)}=\lim_{x\to 0}\left(\frac{2xg'}{g}-1\right)\left(1-\frac{R_0}{f}\right)+\frac{R_0}{f},$$

which, together with Theorem I, completes the proof of the theorem.

We now consider the case in which the particle is projected with a non-zero velocity in the direction of the force. Kasner has obtained a theorem, assuming that the line of force has integral order of contact with its tangent, which we generalize.

THEOREM IX. *If the line of force has generalized contact of order α with the tangent line, any trajectory obtained by projecting a particle with a non-zero speed in the direction of the force will have generalized contact of order $\alpha+1$; and the departure from the common tangent of these trajectories will vary inversely as the square of the speed. If the line of force has ordinary contact of order α with the tangent line, any trajectory obtained by projection will have ordinary contact of order $\alpha+1$.*

The proof is similar to that of the preceding theorem. Again let (1) and (7) be the equations of the trajectory and the line of force respectively and (2) the components of the force. Then the trajectory is a solution of (4), having initial velocity $v\neq 0$. Its equation may be written in the parametric form

$$x=vt+\tfrac{1}{2}ft^2+k(t),\qquad y=y(t),$$

where $k(t)$ and $y(t)$ and their first two derivatives vanish at the origin. Proceeding as in the derivation of (5),

$$(25)\qquad \frac{\ddot y}{\ddot x}=\left[\frac{v^2}{f}+m(x)\right]g''+g',$$

where $m(x)\to 0$ as $x\to 0$. From (3), (4), and (25)

$$(26)\qquad \left[\frac{v^2}{f}+m(x)\right]g''+g'=D(x)+A(x,g)\cdot g.$$

We apply the method in Lemmas 1 and 2 of Theorem I to (8) and (26) and find that

$$\lim_{x \to 0} \frac{\dfrac{v^2}{f} g'(x) + g(x)}{h(x)} = 1.$$

By a proof analogous to that used in the derivation of (15) from (13), it can be shown that

$$\lim_{x \to 0} \frac{\int_0^x h(x)dx}{\dfrac{v^2}{f} g(x)} = 1. \tag{27}$$

By hypothesis, $h(x)$ has generalized contact of order α. Hence, by L'Hospital's Rule, $\int_0^x h(x)dx$ has generalized contact of order $\alpha+1$. From (27), it follows that $g(x)$ has generalized contact of order $\alpha+1$ and that its departure from the common tangent varies inversely as the square of the speed of projection. The case of ordinary contact is treated similarly. This completes the proof of Theorem IX.

In a later paper, we shall extend these results to fields of force which fluctuate with the time.

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THE DIRICHLET PROBLEM FOR DOMAINS WITH MULTIPLE BOUNDARY POINTS*

BY

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Introduction

Several years ago Professor Kellogg called my attention to the desirability of extending the theory of the Dirichlet problem so as to include the case in which the domain has multiple boundary points with boundary values depending upon the manner of approach. In the two-dimensional case conformal mapping may sometimes be used. Also, a paper by Perron† contains results related to this subject. It seems desirable, however, to develop a general theory for this extended form of the problem.

Professor Kellogg noted that a spatial analogue of Carathéodory's‡ theory of prime ends would be of value here, since this would render possible in some cases the definition of functions corresponding to barriers.§ He communicated his ideas on this topic to me, and invited me to collaborate with him on the problem. Later he suggested that I develop the subject alone, a procedure which unfortunately was made necessary by his death. This paper contains the results of the ensuing study of the problem.

The discussion is formulated for a general finite domain of three-dimensional space, except in the case of a few theorems where special restrictions are imposed. It is readily seen that corresponding results are valid in the plane.

In Part I we introduce the notions of component and boundary element. These correspond to Carathéodory's ends and prime ends, respectively,

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† O. Perron, *Eine neue Behandlung der ersten Randwertaufgabe für $\Delta u=0$*, Mathematische Zeitschrift, vol. 18 (1923), pp. 42–55. See also N. Wiener, *Note on a paper of O. Perron*, Journal of Mathematics, and Physics of the Massachusetts Institute of Technology, vol. 4, No. 1 (January, 1925), p. 21 ff.

‡ C. Carathéodory, *Über die Begrenzung einfach zusammenhängender Gebiete*, Mathematische Annalen, vol. 73 (1913), pp. 323–370. I am indebted to Dr. Seidel for calling my attention to a group of papers containing a spatial generalization of this theory, though not of the type desired for the present problem: B. Kaufmann, *Über die Berandung ebener und räumlicher Gebiete* (*Primendentheorie*), Mathematische Annalen, vol. 103 (1930), pp. 70–144; *Über die Struktur der Komplexe erster Ordnung in der Theorie der Primenden*, ibid., vol. 106 (1932), pp. 308–333; *Über die Bestimmung der Primenden durch reguläre Komplexe*, ibid., vol. 106 (1932), pp. 334–342.

§ See Theorem 24 of this paper. Professor Kellogg suggested the construction of a barrier by means of an infinite series, as is done here.

though the analogy is not as close as that which I believe Professor Kellogg had in mind. Like the prime ends of Carathéodory's theory, boundary elements furnish a means of distinguishing between the various modes of approach to a multiple boundary point, which (for our purposes) may be defined as a point contained in more than one boundary element.

Part II is devoted largely to the study of functions of boundary elements. We introduce various concepts corresponding to familiar notions in the theory of functions of a real variable. We give in §2 a theorem which is used later as a substitute for the Weierstrass polynomial approximation theorem.

In Part III we attack the Dirichlet problem. It is shown that the sequence solution of the generalized Dirichlet problem studied by Wiener* and by Kellogg† has a direct analogue in the new theory. The discussion here given corresponds quite closely to Kellogg's treatment of the earlier form of the problem, except that we confine our attention to finite domains. We also introduce the idea of a "pseudo-barrier" and discuss its properties briefly.

I. Components and boundary elements

1. Preliminary ideas

In addition to a number of special propositions necessary in connection with the later theory, we include here explicit definitions of certain familiar concepts, in order to avoid ambiguity in our use of these terms.

Definition 1. *By a finite domain, or a domain T, we mean a proper‡ bounded, open, connected§ point set in a three-dimensional euclidean space.*

Except in the case of a few propositions where special restrictions are desirable, we will find it convenient to think of the domain T as chosen initially in an arbitrary fashion, and then held fast throughout the discussion.

* N. Wiener, *Certain notions of potential theory*, Journal of Mathematics and Physics, vol. 3, No. 1 (January, 1924), p. 24 ff.

† O. D. Kellogg, *Foundations of Potential Theory*, Berlin, 1929, pp. 322–326. (See also Proceedings of the American Academy, vol. 58 (1923), pp. 528, 529.)

‡ A proper point set is one which contains at least one point. A proper subset of a given set is a proper set contained in the given set but not identical with it.

§ A point set satisfying the other requirements will be called connected if, given any two points of the set, it is possible to join them by a continuous curve lying entirely in the set. By a continuous curve we mean a closed point set which with reference to some (and therefore any) Cartesian coordinate system, may be represented parametrically by equations of the form

$$x=X(\theta), \qquad y=Y(\theta), \qquad z=Z(\theta), \qquad 0\leqq\theta\leqq 1,$$

where the functions $X(\theta)$, $Y(\theta)$, and $Z(\theta)$ are each defined and continuous on the interval $0\leqq\theta\leqq 1$. The ends of the curve are the points corresponding to $\theta=0$ and $\theta=1$. It will be noted that we make no restriction that such a curve may not cross itself.

Definition 2. *By a finite closed region T' we mean a point set which may be obtained by adjoining to a finite domain T all points of its boundary,* t.*

Definition 3. *Given any proper subset E of the closed finite region $T'=T+t$, the closed set comprising all boundary points of E which are not in t, and all limit points of such boundary points, is called the auxiliary boundary of E.*

Definition 4. *By a partial domain G of T we mean a finite domain having the following properties:*

(1) *The set G is a proper subset of T.*

(2) *At least one point of t is a boundary point of G.*

(3) *The auxiliary boundary, g, of G contains only frontier points of G.*

Definition 5. *By a closed partial region G' of T' we mean a finite closed region which may be formed by adjoining to a partial domain G of T all boundary points of G.*

Theorem 1. *The set of all points of T which are interior points of a closed partial region G' of T' is identical with the partial domain G of T from which G' is formed. Moreover, the auxiliary boundaries of G and G' are proper sets and are identical.*

A point of T interior to G' is not an exterior point of G; moreover it is not a limit point of points exterior to G, and so cannot be a frontier point of G. Hence such a point cannot be a boundary point of G, since all boundary points of G are frontier points of G. Consequently a point of T interior to G' is a point of G. Since a point of G is a point of T interior to G' we see that the first part of the theorem is true.

If the proper set G had no boundary points except points of t, we could infer that every point of T is a point of G, contrary to the definition of G. Since T contains points not in G, it must contain a boundary point of G, and so the auxiliary boundary of G is a proper set. The interior points of G and G' form identical sets; it is also readily seen that the points exterior to G and G', respectively, also form identical sets, and so the auxiliary boundaries of G and G' are identical point sets.

Definition 6. *By the partial domain G of T corresponding to the closed partial region G' of T' we mean the set of all points of T which are interior points of G'.*

Definition 7. *An infinite sequence of closed partial regions of T': G_1', G_2', G_3', $\cdots$ is said to be monotone if the following conditions are satisfied:*

* A boundary point of any given set E is a limit point of points of E which is not interior to E; a frontier point of E is a limit point of exterior points of E which is not exterior to E. The frontier of a finite domain T is always a proper set and is contained in the boundary, t.

(1) *Each point of G'_{i+1} is a point of G'_i, $i=1, 2, 3, \cdots$.*

(2) *The auxiliary boundaries of no two of the given closed partial regions have a point in common.*

2. Components

Definition 8. *A component Γ of T' is a monotone sequence of closed partial regions of T', with the convention that two such sequences $G'_1, G'_2, G'_3, \cdots$, and $\overline{G}'_1, \overline{G}'_2, \overline{G}'_3, \cdots$, determine identical components if and only if to every positive integer i there corresponds a pair of positive integers j and k such that $G'_j \leqq \overline{G}'_i$ and $\overline{G}'_k \leqq G'_i$.*

Definition 9. *The component $\Gamma: G'_1, G'_2, G'_3, \cdots$ of T' is contained in a given finite closed region E (not necessarily a subregion of T') if and only if there exists a positive integer i such that $G'_i \leqq E$.*

Definition 10. *The component $\Gamma: G'_1, G'_2, G'_3, \cdots$ of T' is contained in the component $\overline{\Gamma}: \overline{G}'_1, \overline{G}'_2, \overline{G}'_3, \cdots$ of T' if and only if Γ is contained in each $\overline{G}'_i$.*

We note that two components Γ and $\overline{\Gamma}$ are identical if and only if each is contained in the other.

Definition 11. *A given point is contained in the component $\Gamma: G'_1, G'_2, G'_3, \cdots$ if and only if that point is contained in each G'_i.*

Theorem 2. *A component Γ of T' contains at least one point of t.*

Let $G'_1, G'_2, G'_3, \cdots$ be a monotone sequence of closed partial regions of T' determining Γ. Now every point of the proper closed point set $t \cdot G'_{+1}$ is also a point of the set $t \cdot G'_i$, $i=1, 2, 3, \cdots$. Hence there is at least one point common to the proper closed sets $t \cdot G'_1, t \cdot G'_2, t \cdot G'_3, \cdots$. Such a point belongs to t and is contained in Γ.

Definition 12. *Let $T^{(1)}$ and $T^{(2)}$ be two given finite domains. If there exists an infinite sequence, $G_1, G_2, G_3, \cdots$, of common partial domains of $T^{(1)}$ and $T^{(2)}$ such that the corresponding closed finite regions, $G'_1, G'_2, G'_3, \cdots$, form a monotone sequence of closed partial regions of $T^{(1)\prime}$ and of $T^{(2)\prime}$, and thus determine components $\Gamma^{(1)}$ of $T^{(1)\prime}$ and $\Gamma^{(2)}$ of $T^{(2)\prime}$, then $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are called equivalent components.*

3. Boundary elements

Definition 13. *An element γ of the boundary t of T is a component of T' which may be determined by a monotone sequence of closed partial regions, $G'_1, G'_2, G'_3, \cdots$, having the property that the diameter* of G'_i approaches the limit zero as i becomes infinite.*

* We adopt the usual definition of the diameter of a closed point set, namely, the maximum distance between two points of the set.

THEOREM 3. *A necessary and sufficient condition that a component* Γ *of* T' *be a boundary element is that* Γ *contain one point of* t, *and no other point.*

The necessity of the condition is immediately obvious. To establish the sufficiency we construct a sphere S of arbitrary positive radius, with center at p, the single point contained in the component Γ determined by the monotone sequence $G_1', G_2', G_3', \cdots$. We denote by E_k the set of all points of G_k' which lie outside or on the boundary of the sphere S. Now E_k is either a null set or a closed proper set; we note also that $E_1 \geqq E_2 \geqq E_3 \geqq \cdots$. Hence if each E_k is a proper set, there is a point P, without or on the boundary of S, which belongs to each E_k and so to each G_k'. But such a point would be a point distinct from p contained in Γ, contrary to hypothesis. We infer that for some positive integral k, E_k is a null set, and that each of the closed partial regions G_k', G_{k+1}', G_{k+2}', $\cdots$ contains only points interior to S. Since the radius of S is arbitrary, this means that the diameter of G_i' approaches the limit zero as i becomes infinite, and so Γ is a boundary element.

THEOREM 4. *If a boundary element* γ *contains a boundary element* $\bar{\gamma}$ *then* γ *and* $\bar{\gamma}$ *are identical.*

Let $G_1', G_2', G_3', \cdots$ and $\overline{G}_1', \overline{G}_2', \overline{G}_3', \cdots$ be monotone sequences determining the boundary elements γ and $\bar{\gamma}$ respectively. We will denote the partial domains of T corresponding to G_i' and $\overline{G}_i'$ by G_i and $\overline{G}_i$ respectively. Clearly, γ and $\bar{\gamma}$ contain the same point p of t. We may, without loss of generality, assume that p does not lie on the auxiliary boundary of any of the closed partial regions of the second sequence. For, from the definition of a monotone sequence, it could not lie on more than one $\overline{G}_i'$, and if such an exceptional closed partial region were deleted, we should still have a monotone sequence determining a boundary element identical with $\bar{\gamma}$.

Given an arbitrary $\overline{G}_i'$, we know that the distance from p to $\bar{g}_i$ (the auxiliary boundary of $\overline{G}_i'$) is positive. Consequently there exists a G_j' which has no point in common with $\bar{g}_i$: $G_j' \cdot \bar{g}_i = 0$. Since γ contains $\bar{\gamma}$, it is readily seen that G_j contains all points of some $\overline{G}_k$ and so some point Q of $\overline{G}_i$. Suppose now that there exists a point Q' in the interior of G_j but exterior to $\overline{G}_i$. Then Q and Q' can be joined by a continuous curve in G_j. Such a curve must pass from the interior of $\overline{G}_i$ to the exterior of $\overline{G}_i$, and so must pass through a point of $\bar{g}_i$. But this contradicts the relation $G_j' \cdot \bar{g}_i = 0$, and so shows that any point interior to G_j is not exterior to $\overline{G}_i$, and so is a point of $\overline{G}_i'$; we infer that given any $\overline{G}_i'$, there exists a G_j' such that $G_j' \leqq \overline{G}_i'$. This means that γ is contained in every closed partial region of a monotone sequence determining $\bar{\gamma}$; that is, γ is contained in $\bar{\gamma}$. Since γ contains $\bar{\gamma}$ by hypothesis, we conclude that γ and $\bar{\gamma}$ are identical.

DEFINITION 14. *Let P_1 be a point of T, and let ρ be any positive quantity. Then $\mathfrak{S}(P_1, \rho)$, the pseudo-spherical domain of radius ρ with center at P_1, is defined as the finite domain containing each point of T in the interior of the sphere of radius ρ with center at P_1 which can be joined to P_1 by a continuous curve made up entirely of such points. The set $\mathfrak{S}'(P_1, \rho)$ obtained by adjoining to $\mathfrak{S}(P_1, \rho)$ all its limit points, is called the closed pseudo-spherical region of radius ρ with center at P_1.*

The pseudo-spherical domain $\mathfrak{S}(P_1, \rho)$ may or may not have an auxiliary boundary. If it has, all points of the auxiliary boundary lie on the surface of the associated sphere. If $\mathfrak{S}(P_1, \rho)$ has an auxiliary boundary, then $\mathfrak{S}'(P_1, \rho)$ has the same auxiliary boundary.

DEFINITION 15. *Let G_1', G_2', G_3', $\cdots$ be a monotone sequence of closed partial regions of T' determining an element γ of t containing the point p. Given any positive ρ, let i be the smallest integer such that the interior S of the sphere of radius ρ with center at p contains G_i. Let P be a point of the corresponding G_i. Then $\mathfrak{S}(\gamma, \rho)$, the pseudo-spherical domain of radius ρ corresponding to γ, is defined as the finite domain consisting of those points which can be joined to P by a continuous curve lying in $T \cdot S$. Also, $\mathfrak{S}'(\gamma, \rho)$, the closed pseudo-spherical region of radius ρ corresponding to γ, is defined as the closed finite region obtained by adjoining to $\mathfrak{S}(\gamma, \rho)$ all its limit points.*

THEOREM 5. *If, in the definition of $\mathfrak{S}(\gamma, \rho)$ and $\mathfrak{S}'(\gamma, \rho)$, the sequence G_1', G_2', G_3', $\cdots$ is replaced by another monotone sequence determining an element of t identical with γ, the sets $\mathfrak{S}(\gamma, \rho)$ and $\mathfrak{S}'(\gamma, \rho)$ are unaltered; these sets are also independent of the choice of P in G_i. If ρ is less than ρ_0, the distance from p to the farthest point or points of t, then $\mathfrak{S}(\gamma, \rho)$ is a partial domain of T and $\mathfrak{S}'(\gamma, \rho)$ is a closed partial region of T'. Moreover, if $\rho_0, \rho_1, \rho_2, \rho_3, \cdots$ is an infinite monotone decreasing sequence of positive numbers tending to zero, then $\mathfrak{S}'(\gamma, \rho_1)$, $\mathfrak{S}'(\gamma, \rho_2)$, $\mathfrak{S}'(\gamma, \rho_3)$, $\cdots$ is a monotone sequence of closed partial regions of T' determining an element of t identical with γ.*

THEOREM 6. *A necessary and sufficient condition that a given point p of t be contained in at least one element of t is that p be an accessible boundary point of T.*

To establish the sufficiency of the condition we assume that there exists a curve C given parametrically by the equations*

$$x = X(\theta), \qquad y = Y(\theta), \qquad z = Z(\theta), \qquad 0 \leqq \theta \leqq 1,$$

* It is not necessary to assume that every pair of distinct values of θ on the interval $0 \leqq \theta \leqq 1$ correspond to distinct points of C. However, the definition of accessibility which we use is in reality no more general than that in which this further restriction is imposed on the curve C.

where $X(\theta)$, $Y(\theta)$, and $Z(\theta)$ are continuous functions, such that each point of C except that corresponding to $\theta=1$ is in T, and the point corresponding to $\theta=1$ is the given point p.

Consider the distance from p to the variable point P of C. This distance is a continuous function of θ, the parametric coordinate of P on C, and assumes a positive maximum value ρ_1 for a finite or a closed infinite set of values of θ on the interval $0\leqq\theta\leqq 1$. In any case there is a largest value of θ, say $\theta=\theta_1$, for which this maximum value ρ_1 is attained. We denote by G_1 the set containing each point of T which can be joined to the point of C with parametric coordinate $\theta_1'=(1+\theta_1)/2$ by a continuous curve lying entirely in T and in the interior of the sphere S_1 with center at p and radius ρ_1. We note that all points of that part of C for which $\theta_1<\theta<1$ are points of G_1. All points of the auxiliary boundary of G_1 are on the surface of the sphere S_1. The set G_1 is a partial domain of T.

Given any integer i greater than unity, we consider the sphere S_i of radius $\rho_i=\rho_1/2^i$. There exists a constant θ_i, less than unity, such that all points of C for which $\theta_i<\theta<1$ lie in the interior of S_i and the point of C for which $\theta=\theta_i$ lies on S_i. We denote by G_i the partial domain* of T which contains each point of T which can be joined by a continuous curve lying in T and in the interior of S_i to the point of C with parametric coordinate $\theta_i'=(1+\theta_i)/2$.

We now form the sequence of the corresponding closed partial regions of T': $G_1', G_2', G_3', \cdots$. This sequence is monotone, and determines a boundary element containing the given point p. This establishes the sufficiency of the given condition.

To prove that the condition is necessary, we assume that p is contained in an element γ of t and define a continuous curve approaching p from the interior of T. Let ρ be a positive constant less than the distance from p to the farthest point or points of t, and let P be an arbitrary point of the pseudo-spherical domain $\mathfrak{S}(\gamma, \rho)$. The distance from P to the boundary of $\mathfrak{S}(\gamma, \rho)$ attains its maximum value on a finite or closed infinite subset $\mathfrak{S}_1$ of points in $\mathfrak{S}(\gamma, \rho)$. In either case it is possible to give a law whereby we may select uniquely† (relatively to an arbitrarily preassigned Cartesian coordinate sys-

* It may be noted that the determination of G_i depends on C, but is independent of the parametric representation of this curve, provided the point p corresponds to $\theta=1$ and the other end of C to $\theta=0$.

† For instance, we may discard all points of the set $\mathfrak{S}_1$ except those for which the x-coordinate attains its maximum value. If this does not restrict us to a single point, we may then discard from the remaining points all except those on which the y-coordinate attains its maximum value. If more than one point remains, we may choose as Q_1 the unique point on which the z-coordinate attains its maximum value. This method may be used to select uniquely a point from any bounded closed set. The purpose of prescribing a law for the choice of Q_1 is to show that it is not necessary to make an infinite number of arbitrary choices to obtain the set of points Q_i, $i=1, 2, 3, \cdots$.

tem) a particular point Q_1 from $\mathfrak{S}_1$. There exists a largest integer k_1 such that Q_1 is a point of $\mathfrak{S}(\gamma, 2^{-k_1}\rho)$. We now determine uniquely a particular point Q_2 of $\mathfrak{S}(\gamma, 2^{-(k_1+1)}\rho)$ by the method that we used to select Q_1 from $\mathfrak{S}(\gamma, \rho)$. There exists a largest integer k_2 such that Q_2 is a point of $\mathfrak{S}(\gamma, 2^{-k_2}\rho)$. We note that $k_2 > k_1$. By repetitions of this process we obtain an infinite sequence of points $Q_1, Q_2, Q_3, \cdots$, and an infinite sequence of integers

$$k_1 < k_2 < k_3 < \cdots,$$

such that Q_i is a point of $\mathfrak{S}(\gamma, 2^{-k_i}\rho)$ but not a point of $\mathfrak{S}(\gamma, 2^{-k_{i+1}}\rho)$, where $i = 1, 2, 3, \cdots$.

We now show that it is possible to state a law whereby, given any positive integer i, we may determine uniquely a broken line of a finite number of segments lying entirely in $\mathfrak{S}(\gamma, 2^{-k_i}\rho)$ and joining Q_i and Q_{i+1}. There exists a smallest integer m_i such that Q_i and Q_{i+1} can be joined by broken line of m_i segments lying entirely in $\mathfrak{S}(\gamma, 2^{-k_i}\rho)$. If $m_i > 1$ the set of all points Q such that $\overline{Q_iQ}$ is a segment of such a broken line of m_i segments is a proper open subset of $\mathfrak{S}(\gamma, 2^{-k_i}\rho)$. We may select uniquely a particular point $Q_{i,1}$ from this subset by restricting ourselves first to those points at a maximum distance from the boundary of the subset, and then using the device employed to choose the point Q_1 from $\mathfrak{S}_1$. Now $Q_{i,1}$ can be joined to Q_{i+1} by a broken line of $m_i - 1$ segments, lying entirely in $\mathfrak{S}(\gamma, 2^{-k_i}\rho)$. Using the method employed above, we may select uniquely a point $Q_{i,2}$ in such a way that $Q_{i,1}Q_{i,2}$ may be used as a segment of this broken line. By successive repetitions of this process we determine uniquely a broken line $Q_i, Q_{i,1}, Q_{i,2}, \cdots, Q_{i,m_i-1}, Q_{i+1}$, lying entirely in $\mathfrak{S}(\gamma, 2^{-k_i}\rho)$ and joining Q_i and Q_{i+1}.

By applying this procedure to each pair of successive points in the sequence $Q_1, Q_2, Q_3, \cdots$, and adjoining to the set of all points on all the line segments the point p, we obtain a continuous curve terminating at p, but otherwise lying entirely in T. It is a simple matter to establish a parametric representation for the curve, if desired. This shows that p is an accessible boundary point of T and completes the proof of the theorem.

Corollary. *Given a point P of T at a distance $\bar{p}$ from t, there exists at least one element γ of t such that P is a point of every pseudo-spherical domain $\mathfrak{S}(\gamma, \rho)$ such that $\rho > \bar{p}$. Moreover, it is possible to prescribe a law whereby such an element γ of t is determined uniquely by a given point P of T and a given Cartesian coordinate system.*

Let p be a point of t on which the distance from P to a variable point of t attains its minimum value, $\bar{p}$. The set of points on t satisfying this requirement form a proper finite or a closed infinite set, and a unique choice may

be made by the law described in connection with a similar situation in the discussion of Theorem 6. From the discussion of this theorem, we know that there exists a unique element γ of t determined by the line segment C joining P and p. Clearly P is a point of $\mathfrak{S}(\gamma, \rho)$, provided $\rho > \bar{\rho}$.

II. Functions of boundary elements

1. Pseudo-continuity

Definition 16. *A function of a variable element of the boundary t of T is defined when a law* is given whereby to each element γ of t there corresponds a uniquely determined real number, $f(\gamma)$.*

Definition 17. *A function $f(\gamma)$ of a variable element γ of t is said to be pseudo-continuous at the element $\gamma = \gamma_1$, if to each positive quantity ϵ there corresponds a positive quantity δ such that*

$$|f(\gamma) - f(\gamma_1)| < \epsilon$$

for all elements γ of t contained in the closed pseudo-spherical region $\mathfrak{S}'(\gamma_1, \delta)$.

Definition 18. *A function $f(\gamma)$ of a variable element γ of t is pseudo-continuous on t if it is pseudo-continuous at each element of t.*

Definition 19. *A function $f(\gamma)$ of a variable element γ of t is said to be uniformly pseudo-continuous on t if, given γ_1 and an arbitrary positive quantity ϵ, there exists a positive quantity δ, independent of γ_1, such that*

$$|f(\gamma) - f(\gamma_1)| < \epsilon,$$

for all elements γ of t contained in the closed pseudo-spherical region $\mathfrak{S}'(\gamma_1, \delta)$.

To any given function $\phi(p)$ of a variable point p of t there corresponds a function† $f(\gamma)$ of the variable element γ of t obtained by assigning to an arbitrary element γ of t the value assumed by the function $\phi(p)$ at the point p contained in γ. If the function $\phi(p)$ is continuous at a particular point p_1 of t,

* The definition of a function of a variable element of t is analogous to Dirichlet's well known definition of a function of a real variable. An alternative form of the definition of the new concept is given below. Let $\mathfrak{F}[X(\theta), Y(\theta), Z(\theta)]$ be any functional of the real functions $X(\theta)$, $Y(\theta)$, $Z(\theta)$, $0 \leqq \theta \leqq 1$, which (1) is defined, single-valued and real for each choice of this triple of functions determining a continuous curve $C: x = X(\theta)$, $y = Y(\theta)$, $z = Z(\theta)$ having the end $\theta = 1$ on t but otherwise lying in T, and which (2) has the property that if any two triples of such functions $X_1(\theta)$, $Y_1(\theta)$, $Z_1(\theta)$ and $X_2(\theta)$, $Y_2(\theta)$, $Z_2(\theta)$ determine curves yielding (by application of the process described in the first part of the proof of Theorem 6) identical elements of t, then

$$\mathfrak{F}[X_1(\theta), Y_1(\theta), Z_1(\theta)] = \mathfrak{F}[X_2(\theta), Y_2(\theta), Z_2(\theta)].$$

Any such functional $\mathfrak{F}[X(\theta), Y(\theta), Z(\theta)]$ gives rise to a function $f(\gamma)$ of the variable element γ of t which assumes at a given element γ of t the value of the functional for functions $X(\theta)$, $Y(\theta)$ and $Z(\theta)$ determining any curve yielding the boundary element γ.

† This is, of course, a very special type of function of γ.

then the corresponding function $f(\gamma)$ is pseudo-continuous at each element which contains p_1. If $\phi(p)$ is continuous at each point of t, and therefore uniformly continuous on t, then $f(\gamma)$ is uniformly pseudo-continuous on t.

We note, however, that even though $\phi(p)$ be discontinuous on t, in fact unbounded on t, the function $f(\gamma)$ may nevertheless be uniformly pseudo-continuous on t. Consider, for instance, the case in which T is the domain bounded by the surfaces

$$\begin{aligned}
x^2+y^2&=9, && 0<z<1;\\
z&=0, && x^2+y^2\leqq 9;\\
z&=1, && x^2+y^2\leqq 9;\\
z&=(2i)^{-1/2}, && x^2+y^2\leqq 9,\ (x-2)^2+y^2\geqq \tfrac{1}{4},\quad i=1,2,3,\cdots,\\
z&=(2i+1)^{-1/2}, && x^2+y^2\leqq 9,\ (x+2)^2+y^2\geqq \tfrac{1}{4},\quad i=1,2,3,\cdots.
\end{aligned}$$

We define $\phi(p)$ at the boundary point $p:(x, y, z)$ so that $\phi(p)=z^{-1}$ when $z\neq 0$ and $\phi(p)=1$ when $z=0$.

A boundary point for which $z=0$ is not contained in any boundary element. Given a boundary element γ_1 of t containing a point $p_1:(x_1, y_1, z_1)$ such that $z_1\leqq\frac{1}{2}$, there exists an integer $m_1>4$ such that $m_1^{-1/2}<z_1\leqq(m_1-1)^{-1/2}$. For each point $p:(x, y, z)$ contained in an element γ contained in $\mathfrak{S}'(\gamma_1, 1)$ we have $(m_1+2)^{-1/2}\leqq z\leqq(m_1-3)^{-1/2}$, and so

$$(m_1-3)^{1/2}-m_1^{1/2}<z^{-1}-z_1^{-1}\leqq(m_1+2)^{1/2}-(m_1-1)^{1/2},$$

whence,

$$|f(\gamma)-f(\gamma_1)|<3m_1^{-1/2}<3z_1.$$

Given any positive ϵ, we infer that if γ_1 is so chosen that $z_1\leqq\epsilon/3$ and $z_1\leqq\frac{1}{2}$, then throughout $\mathfrak{S}'(\gamma_1, 1)$ we have

$$|f(\gamma)-f(\gamma_1)|<\epsilon.$$

Let $\bar{t}$ be the subset of t each point of which is contained in a pseudo-spherical region of unit radius corresponding to some boundary element containing a point on or above at least one of the planes $z=\epsilon/3$ and $z=\frac{1}{2}$. To the given ϵ there corresponds a positive quantity $\bar{\delta}$, independent of the choice of the point $\bar{p}:(\bar{x}, \bar{y}, \bar{z})$ of $\bar{t}$, such that for each point $p:(x, y, z)$ of t at a distance from $\bar{p}$ less than $\bar{\delta}$ we have

$$|\phi(p)-\phi(\bar{p})|<\epsilon.$$

If, now, we choose δ as the smaller of the two numbers $\bar{\delta}$ and unity, or their common value if $\bar{\delta}=1$, then, given any element γ_2 of t and any element γ of t contained in $\mathfrak{S}'(\gamma_2, \delta)$, we may write

$$|f(\gamma) - f(\gamma_2)| < \epsilon.$$

Hence the function $f(\gamma)$ is uniformly pseudo-continuous on t, even though it has no upper bound on t. It is a simple matter to extend the method used in the construction of this example so as to obtain a function with neither an upper nor a lower bound on the boundary of a certain finite domain, but which is nevertheless uniformly pseudo-continuous on the boundary of that domain.

DEFINITION 20. *A function $F(P)$ of the variable point P of T is said to approach a given value c at a given boundary element γ of t if to each positive quantity ϵ there corresponds a positive quantity δ such that*

$$|F(P) - c| < \epsilon,$$

throughout the pseudo-spherical domain $\mathfrak{S}(\gamma, \delta)$.

DEFINITION 21. *Given a function $F(P)$ of a variable point P of T, and a function $f(\gamma)$ of a variable element γ of t, the function $F(P)$ is said to approach the boundary values $f(\gamma)$ with uniform pseudo-continuity if, given an arbitrary positive quantity ϵ, there exists a positive quantity δ, independent of γ, such that*

$$|F(P) - f(\gamma)| < \epsilon,$$

throughout the pseudo-spherical domain $\mathfrak{S}(\gamma, \delta)$.

DEFINITION 22. *A function $F(P)$ of a variable point P of T is said to be pseudo-uniformly continuous in T if, given an arbitrary positive quantity ϵ, there exists a positive quantity δ, independent of P_1, such that*

$$|F(P) - F(P_1)| < \epsilon,$$

for all points P of the pseudo-spherical domain $\mathfrak{S}(P_1, \delta)$.

A slightly different formulation of the property here involved is useful. It is embodied in the following proposition:

THEOREM 7. *A necessary and sufficient condition that a function $F(P)$ be pseudo-uniformly continuous in T is that given any positive quantity ϵ, there exists a positive quantity δ such that*

$$|F(P) - F(P')| < \epsilon,$$

for every pair of points P and P' in T that can be joined by a continuous curve (lying entirely in T) the maximum distance between two points of which is less than δ.

Let $F(P)$ be a function of the point P of T. If to a given positive ϵ there corresponds a δ for which the condition given in Definition 22 holds, then

the condition given in this theorem also holds for the same ϵ and δ. If to a given positive ϵ there corresponds a δ such that the condition given in this theorem holds, the condition given in Definition 22 holds for the given ϵ provided $\mathfrak{S}(P_1, \delta)$ is replaced by $\mathfrak{S}(P_1, \delta/2)$.

THEOREM 8. *A necessary and sufficient condition that a function $F(P)$ of a variable point P of T be continuous in T and approach bounded and uniformly pseudo-continuous boundary values on t with uniform pseudo-continuity is that $F(P)$ be bounded and pseudo-uniformly continuous in T.*

If $F(P)$ is continuous in T and approaches with uniform pseudo-continuity the bounded and uniformly pseudo-continuous boundary values $f(\gamma)$ on t, then corresponding to any given positive ϵ there exists a positive δ_1, independent of the arbitrary element γ of t, such that

$$|F(P) - f(\gamma)| < \frac{\epsilon}{2},$$

throughout $\mathfrak{S}(\gamma, 3\delta_1)$. Now the function $F(P)$ is bounded and uniformly continuous on the closed set comprising points of T at a distance at least δ_1 from t; hence to the given ϵ there corresponds a positive δ_2 such that

$$|F(P_1) - F(P_2)| < \epsilon,$$

for all pairs of points P_1 and P_2 of this subset of T which are so located that $\overline{P_1P_2} < \delta_2$. From these relations and the Corollary of Theorem 6, it follows that if δ is the smaller of δ_1 and δ_2 (or their common value if they are equal) then

$$|F(P) - F(P')| < \epsilon,$$

for all points P in $\mathfrak{S}(P', \delta)$, where P' is an arbitrary point of T. Hence the given condition is necessary.

Assuming now that $F(P)$ is bounded and pseudo-uniformly continuous in T, let γ be an arbitrary element of t, and ρ an arbitrary positive quantity. We denote by $H(\gamma, \rho)$ and $h(\gamma, \rho)$ the least upper and greatest lower bounds, respectively, of $F(P)$ in $\mathfrak{S}(\gamma, \rho)$. We note that as ρ decreases in value, $H(\gamma, \rho)$ never increases and $h(\gamma, \rho)$ never decreases. Consequently, for any fixed element γ, $\lim_{\rho\to 0} H(\gamma, \rho)$ and $\lim_{\rho\to 0} h(\gamma, \rho)$ exist, and

$$h(\gamma, \rho) \leqq \lim_{\rho\to 0} h(\gamma, \rho) \leqq \lim_{\rho\to 0} H(\gamma, \rho) \leqq H(\gamma, \rho).$$

Theorem 7 implies that, given any positive ϵ, there exists a positive δ, independent of γ, such that

$$0 \leqq H(\gamma, \delta) - h(\gamma, \delta) < \epsilon.$$

Hence,

$$\lim_{\rho\to 0} H(\gamma, \rho) = \lim_{\rho\to 0} h(\gamma, \rho).$$

The common value of these limits is a bounded function of γ on t; we represent this function by $f(\gamma)$.

Using Theorem 7 it is easy to show that, given any positive ϵ, there exists a positive δ independent of P, P' and the arbitrary element γ_1 of t, such that

$$|F(P) - F(P')| < \frac{\epsilon}{3},$$

for every pair of points P and P' in $\mathfrak{S}(\gamma_1, \delta)$. If $0<\rho<\delta$, we have then, in $\mathfrak{S}(\gamma_1, \rho)$,

$$|F(P) - H(\gamma_1, \rho)| \leqq \frac{\epsilon}{3},$$

whence,

$$|F(P) - f(\gamma_1)| \leqq \frac{\epsilon}{3} < \epsilon,$$

which proves that $F(P)$ approaches the boundary values $f(\gamma)$ with uniform pseudo-continuity. Moreover, if γ is an element of t contained in $\mathfrak{S}'(\gamma_1, \delta)$ there exists a positive ρ' so small that $\mathfrak{S}(\gamma, \rho')$ is a subset of $\mathfrak{S}(\gamma_1, \delta)$; hence the relations

$$|F(P) - F(P')| < \frac{\epsilon}{3}, \quad |F(P) - f(\gamma_1)| \leqq \frac{\epsilon}{3}, \text{ and } |F(P') - f(\gamma)| \leqq \frac{\epsilon}{3}$$

are valid for P in $\mathfrak{S}(\gamma_1, \delta)$ and P' in $\mathfrak{S}(\gamma, \rho')$ and imply that

$$|f(\gamma) - f(\gamma_1)| < \epsilon.$$

This shows that $f(\gamma)$ is uniformly pseudo-continuous on t, and so completes the proof of the theorem.

THEOREM 9. *Given any bounded and uniformly pseudo-continuous function $f(\gamma)$ of a variable element γ of t, there exists a function $F(P)$ of the variable point P of T which is bounded and continuous in T and approaches the boundary values $f(\gamma)$ with uniform pseudo-continuity.*

This proposition corresponds to part of a theorem regarding functions

continuous on the boundary of a domain established by Lebesgue.* The proof given below is an adaptation of a proof of Lebesgue's theorem due to Carathéodory.†

Let P be any point of T. Let $\rho(P)$ be the distance from P to t. We define $u_P(r)$ for $r>\rho(P)$ as the least upper bound of $f(\gamma)$ for all elements of t contained in $\mathfrak{S}'(P, r)$; as r increases $u_P(r)$ never decreases, and we may define a function $F(P)$ in T by the formula

$$F(P) = \frac{1}{\rho(P)}\int_{\rho(P)}^{2\rho(P)} u_P(r)dr.$$

This function has the desired properties, as is proved below.

Given any positive ϵ there exists a positive δ, independent of the arbitrarily chosen element γ_1 of t, such that

$$| f(\gamma) - f(\gamma_1) | < \frac{\epsilon}{2}$$

for each γ contained in $\mathfrak{S}'(\gamma_1, \delta)$. If we choose P in $\mathfrak{S}(\gamma_1, \delta/3)$ and restrict r by the inequality $\rho(P)<r<2\rho(P)$, then $\mathfrak{S}'(P, r)$ is a subset of $\mathfrak{S}'(\gamma_1, \delta)$ and consequently

$$| u_P(r) - f(\gamma_1) | \leqq \frac{\epsilon}{2} < \epsilon.$$

Accordingly,

$$| F(P) - f(\gamma_1) | \leqq \frac{1}{\rho(P)}\int_{\rho(P)}^{2\rho(P)} \frac{\epsilon}{2}\, dr < \epsilon,$$

for any P in $\mathfrak{S}(\gamma_1, \delta/3)$. Hence $F(P)$ approaches the boundary values $f(\gamma)$ with pseudo-uniform continuity.

The function $F(P)$ is clearly bounded in T. Since $\rho(P)$ is plainly a non-vanishing continuous function of P in T, we may infer the continuity of $F(P)$ in T if we can show that the function

$$\bar{F}(P) = \int_{\rho(P)}^{2\rho(P)} u_P(r)dr$$

is continuous in T. Let P and P' be any two points of T such that $\overline{PP'}$ is less than each of the quantities $\rho(P)$, $\rho(P')$ and $m^{-1}\epsilon/6$ where m is the least upper bound of $|f(\gamma)|$ on t and ϵ is an arbitrarily chosen positive constant. Since

* H. Lebesgue, *Sur le problème de Dirichlet*, Rendiconti del Circolo Matematico di Palermo, vol. 24 (1907), pp. 371–402. See particularly pp. 379, 380.

† C. Carathéodory, *Vorlesungen über reelle Funktionen*, 2d edition, Leipzig, 1918, pp. 617–618.

the value of $u_P(r)$ for any r such that $\rho(P) < r < 2\rho(P)$ depends on the values of $f(\gamma)$ at elements γ of t contained in $\mathfrak{S}'(P, r)$ which is a subset of $\mathfrak{S}'(P', r+\overline{PP'})$, we see that

$$|\overline{F}(P) - \overline{F}(P')| \leqq \left| \int_{\rho(P)}^{2\rho(P)} u_{P'}(r + \overline{PP'})dr - \int_{\rho(P')}^{2\rho(P')} u_{P'}(r)dr \right|$$

$$= \left| \int_{\rho(P)+\overline{PP'}}^{2\rho(P)+2\overline{PP'}} u_{P'}(r)dr - \int_{\rho(P')}^{2\rho(P')} u_{P'}(r)dr \right|,$$

whence,

$$|\overline{F}(P) - \overline{F}(P')| \leqq \int_{\rho(P)+\overline{PP'}}^{\rho(P')} | u_{P'}(r) | dr + \int_{2\rho(P')}^{2\rho(P)+2\overline{PP'}} | u_{P'}(r) | dr$$

$$\leqq 6m\overline{PP'} < \epsilon.$$

Hence $F(P)$ is continuous in T, and the proof is complete.

THEOREM 10. *Let $F(P)$ be a function of the variable point P of T, $\phi(p)$ a continuous function of the variable point p of t, and $f(\gamma)$ the function of the variable element γ of t which at each element γ has the same value as does $\phi(p)$ at the point p contained in γ. Then a necessary and sufficient condition that $F(P)$ approach the boundary values $f(\gamma)$ with uniform pseudo-continuity is that $F(P)$ approach the boundary values $\phi(p)$ continuously.*

The sufficiency of the condition is immediately obvious. The necessity is readily established by observing that the definition of the function $f(\gamma)$ implies that $f(\gamma)$ is uniformly pseudo-continuous on t, and by using the first part of the Corollary of Theorem 6.

COROLLARY. *Given any uniformly pseudo-continuous function $f(\gamma)$ of a variable element of t, and two functions, $F_1(P)$ and $F_2(P)$ of a variable point P of T, each of which approaches the boundary values $f(\gamma)$ with uniform pseudo-continuity, then the function $F(P) = F_1(P) - F_2(P)$ approaches zero continuously on t.*

2. AN APPROXIMATION THEOREM

The theorem to which this section is devoted does not itself directly involve functions of boundary elements, but is essential for later work.

THEOREM 11. *Given an arbitrary Cartesian coordinate system, a positive constant ϵ, and a function $F(P)$ (of the variable point $P:(x, y, z)$ of T) bounded and pseudo-uniformly continuous in T, then there exists a function $\mathfrak{F}(P)$, defined in T, which has the following properties:*

(1) *The function $\mathfrak{F}(P)$ is bounded and pseudo-uniformly continuous in T.*

(2) *The function $\mathfrak{F}(P)$ has continuous partial derivatives of the first and second order in T, and a bounded Laplacian in T.*

(3) *Throughout T,*

$$|\mathfrak{F}(P) - F(P)| < \epsilon.$$

Since $F(P)$ is bounded and pseudo-uniformly continuous in T, we know that corresponding to the given positive ϵ there exists a positive δ such that the relation

$$|F(P) - F(Q)| < \frac{\epsilon}{2}$$

holds for every pair of points P and Q in T that can be joined by a continuous curve (lying entirely in T) the maximum distance between two points of which is less than 10δ. We also know that there exists a positive constant M such that $|F(P)| \leqq M$ throughout T.

We shall find it convenient to represent by $\mathfrak{D}(\Pi, \rho; P)$ the finite domain containing each point of T which can be joined to the point $P:(x, y, z)$ of T by a continuous curve lying in T and in the interior of the sphere of radius ρ with center at a point $\Pi:(\xi, \eta, \zeta)$ whose distance from P is less than ρ. It is not required that Π be a point of T. Every boundary point of $\mathfrak{D}(\Pi, \rho; P)$ is either a point of t or a boundary point of the sphere described above.

We now define a function $F_0(\Pi, P)$, for any point P of T and any point Π whose distance from P is less than 5δ, as the greatest lower bound of $F(Q)$ for all points Q in $\mathfrak{D}(\Pi, 5\delta; P)$. Clearly,

$$|F_0(\Pi, P)| \leqq M,$$

for all admissible pairs of points, Π and P.

We consider now the behavior of $F_0(\Pi, P')$, where P' is an arbitrarily chosen fixed point of T and Π a variable point in the interior of the sphere S_4 of radius 4δ with center at P'. Given any positive ϵ_1 there exists a continuous curve C lying in $\mathfrak{D}(\Pi, 5\delta; P')$ and joining P' to a point $\overline{P}$ such that

$$F(\overline{P}) < F_0(\Pi, P') + \epsilon_1.$$

Let δ_1 be the distance from C to the surface of the sphere of radius 5δ with center at Π. Let Π' be any point whose distance from P' is less than 4δ and whose distance from Π is less than δ_1. Then $\overline{P}$ is a point of $\mathfrak{D}(\Pi', 5\delta; P')$, and

$$F_0(\Pi', P') \leqq F(\overline{P}) < F_0(\Pi, P') + \epsilon_1.$$

This relation shows that for fixed P' in T, $F_0(\Pi, P')$ is an upper semi-continuous function of Π within the sphere S_4.

We now define the function $F_1(\Pi_1, P)$ for certain pairs of points by requiring that at each point $\Pi_1:(\xi_1, \eta_1, \zeta_1)$ (not necessarily in T) whose distance from the arbitrary point $P:(x, y, z)$ of T is less than 3δ we have

$$F_1(\Pi_1, P) = \frac{1}{\delta^3}\int_{\zeta_1-\delta/2}^{\zeta_1+\delta/2} d\zeta \int_{\eta_1-\delta/2}^{\eta_1+\delta/2} d\eta \int_{\xi_1-\delta/2}^{\xi_1+\delta/2} F_0(\Pi, P)d\xi.$$

In the integral above, the point $P:(x, y, z)$ is held fixed in T, and the integration is performed with respect to the coordinates ξ, η, ζ of the variable point Π over the closed cube

$$\mathfrak{R}_1': \ \xi_1 - \frac{\delta}{2} \leqq \xi \leqq \xi_1 + \frac{\delta}{2}, \ \eta_1 - \frac{\delta}{2} \leqq \eta \leqq \eta_1 + \frac{\delta}{2}, \ \zeta_1 - \frac{\delta}{2} \leqq \zeta \leqq \zeta_1 + \frac{\delta}{2}.$$

Every point of this closed cube is within the open sphere of radius 4δ with center at P, and so we know, from properties of the integrand derived above, that the integral has a meaning. Moreover $|F_1(\Pi_1, P)| \leqq M$, for all admissible pairs of points Π_1 and P.

We shall establish now an inequality which will prove useful later. For all pairs of points Π_1 and P for which $F_1(\Pi_1, P)$ is defined, we have

$$|F_1(\Pi_1, P) - F(P)| \leqq \frac{\epsilon}{2}.$$

For

$$|F_1(\Pi_1, P) - F(P)| \leqq \frac{1}{\delta^3}\int_{\zeta_1-\delta/2}^{\zeta_1+\delta/2} d\zeta \int_{\eta_1-\delta/2}^{\eta_1+\delta/2} d\eta \int_{\xi_1-\delta/2}^{\xi_1+\delta/2} |F_0(\Pi, P) - F(P)|\, d\xi,$$

and it is easily seen that, for all points Π in the region of integration,

$$|F_0(\Pi, P) - F(P)| \leqq \frac{\epsilon}{2},$$

whence,

$$|F_1(\Pi_1, P) - F(P)| \leqq \frac{\epsilon}{2}.$$

If P is any fixed point of T, then $F_1(\Pi_1, P)$ is a continuous function of Π_1, throughout the interior of the sphere of radius 3δ with center at P. We may therefore define a function $\mathfrak{F}(P)$, for an arbitrary point $P:(x, y, z)$ of T, by setting

$$\mathfrak{F}(P) = \frac{1}{\delta^6}\int_{z-\delta/2}^{z+\delta/2} d\zeta_2 \int_{y-\delta/2}^{y+\delta/2} d\eta_2 \int_{x-\delta/2}^{x+\delta/2} d\xi_2 \int_{\zeta_2-\delta/2}^{\zeta_2+\delta/2} d\zeta_1 \int_{\eta_2-\delta/2}^{\eta_2+\delta/2} d\eta_1 \int_{\xi_2-\delta/2}^{\xi_2+\delta/2} F_1(\Pi_1, P)d\xi_1.$$

For if P is a given fixed point of T, the existence of $\mathfrak{F}(P)$ depends only on the behavior of the integrand, $F_1(\Pi_1, P)$, for points Π_1 in the closed cube

$$\mathfrak{R}_2': \qquad x-\delta \leqq \xi_1 \leqq x+\delta, \quad y-\delta \leqq \eta_1 \leqq y+\delta, \quad z-\delta \leqq \zeta_1 \leqq z+\delta,$$

no point of which is at a distance from P greater than $3^{1/2}\delta$. The function $F_1(\Pi_1, P)$ is continuous throughout the range of integration.

For any point P of T, we have $|\mathfrak{F}(P)| \leqq M$, and also

$$| \mathfrak{F}(P) - F(P) | \leqq \frac{1}{\delta^6} \int_{z-\delta/2}^{z+\delta/2} d\zeta_2 \int_{y-\delta/2}^{y+\delta/2} d\eta_2 \int_{x-\delta/2}^{x+\delta/2} d\xi_2 \int_{\zeta_2-\delta/2}^{\zeta_2+\delta/2} d\zeta_1 \int_{\eta_2-\delta/2}^{\eta_2+\delta/2} d\eta_1 \int_{\xi_2-\delta/2}^{\xi_2+\delta/2} | F_1(\Pi_1, P) - F(P) | \, d\xi_1,$$

from which we obtain

$$| \mathfrak{F}(P) - F(P) | \leqq \frac{\epsilon}{2} < \epsilon.$$

Hence $\mathfrak{F}(P)$ has the properties given in the first part of item (1) and in item (3) of the conclusion of the theorem.

Let P' be any point of T, and let S_4 be the set of all points within the sphere of radius 4δ with center at P'. If Π is a fixed point of S_4 and P a variable point of $\mathfrak{S}(P', \delta)$, then $F_0(\Pi, P)$ is constant. Hence, if a variable point P is restricted to $\mathfrak{S}(P', \delta)$ then $F_0(\Pi, P)$ is a function of Π alone throughout S_4. If Π_1 is a variable point of the sphere S_3 of radius 3δ with center at P', then the cube $\mathfrak{R}_1'$ lies in the interior of S_4. Hence if P is restricted to $\mathfrak{S}(P', \delta)$ then $F_1(\Pi_1, P)$ is a function of Π_1 alone throughout S_3.

We now define the function $\Phi_1(\Pi_1, P')$ for any point P' of T and any point Π_1 in the interior of the corresponding sphere S_3 by requiring that, for any admissible P', $\Phi_1(\Pi_1, P')$ coincide with the function of Π_1 to which $F_1(\Pi_1, P)$ reduces when P is restricted to $\mathfrak{S}(P', \delta)$. For fixed P' in T, $\Phi_1(\Pi_1, P')$ is a continuous function of Π_1 in S_3, and $|\Phi_1(\Pi_1, P')| \leqq M$. We also define a function $\Phi(P, P')$ for any point P' of T and any point P (not necessarily in T) in the interior of the sphere S_1 of radius δ with center at P', by setting

$$\Phi(P, P') = \frac{1}{\delta^6} \int_{z-\delta/2}^{z+\delta/2} d\zeta_2 \int_{y-\delta/2}^{y+\delta/2} d\eta_2 \int_{x-\delta/2}^{x+\delta/2} d\xi_2 \int_{\zeta_2-\delta/2}^{\zeta_2+\delta/2} d\zeta_1 \int_{\eta_2-\delta/2}^{\eta_2+\delta/2} d\eta_1 \int_{\xi_2-\delta/2}^{\xi_2+\delta/2} \Phi_1(\Pi_1, P') d\xi_1.$$

Let P' be any fixed point of T. Then $\Phi(P, P')$ is a continuous function of P throughout the interior of the sphere S_1, and $|\Phi(P, P')| \leqq M$; moreover $\Phi(P, P')$ coincides with $\mathfrak{F}(P)$ throughout $\mathfrak{S}(P', \delta)$.

We shall now prove that the function $\mathfrak{F}(P)$ is pseudo-uniformly continuous in T. In order to do this, we shall show that, given any positive ϵ', there exists a positive δ', independent of the arbitrarily chosen point $P':(x', y', z')$ of T, such that

$$|\mathfrak{F}(P) - \mathfrak{F}(P')| < \epsilon',$$

for each point $P:(x, y, z)$ of $\mathfrak{S}(P', \delta')$. As a first restriction on δ' we require that $\delta' < 3^{-1/2}\delta$. Then every point of $\mathfrak{S}(P', \delta')$ is a point of the cube

$$\mathfrak{R}_3': \quad x' - \delta' \leqq x \leqq x' + \delta', \quad y' - \delta' \leqq y \leqq y' + \delta', \quad z' - \delta' \leqq z \leqq z' + \delta',$$

and every point of this cube is a point of the sphere S_1 with center at P' and radius δ. Now if $P(x, y, z)$ is a point of $\mathfrak{R}_3'$ then the points (x', y, z), (x', y', z) and (x', y', z') are also points of this cube, and so we have*

$$|\Phi(x, y, z; x', y', z') - \Phi(x', y, z; x', y', z')| \leqq \frac{2M|x' - x|}{\delta} \leqq \frac{2M\delta'}{\delta}.$$

Similarly,

$$|\Phi(x', y, z; x', y', z') - \Phi(x', y', z; x', y', z)| \leqq \frac{2M\delta'}{\delta},$$

$$|\Phi(x', y', z; x', y', z') - \Phi(x', y', z'; x', y', z')| \leqq \frac{2M\delta'}{\delta}.$$

Adding these inequalities we obtain

$$|\Phi(P, P') - \Phi(P', P')| \leqq \frac{6M\delta'}{\delta},$$

for an arbitrary point P of $\mathfrak{R}_3'$. We infer that $|\mathfrak{F}(P) - \mathfrak{F}(P')| \leqq 6M\delta'/\delta$, throughout $\mathfrak{S}(P', \delta')$. If we choose a positive δ' less than each of the quantities $3^{-1/2}\delta$ and $M^{-1}\delta\epsilon'/6$, then $|\mathfrak{F}(P) - \mathfrak{F}(P')| < \epsilon'$ for every point P of $\mathfrak{S}(P', \delta')$. Since the restrictions imposed on δ' are independent of the position of P' in T this means that $\mathfrak{F}(P)$ is pseudo-uniformly continuous in T.

It remains to establish item (2) of the conclusion of the theorem. That $\mathfrak{F}(P)$ has continuous partial derivatives of the first and second order in the neighborhood of an arbitrarily chosen point P' of T becomes obvious when we replace $\mathfrak{F}(P)$ by $\Phi(P, P')$ in $\mathfrak{S}(P', \delta)$. We have, in particular, the following formula, valid in $\mathfrak{S}(P', \delta)$:

$$\frac{\partial^2}{\partial x^2}\mathfrak{F}(P) = \frac{1}{\delta^6}\int_{z-\delta/2}^{z+\delta/2} d\zeta_2 \int_{y-\delta/2}^{y+\delta/2} d\eta_2 \int_{\zeta_2-\delta/2}^{\zeta_2+\delta/2} d\zeta_1 \int_{\eta_2-\delta/2}^{\eta_2+\delta/2} \{\Phi_1(x + \delta, \eta_1, \zeta_1; x', y', z')$$
$$- 2\Phi_1(x, \eta_1, \zeta_1; x', y', z') + \Phi_1(x - \delta, \eta_1, \zeta_1; x', y', z')\} d\eta_1.$$

* It will be convenient in some cases to replace the symbol $\Phi(P, P')$ by $\Phi(x, y, z; x', y', z')$. Similarly, $\Phi_1(P, P')$ may be replaced by $\Phi_1(x, y, z; x', y', z')$.

From this we have

$$\left| \frac{\partial^2}{\partial x^2} \mathfrak{F}(P) \right|_{P=P'} \leqq \frac{4M}{\delta^2}.$$

Similarly,

$$\left| \frac{\partial^2}{\partial y^2} \mathfrak{F}(P) \right|_{P=P'} \leqq \frac{4M}{\delta^2}, \text{ and } \left| \frac{\partial^2}{\partial z^2} \mathfrak{F}(P) \right|_{P=P'} \leqq \frac{4M}{\delta^2}.$$

Since P' is an arbitrarily chosen point of T, we infer that

$$\left| \nabla^2 \mathfrak{F}(P) \right| \leqq \frac{12M}{\delta^2},$$

throughout T. This completes the proof of the theorem.

3. Ordinary domains

We now consider briefly a special type of finite domain.

Definition 23. *A finite domain T is said to be ordinary if every function of the variable point P of T which approaches the value zero pseudo-continuously at each element γ of t approaches the boundary values $f(\gamma)=0$ on t with uniform pseudo-continuity.*

Theorem 12. *Let $f(\gamma)$ be any bounded and uniformly pseudo-continuous function of the variable element γ of the boundary of an ordinary domain T. Then any function $F(P)$ which approaches the boundary values $f(\gamma)$ with pseudo-continuity approaches these boundary values with uniform pseudo-continuity.*

From Theorem 9 we know that there exists in T a function $F_1(P)$ which approaches the boundary values $f(\gamma)$ with uniform pseudo-continuity. Now the difference between the functions $F(P)$ and $F_1(P)$ approaches the value zero pseudo-continuously at each element of t; therefore (since T is ordinary) this difference vanishes on t with uniform pseudo-continuity. It follows that $F(P)$ approaches the boundary values $f(\gamma)$ with uniform pseudo-continuity.

Theorem 13. *There exist finite domains T, having only accessible boundary points, which are not ordinary.*

We shall prove this theorem by exhibiting an example. Consider the domain T which is constructed by deleting from the open sphere

$$x^2 + y^2 + z^2 < 1$$

all points which lie on the following surfaces:

$$(1)\quad x^2+y^2=\tfrac{1}{4},\qquad 0\leqq z\leqq \tfrac{1}{2};$$

$$(2)\quad z=0,\qquad x^2+y^2\leqq \tfrac{1}{4};$$

$$(3)\quad z=\frac{1}{i+2},\quad \frac{1}{(i+2)^2}\leqq x^2+y^2\leqq\frac{1}{4},\quad \text{where } i=1,2,3,\cdots.$$

Consider now the function $F(P)$ defined at each point $P:(x, y, z)$ of T requiring that when

$$x^2+y^2<\tfrac{1}{4},\ \text{and}\ 0<z\leqq\tfrac{1}{2},$$

we have

$$F(P)=(x^2+y^2)(\tfrac{1}{4}-x^2-y^2)\sin\frac{\pi}{z},$$

while at all other points of T we have $F(P)=0$. The function $F(P)$ approaches the value zero at each boundary element. But $F(P)$ does not approach the boundary value zero with uniform pseudo-continuity, for if it did we could infer from Theorem 10 that $F(P)$ approached zero continuously on the boundary, which would be inconsistent with the behavior of $F(P)$ at the points

$$P_j:\qquad \left(\frac{1}{3},\frac{1}{3},\frac{2}{2j+5}\right)\qquad (j=1,2,3,\cdots).$$

THEOREM 14. *If there exists a function $\alpha(p)$ having a positive value at each point p of t and such that every sphere with center at any point p of t and radius less than $\alpha(p)$ yields, as the subset of its interior points which belong to T, a finite sum of finite domains, then T is an ordinary domain.*

Suppose that there exists a finite domain, T, which satisfies the hypothesis of the theorem but is not ordinary. Then there exists in T a function $F(P)$ which approaches the boundary values $f(\gamma)=0$ on t with non-uniform pseudo-continuity. There exists a positive constant ϵ such that the set E of all points of T at which $|F(P)|\geqq\epsilon$ has at least one limit point on t. Let $\bar{p}$ be any such point of t. Let S_1 be a sphere with center at $\bar{p}$ and radius $\alpha(\bar{p})/2$. The set of points common to T and the interior of S_1 consists of a finite sum of finite domains, of which at least one has $\bar{p}$ as a limit point. Among those which have $\bar{p}$ as a limit point, at least one contains infinitely many points of E. We choose arbitrarily a Cartesian coordinate system; it is then possible to specify uniquely* a particular domain, G_1, having these properties.

* Consider the point set comprising all points belonging to domains eligible for choice as G_1. A variable point of this set attains its maximum distance from the boundary of this set at one or more fixed points of which a particular one, Q, may be specified by the method explained in connection with the proof of Theorem 6. We select as G_1 that eligible domain which contains Q.

We now replace S_1 by a concentric sphere S_2 of radius $\alpha(\bar{p})/2^2$. The set of points common to G_1 and the interior of S_2 consists of a finite sum of finite domains of which at least one has $\bar{p}$ as a limit point, and contains infinitely many points of E. As before we may select uniquely (relative to the previously chosen coordinate system) a particular domain G_2 having these properties.

Proceeding in this way we obtain an infinite sequence of finite domains: $G_1, G_2, G_3, \cdots$. There exists a positive integer k such that for all integers i greater than k, G_i is a partial domain of T. Denoting by G_i' the closed partial region of T' corresponding to the partial domain G_i we readily verify that the sequence of closed partial regions of T': $G_{k+1}', G_{k+2}', G_{k+3}', \cdots$ is monotone, and so determines a component of T'. Since the diameter of G_i' is at most $\alpha(\bar{p})/2^{i-1}$ this component is an element $\bar{\gamma}$ of t.

Every pseudo-spherical domain corresponding to $\bar{\gamma}$ contains points of E, at which $|F(P)| \geqq \epsilon$. Hence $F(P)$ does not approach the value zero at $\bar{\gamma}$. This is inconsistent with the hypothesis of the theorem, and so establishes the falsity of the assumption that there exists a finite domain T, satisfying the condition given in the theorem, which is not ordinary.

III. An extension of the Dirichlet problem

1. The pseudo-classical Dirichlet problem

We first consider the following proposition:

Theorem 15. *Given a bounded and uniformly pseudo-continuous function $f(\gamma)$ of a variable element γ of the boundary of a finite domain T, there exists at most one function $U(P)$ of the variable point P of T which is single-valued and harmonic throughout T and approaches the boundary values $f(\gamma)$ with uniform pseudo-continuity.*

The difference between any two functions which satisfy the conditions imposed on $U(P)$ is a harmonic function in T, and approaches zero continuously on t. Such a function is identically zero in T, and so the two given functions are identical.

Definition 24. *Let $f(\gamma)$ be a bounded and uniformly pseudo-continuous function of the variable element γ of the boundary of a finite domain T. If there exists a function $U(P)$ of the variable point P of T which is harmonic in T and approaches the boundary values $f(\gamma)$ with uniform pseudo-continuity, then this function $U(P)$ is called the solution of the Dirichlet problem, in the pseudo-classical sense, for the domain T and the boundary values $f(\gamma)$.*

We note that (by virtue of Theorem 10) a solution of the Dirichlet prob-

lem in the classical sense for continuous boundary values $\phi(p)$ on t may be regarded as a solution in the pseudo-classical sense for boundary values defined by the corresponding function $f(\gamma)$ of the variable element γ of t.

DEFINITION 25. *If a finite domain T is such that for every bounded and uniformly pseudo-continuous function $f(\gamma)$ of a variable element γ of t there exists a solution of the Dirichlet problem in the pseudo-classical sense for T and $f(\gamma)$, then T is said to be pseudo-normal.*

A pseudo-normal finite domain is necessarily normal, i.e., one for which the Dirichlet problem is possible in the classical sense.

THEOREM 16. *Let $U(P)$ be a function which is harmonic in T and approaches, with uniform pseudo-continuity, boundary values $f(\gamma)$ such that*

$$c_1 \leqq f(\gamma) \leqq c_2,$$

on t, where c_1 and c_2 are constants. Then

$$c_1 \leqq U(P) \leqq c_2,$$

throughout T; moreover if $U(P)$ is not a constant we have

$$c_1 < U(P) < c_2,$$

throughout T.

Suppose that there exists a point Q of T such that $U(Q) > c_2$. Then there exists a quantity δ, independent of the arbitrarily chosen element γ_1 of t, such that

$$| U(P) - f(\gamma_1) | < \frac{U(Q) - c_2}{2},$$

at each point P of $\mathfrak{S}(\gamma_1, \delta)$. Therefore at each point P' of T whose distance from t is less than δ,

$$U(P') < \frac{U(Q) + c_2}{2} < U(Q).$$

Consider the set E of points of T at which

$$U(P) > \frac{U(Q) + c_2}{2}.$$

Now Q is an interior point of E, and the set $\mathfrak{T}$ containing each point of E which can be joined to Q by a continuous curve lying in E is a finite domain. Each boundary point of $\mathfrak{T}$ is at a distance at least δ from t. Hence $U(P)$ is continuous on the boundary of $\mathfrak{T}$; moreover it has there the constant value $[U(Q)+c_2]/2$. But in the interior of $\mathfrak{T}$ the function $U(P)$ is harmonic and

$$U(P) > \frac{U(Q) + c_2}{2}.$$

This inconsistency establishes the falsity of the assumption that there exists a point Q of T such that $U(Q) > c_2$. Similarly it may be shown that there exists no point Q' of T such that $U(Q') < c_1$.

Suppose that T contains a point Q_1 such that $U(Q_1) = c_1$. Let P_1 be any other point of T, and let $\mathfrak{T}_1$ be a domain which contains P_1 and Q_1 and lies (together with its boundary) in the interior of T. From a well known property of harmonic functions we see that $U(P) = c_1$ throughout $\mathfrak{T}_1$. Hence $U(P_1) = c_1$, and since P_1 is an arbitrary point of T, we infer that $U(P)$ is constant in T. Similarly if T contains a point Q_2 such that $U(Q_2) = c_2$, then $U(P)$ is constant in T.

2. Subharmonic and superharmonic functions

In the discussions of some of the theorems given below, we shall have frequent occasion to use properties of functions which are continuous and subharmonic or superharmonic in T. We use these terms in a sense similar to that adopted by Kellogg*:

Definition 26. *Let $W(P)$ be a function which is continuous in T. Let $\mathfrak{T}$ be a finite domain contained (together with its boundary) in T, and let $u(P)$ be a function which is harmonic in $\mathfrak{T}$ and continuous on the boundary of $\mathfrak{T}$. If for every $\mathfrak{T}$ and every $u(P)$ such that $W(P) \leqq u(P)$ on the boundary of $\mathfrak{T}$ we have $W(P) \leqq u(P)$ in the interior of $\mathfrak{T}$, then $W(P)$ is said to be subharmonic in T; if, on the other hand, for every $\mathfrak{T}$ and every $u(P)$ such that $W(P) \geqq u(P)$ on the boundary of $\mathfrak{T}$ we have $W(P) \geqq u(P)$ in the interior of $\mathfrak{T}$, then $W(P)$ is said to be superharmonic in T.*

Theorem 17. *Let $W(P)$ be a continuous superharmonic function of the variable point P of T. If to every positive constant ϵ there corresponds a positive constant δ, independent of P, such that $-\epsilon < W(P)$ for each point P of T which is at a distance less than δ from t, then $0 \leqq W(P)$ throughout T.*

Let $\overline{P}$ be an arbitrarily chosen fixed point of T. There exists a domain $\mathfrak{T}$ containing $\overline{P}$, which lies in T and has a boundary each point of which is at a distance from t which is positive and less than the δ corresponding to a preassigned positive ϵ. Now the function $W_1(P) = W(P) + \epsilon$ is superharmonic in $\mathfrak{T}$ and has a positive minimum on the boundary of $\mathfrak{T}$, and therefore a positive lower bound in the interior of $\mathfrak{T}$. It follows that $-\epsilon < W(\overline{P})$. Since ϵ is

* O. D. Kellogg, loc. cit., p. 315 ff. Kellogg formulates the definition for a region R, which may be interpreted as a domain. We shall need some of the properties given in Kellogg's treatment of these functions.

any positive number, and $\bar{P}$ any point of T, we infer that $0 \leqq W(P)$ throughout T.

COROLLARY. *Let $W(P)$ be a bounded, superharmonic and pseudo-uniformly continuous function in T, determining boundary values* $f(\gamma)$ on t. If $0 \leqq f(\gamma)$ on t, then $0 \leqq W(P)$ throughout T.*

THEOREM 18. *Let $W(P)$ be a function which is bounded and pseudo-uniformly continuous in T. Let $\mathfrak{T}$ be a domain (containing† only points of T) such that there exists a function $u(P)$ which is harmonic in $\mathfrak{T}$ and approaches with uniform pseudo-continuity the same boundary values on the boundary of $\mathfrak{T}$ as does $W(P)$. Let $\mathfrak{W}(P)$ be the function which coincides with $u(P)$ in $\mathfrak{T}$ and with $W(P)$ in $T-\mathfrak{T}$. Then $\mathfrak{W}(P)$ is bounded and pseudo-uniformly continuous in T and approaches with pseudo-uniform continuity the same boundary values on t as does $W(P)$. Moreover, if $W(P)$ is superharmonic in T, then $\mathfrak{W}(P)$ is superharmonic in T, and $\mathfrak{W}(P) \leqq W(P)$ in T.*

From Theorem 10 we see that the function $w(P) = W(P) - \mathfrak{W}(P)$ approaches the boundary value zero continuously on the boundary of $\mathfrak{T}$ and vanishes identically in $T-\mathfrak{T}$. Hence $w(P)$ is uniformly continuous on $T+t$ (when defined as zero on t) and therefore is bounded and pseudo-uniformly continuous in T. Hence $\mathfrak{W}(P) = W(P) - w(P)$ is also bounded and pseudo-uniformly continuous in T, and approaches with pseudo-uniform continuity the same boundary values on t as does $W(P)$.

If $W(P)$ is superharmonic in T, then it follows from the Corollary of Theorem 17 that $\mathfrak{W}(P) \leqq W(P)$ in $\mathfrak{T}$, and so in T. That $\mathfrak{W}(P)$ is superharmonic in T can be proved by the same reasoning as that given by Kellogg‡ in the demonstration of an analogous theorem: the value of $\mathfrak{W}(P)$ at any point Q of T is readily shown to be greater than or equal to the average value of $\mathfrak{W}(P)$ on the surface of any sufficiently small sphere with center at Q.

THEOREM 19. *Let $F(P)$ be a function which is bounded and pseudo-uniformly continuous in T, and let ϵ be any positive constant. Then there exists a function $\mathfrak{F}(P)$ which has the following properties:*

(1) *The function $\mathfrak{F}(P)$ may be expressed as the difference of two functions, each of which is bounded, pseudo-uniformly continuous, and subharmonic in T.*

(2) *The inequality $|\mathfrak{F}(P) - F(P)| < \epsilon$ holds throughout T.*

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* That $W(P)$ approaches (with uniform pseudo-continuity) bounded and uniformly pseudo-continuous boundary values on t, is evident from Theorem 8.

† It should be noted that in this theorem there is no restriction that the boundary of $\mathfrak{T}$ should be contained in T.

‡ O. D. Kellogg, loc. cit., p. 317. (Proof of property 4 of superharmonic functions.)

We introduce a Cartesian coordinate system; then the function $\mathfrak{F}(P)$, the existence of which has been established in Theorem 11, has the required properties. This function satisfies the relation

$$|\mathfrak{F}(P) - F(P)| < \epsilon$$

throughout T. Furthermore there exists a positive constant K such that $|\nabla^2\mathfrak{F}(P)| < K$, throughout T.

Let Q be an arbitrarily chosen fixed point exterior to T, and denote by r_0 the maximum distance from Q to a point of t, and by r the distance from Q to the variable point P of T. We set* $\mathfrak{F}(P) = \mathfrak{F}'(P) - \mathfrak{F}''(P)$ where

$$\mathfrak{F}'(P) = \mathfrak{F}(P) + \frac{Kr_0}{2} r, \quad \text{and} \quad \mathfrak{F}''(P) = \frac{Kr_0}{2} r.$$

The functions $\mathfrak{F}'(P)$ and $\mathfrak{F}''(P)$ are pseudo-uniformly continuous in T, and have continuous partial derivatives of the first and second order, throughout T. Moreover at any point P of T

$$\nabla^2\mathfrak{F}'(P) = \nabla^2\mathfrak{F}(P) + \frac{Kr_0}{r} > 0, \quad \text{and} \quad \nabla^2\mathfrak{F}''(P) = \frac{Kr_0}{r} > 0.$$

This shows that $\mathfrak{F}'(P)$ and $\mathfrak{F}''(P)$ are subharmonic in T.

DEFINITION 27. *By $\Psi(\gamma, P)$, the pseudo-distance from an element γ of t to a point P of T, we mean the least upper bound of the radii of all pseudo-spherical domains corresponding to γ which do not contain P.*

THEOREM 20. *If γ_1 is a fixed element of t and P a variable point of T, then $\Psi(\gamma_1, P)$ has the following properties:*

(1) *$\Psi(\gamma_1, P)$ is a bounded and pseudo-uniformly continuous function of P throughout T.*

(2) *In the part of T outside any given pseudo-spherical domain (not identical with T) corresponding to γ_1, $\Psi(\gamma_1, P)$ has a positive lower bound.*

(3) *$\Psi(\gamma_1, P)$ approaches (with uniform pseudo-continuity) bounded and uniformly pseudo-continuous boundary values $\psi(\gamma)$ such that $\psi(\gamma) > 0$ on t, except at $\gamma = \gamma_1$, and $\psi(\gamma_1) = 0$.*

(4) *$\Psi(\gamma_1, P)$ is subharmonic in T.*

We denote by r the distance from the point p_1 contained in γ_1 to the variable point P of T. Let ρ be any constant such that $\mathfrak{S}(\gamma_1, \rho)$ is not identical with T. Then $\rho < \bar{r}$, where $\bar{r}$ is the least upper bound of r in T. If P is a point of $\mathfrak{S}(\gamma_1, \rho)$, and therefore a point of $\mathfrak{S}(\gamma_1, \rho')$ for some ρ' less than ρ, then

* In the corresponding problem in two dimensions we set $\mathfrak{F}'(P) = \mathfrak{F}(P) + Kr_0r$ and $\mathfrak{F}''(P) = Kr_0r$.

$$\Psi(\gamma_1, P) < \rho.$$

If P is a point of $T - \mathfrak{S}(\gamma_1, \rho)$, then

$$\rho \leqq \Psi(\gamma_1, P) < \bar{r}.$$

These relations show that $\Psi(\gamma_1, P)$ has property (2), and also that $\Psi(\gamma_1, P)$ is bounded, and approaches pseudo-continuously the value zero at the boundary element γ_1.

The function $\Psi(\gamma_1, P)$ is pseudo-uniformly continuous in T. To prove this we shall show that, given any positive ϵ, the relation

$$|\Psi(\gamma_1, P') - \Psi(\gamma_1, P'')| < \epsilon$$

holds for every pair of points P' and P'' that can be joined by a continuous curve in T, the maximum distance between two points of which is less than $\epsilon/3$. Given such a pair of points, P' belongs to

$$\mathfrak{S}\left(\gamma_1, \Psi(\gamma_1, P') + \frac{\epsilon}{3}\right),$$

and consequently P'' belongs to

$$\mathfrak{S}\left(\gamma_1, \Psi(\gamma_1, P') + \frac{2\epsilon}{3}\right).$$

We infer that

$$\Psi(\gamma_1, P'') < \Psi(\gamma_1, P') + \epsilon.$$

Similarly,

$$\Psi(\gamma_1, P') < \Psi(\gamma_1, P'') + \epsilon,$$

whence,

$$|\Psi(\gamma_1, P') - \Psi(\gamma_1, P'')| < \epsilon.$$

Since $\Psi(\gamma_1, P)$ is bounded and pseudo-uniformly continuous in T, it approaches, with uniform pseudo-continuity, bounded and pseudo-uniformly continuous boundary values $\psi(\gamma)$ on t. That $\psi(\gamma) > 0$ on t, except at $\gamma = \gamma_1$ where $\psi(\gamma) = 0$, is immediately obvious from properties of $\Psi(\gamma_1, P)$ already established.

In order to show that $\Psi(\gamma_1, P)$ is subharmonic in T, we now define a function $\Omega(\gamma_1, \rho, P)$ for every positive value of ρ and every point P of T. If P is a point of T for which $r < \rho$, but which does not belong to $\mathfrak{S}(\gamma_1, \rho)$, then $\Omega(\gamma_1, \rho, P) = \rho$; for every other point P of T we set $\Omega(\gamma_1, \rho, P) = r$. If ρ is held fast then $\Omega(\gamma_1, \rho, P)$ is a continuous function of P in T, subharmonic in a

sufficiently small neighborhood of each point of T, and therefore subharmonic* in T. Moreover at each point P of T the least upper bound of $\Omega(\gamma_1, \rho, P)$ for all positive ρ is $\Psi(\gamma_1, P)$.

Let $\mathfrak{T}$ be any finite domain, contained (together with its boundary) in T. Let $u(P)$ be any function harmonic in $\mathfrak{T}$ such that

$$u(P) \geqq \Psi(\gamma_1, P)$$

on the boundary of $\mathfrak{T}$. Then

$$u(P) \geqq \Omega(\gamma_1, \rho, P),$$

for any $\rho > 0$. Since $\Omega(\gamma_1, \rho, P)$ is subharmonic in T the second relation holds in the interior as well as on the boundary of $\mathfrak{T}$. Inasmuch as $\Psi(\gamma_1, P)$ is the least upper bound of $\Omega(\gamma_1, \rho, P)$ for all $\rho > 0$, we conclude that

$$u(P) \geqq \Psi(\gamma_1, P),$$

throughout the interior of $\mathfrak{T}$. Hence $\Psi(\gamma_1, P)$ is a subharmonic function of P throughout T.

3. The sequence solution of the extended Dirichlet problem

Definition 28. *An infinite sequence of finite domains,*

$$\{T_i\}: \qquad T_1, T_2, T_3, \cdots,$$

is said to be a pseudo-normal sequence† *of domains in T if the following conditions are satisfied:*

(1) *Each T_i is contained in T.*

(2) *Each point of T is the center of a sphere which is contained in infinitely many of the domains T_i.*

(3) *Given any domain T_i and any function $F(P)$, bounded and pseudo-uniformly continuous in T, there exists a function which is harmonic in T_i and approaches (with uniform pseudo-continuity) the same boundary values on the boundary of T_i as does $F(P)$.*

Condition (3) is satisfied (for a $T_i < T$) if T_i is pseudo-normal, or if T_i is normal and has a boundary contained in T. Given any finite domain T there exist pseudo-normal sequences of domains in T, for any nested sequence of normal domains approximating to T satisfies all the conditions given above.

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* We use here theorems relating to subharmonic functions analogous to those given with respect to superharmonic functions by Kellogg (loc. cit.) in the exercise on p. 317, and in property 2 on p. 316.

† Such a sequence corresponds to the sequence of closed regions $R_1, R_2, R_3, \cdots$, used by Kellogg (loc. cit., p. 322 ff.) in the study of the sequence solution of the generalized Dirichlet problem.

DEFINITION 29. *By the sequence of functions* $\{U_i(P)\}: U_0(P), U_1(P), U_2(P), \cdots$, *associated with* T, $\{T_i\}$ *(a given pseudo-normal sequence of domains in* T*), and* $F(P)$ *(a given function, bounded and pseudo-uniformly continuous in* T*), we mean the infinite sequence of functions uniquely determined* by the following conditions*:

(1) $U_0(P)$ *is identical in* T *with* $F(P)$.

(2) *For every positive integral* i, $U_i(P)$ *is identical in* $T-T_i$ *with* $U_{i-1}(P)$, *and in* T_i *with that function which is harmonic in* T_i *and approaches (with uniform pseudo-continuity) the same values on the boundary of* T_i *as does* $U_{i-1}(P)$.

THEOREM 21. *Given a bounded and uniformly pseudo-continuous function* $f(\gamma)$, *defined on the boundary of a finite domain* T, *let* $\{T_i\}$ *be an arbitrarily chosen pseudo-normal sequence of domains in* T, *and let* $F(P)$ *be any function which is bounded and continuous in* T *and approaches the boundary values* $f(\gamma)$ *with uniform pseudo-continuity. Then the sequence of functions* $\{U_i(P)\}$ *associated with* T, $\{T_i\}$ *and* $F(P)$ *converges in* T *(uniformly in any closed region contained in* T*) to a function* $U(P)$ *which is harmonic in* T *and depends only on* T *and* $f(\gamma)$.

Let ϵ be an arbitrarily chosen positive constant. By Theorem 19, there exist functions $\mathfrak{F}(P)$, $\mathfrak{F}'(P)$ and $\mathfrak{F}''(P)$ such that

$$\mathfrak{F}(P) = \mathfrak{F}'(P) - \mathfrak{F}''(P),$$

where $\mathfrak{F}'(P)$ and $\mathfrak{F}''(P)$ are each bounded, pseudo-uniformly continuous and subharmonic in T, and

$$|\mathfrak{F}(P) - F(P)| < \frac{\epsilon}{3}.$$

Let $\{\mathfrak{U}_i'(P)\}$ be the sequence of functions associated with T, $\{T_i\}$, and $\mathfrak{F}'(P)$. Now Theorem 18 implies that throughout T each $\mathfrak{U}_i'(P)$ is subharmonic, and $\{\mathfrak{U}_i'(P)\}$ is a monotone increasing sequence of functions. Using Theorem 16 we infer that for each non-negative integral i, the least upper bound of $\mathfrak{U}_{i+1}'(P)$ is no greater than that of $\mathfrak{U}_i'(P)$, and so we have $\mathfrak{U}_i'(P) \leqq \mathfrak{M}'$ where $\mathfrak{M}'$ is the least upper bound of $\mathfrak{U}_0'(P) = \mathfrak{F}'(P)$. Hence the sequence $\{\mathfrak{U}_i'(P)\}$ converges, throughout T.

Any given point P of T is the center of a sphere S which lies in infinitely many of the domains T_i. Hence the sequence $\{\mathfrak{U}_i'(P)\}$ contains a bounded and monotone increasing subsequence of functions which are harmonic within S. The convergence of this subsequence, and so also the convergence of the

* Using the first part of Theorem 18, it is readily seen by mathematical induction that each $U_{i-1}(P)$ determines uniquely a bounded and pseudo-uniformly continuous $U_i(P)$.

monotone sequence $\{U_i'(P)\}$, is uniform* within S. By the Heine-Borel theorem we infer that $\{U_i'(P)\}$ converges uniformly in any closed region contained in T. Moreover the limiting function, $U'(P)$, is harmonic in T.

Let $\{\mathfrak{T}_i\}$ be another pseudo-normal sequence of domains in T, and let $\{\mathfrak{U}_i'(P)\}$ be the sequence of functions associated with T, $\{\mathfrak{T}_i\}$, and $\mathfrak{F}'(P)$. For any non-negative integral i we may write

$$U'(P) - \mathfrak{U}_{i+1}'(P) = [U'(P) - \mathfrak{U}_i'(P)] + [\mathfrak{U}_i'(P) - \mathfrak{U}_{i+1}'(P)].$$

Now $[\mathfrak{U}_i'(P) - \mathfrak{U}_{i+1}'(P)]$ approaches the value zero continuously on t. Hence if

$$0 \leqq U'(P) - \mathfrak{U}_i'(P)$$

throughout T, then to any preassigned positive ϵ there corresponds a positive δ independent of P such that

$$-\epsilon < U'(P) - \mathfrak{U}_{i+1}'(P),$$

at each point P of T which lies at a distance less than δ from t. Using Theorem 17, we infer that if

$$0 \leqq U'(P) - \mathfrak{U}_i'(P),$$

in T, then

$$0 \leqq U'(P) - \mathfrak{U}_{i+1}'(P),$$

in T. Since

$$0 \leqq U'(P) - \mathfrak{U}_0'(P),$$

we see by mathematical induction that throughout T,

$$0 \leqq U'(P) - \mathfrak{U}_i'(P) \qquad (i = 0, 1, 2, \cdots).$$

Denoting by $\mathfrak{U}'(P)$ the limit of the sequence $\{\mathfrak{U}_i'(P)\}$, we have $\mathfrak{U}'(P) \leqq U'(P)$. Similarly, $U'(P) \leqq \mathfrak{U}'(P)$. Hence $\mathfrak{U}'(P) = U'(P)$, in T, and we conclude that the limit of the sequence $\{U_i'(P)\}$ is independent of $\{T_i\}$.

The sequence $\{U_i''(P)\}$ associated with T, $\{T_i\}$, and $\mathfrak{F}''(P)$ has similar properties. Furthermore, if we represent by $\{U_i(P)\}$ the sequence of functions associated with T, $\{T_i\}$, and $\mathfrak{F}(P)$, then

$$U_i(P) = U_i'(P) - U_i''(P) \qquad (i = 0, 1, 2, \cdots).$$

Hence the sequence $\{U_i(P)\}$ also converges (uniformly in any closed region contained in T) to a function $U(P)$ which is harmonic in T, and independent of $\{T_i\}$.

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* This follows from Harnack's second convergence theorem: A. Harnack, *Grundlagen der Theorie des Logarithmischen Potentials*, Leipzig, 1887, p. 67.

Let $\{U_i(P)\}$ be the sequence associated with T, $\{T_i\}$, and $F(P)$. Then the sequence associated with T, $\{T_i\}$, and $\mathfrak{F}(P)-F(P)$ is

$$\{\mathcal{U}_i(P) - U_i(P)\}:\ \mathcal{U}_0(P) - U_0(P),\ \mathcal{U}_1(P) - U_1(P),\ \mathcal{U}_2(P) - U_2(P),\ \cdots .$$

Hence,

$$|\mathcal{U}_i(P) - U_i(P)| < \epsilon/3 \qquad (i = 0,1,2,\cdots).$$

For a bound for the first term of a pseudo-normal sequence is a bound for every term.

Now given any finite closed region $\mathfrak{T}'$ contained in T, there exists a positive integer J, independent of the arbitrary point P of $\mathfrak{T}'$, such that for any point P in $\mathfrak{T}'$, and for any integral j and k such that $j>J$ and $k>J$, we have the following inequalities:

$$|\mathcal{U}_j(P) - \mathcal{U}_k(P)| < \epsilon/3,$$

$$|U_j(P) - \mathcal{U}_j(P)| < \epsilon/3 \text{ and } |\mathcal{U}_k(P) - U_k(P)| < \epsilon/3.$$

Hence, throughout $\mathfrak{T}'$,

$$|U_j(P) - U_k(P)| < \epsilon, \text{ if } j > J \text{ and } k > J.$$

Since the quantity ϵ was originally chosen arbitrarily, we infer that the sequence $\{U_i(P)\}$ converges in T (uniformly in $\mathfrak{T}'$) to a function $U(P)$. Furthermore, any given point of $\mathfrak{T}'$ is the center of a sphere within which the functions of a suitably chosen infinite subsequence of $\{U_i(P)\}$ form a uniformly convergent sequence of harmonic functions. Hence $U(P)$ is harmonic in T. Moreover, from the relations

$$|\mathcal{U}_i(P) - U_i(P)| < \epsilon/3 \qquad (i = 0, 1, 2, \cdots),$$

we see that

$$|\mathcal{U}(P) - U(P)| \leqq \epsilon/3,$$

whence we readily infer that $U(P)$ is independent of $\{T_i\}$.

We shall now show that $U(P)$ is unaltered if $F(P)$ is replaced by another function $F'(P)$ which satisfies the same hypotheses as those imposed on $F(P)$. Let $U'(P)$ be the limit of the sequence of functions $\{U_i'(P)\}$ associated with T, $\{T_i\}$, and $F'(P)$. Then

$$\{U_i'(P) - U_i(P)\}:\ U_0'(P) - U_0(P),\ U_1'(P) - U_1(P),\ U_2'(P) - U_2(P),\ \cdots$$

is the sequence of functions associated with T, $\{T_i\}$ and $F'(P)-F(P)$. Since the limit of this sequence is independent of $\{T_i\}$ we may assume, without

loss of generality, that the finite domains T_i are normal. Since the function $F'(P)-F(P)$ approaches the value zero continuously on t, we then have a situation in which the earlier theory of the sequence solution for continuous boundary values is applicable. The function $F''(P)=0$ approaches continuously the same boundary values on t as does $F'(P)-F(P)$, and so may be substituted for $F'(P)-F(P)$ without affecting the limit of the sequence of functions.* Hence $\{U_i'(P)-U_i(P)\}$ converges to zero, and so $U'(P)=U(P)$. This completes the proof of the theorem.

COROLLARY 1. *If there exists a solution of the Dirichlet problem, in the pseudo-classical sense, for the domain T and given bounded and uniformly pseudo-continuous boundary values $f(\gamma)$ on t, then this solution coincides with the function $U(P)$ obtained by the method described in Theorem* 21.

We may use this solution as the function $F(P)$. Then $U_i(P)=F(P)$, $i=0, 1, 2, \cdots$. Hence $U(P)=F(P)$.

COROLLARY 2. *The sequence solution of the generalized Dirichlet problem*†, *for the finite domain T of three-dimensional space and continuous boundary values $\phi(p)$, coincides in T with the function $U(P)$ (obtained by the method described in Theorem* 21) *corresponding to boundary values defined by the function $f(\gamma)$ which has at each element γ of t the same value as does $\phi(p)$ at the point p contained in γ.*

The proof is immediately obvious.

DEFINITION 30. *The function $U(P)$ the existence of which has been established in the proof of Theorem* 21 *is called the sequence solution of the extended Dirichlet problem for the finite domain T and the boundary values $f(\gamma)$.*

4. PSEUDO-REGULARITY‡

DEFINITION 31. *A function $V(\gamma_1, P)$ of the variable point P of T is said to be a pseudo-barrier for T at a fixed element γ_1 of t if the following conditions are satisfied:*

(1) *$V(\gamma_1, P)$ is a continuous superharmonic function of P in T.*

(2) *$V(\gamma_1, P)$ approaches the value zero pseudo-continuously at γ_1.*

(3) *In the part of T outside any given pseudo-spherical domain (not identical with T) corresponding to γ_1, the function $V(\gamma_1, P)$ has a positive lower bound.*

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* See O. D. Kellogg, loc. cit., Theorem II, p. 325.

† By the generalized Dirichlet problem is meant the form of the Dirichlet problem in which the sequence solution in the form developed by Wiener or Kellogg (loc. cit.) is valid.

‡ Cf. Kellogg, loc. cit., pp. 326–328.

Definition 32. *An element γ_1 of t is said to be pseudo-regular if there exists a pseudo-barrier for T at γ_1.*

Theorem 22. *Let $T^{(1)}$ and $T^{(2)}$ be two finite domains which have equivalent boundary elements $\gamma^{(1)}$ and $\gamma^{(2)}$ respectively. If $\gamma^{(1)}$ is a pseudo-regular boundary element of $T^{(1)}$, then $\gamma^{(2)}$ is a pseudo-regular boundary element of $T^{(2)}$.*

From Definition 12 we know that there exists an infinite sequence $G_1, G_2, G_3, \cdots$ of common partial domains of $T^{(1)}$ and $T^{(2)}$ such that the corresponding finite closed regions form a monotone sequence of closed partial regions determining the boundary elements $\gamma^{(1)}$ of $T^{(1)}$ and $\gamma^{(2)}$ of $T^{(2)}$. Now if a boundary point q of G_2 belongs to the boundary of $T^{(1)}$, then q belongs to the boundary of $T^{(2)\prime}$. Otherwise q would be an auxiliary boundary point of G_1' and of G_2' with respect to $T^{(2)}$. But this is impossible, inasmuch as the sequence $G_1', G_2', G_3', \cdots$ is monotone with respect to $T^{(2)\prime}$.

Let $V_1(\gamma^{(1)}, P)$ be a pseudo-barrier for $T^{(1)}$ at $\gamma^{(1)}$, and let β be the greatest lower bound of $V_1(\gamma^{(1)}, P)$ on the proper point set $T^{(1)}-G_2$. The function $V_1(\gamma^{(1)}, P)$ is defined at every point of $T^{(2)}$ which lies on the boundary of G_2 and at such a point $V_1(\gamma^{(1)}, P) \geqq \beta$.

We now define a function $V_2(\gamma^{(2)}, P)$ in $T^{(2)}$:

(1) If P is in G_2, then $V_2(\gamma^{(2)}, P)$ is the smaller of the numbers $V_1(\gamma^{(1)}, P)$ and β, or their common value if $V_1(\gamma^{(1)}, P)=\beta$.

(2) If P is in $T^{(2)}-G_2$, then $V_2(\gamma^{(2)}, P)=\beta$.

Now $V_2(\gamma^{(2)}, P)$ is a pseudo-barrier for $T^{(2)}$ at $\gamma^{(2)}$. It is clear that $V_2(\gamma^{(2)}, P)$ is continuous and superharmonic in $T^{(2)}$. Moreover, from Theorem 5 it follows that for sufficiently small positive $\bar{\delta}$, the pseudo-spherical domain $\mathfrak{S}(\gamma^{(2)}, \bar{\delta})$ of $T^{(2)}$ is contained in G_2. Consequently

$$V_2(\gamma^{(2)}, P) \leqq V_1(\gamma^{(1)}, P),$$

in $\mathfrak{S}(\gamma^{(2)}, \bar{\delta})$. From this relation, and from the fact that for sufficiently small positive δ the pseudo-spherical domains $\mathfrak{S}(\gamma^{(1)}, \delta)$ of $T^{(1)}$ and $\mathfrak{S}(\gamma^{(2)}, \delta)$ of $T^{(2)}$ are identical sets of points, we infer that $V_2(\gamma^{(2)}, P)$ approaches the value zero at the boundary element $\gamma^{(2)}$ of $T^{(2)}$, and that in the part of $T^{(2)}$ outside any given pseudo-spherical domain of $T^{(2)}$ (not identical with $T^{(2)}$) corresponding to $\gamma^{(2)}$, the function $V_2(\gamma^{(2)}, P)$ has a positive lower bound.

From the existence of the pseudo-barrier $V_2(\gamma^{(2)}, P)$ we conclude that $\gamma^{(2)}$ is a pseudo-regular boundary element of $T^{(2)}$.

Corollary. *The answer to the question whether or not a given boundary element γ_1 of T is pseudo-regular depends only on the character of that part of t which also belongs to the boundary of a pseudo-spherical domain, of arbitrarily small preassigned radius, corresponding to γ_1.*

For such a pseudo-spherical domain has a boundary element $\bar{\gamma}$ equivalent to the boundary element γ_1 of T.

THEOREM 23. *Let γ_1 be any fixed boundary element of a given finite domain T, and let $U(P)$ be the sequence solution of the extended Dirichlet problem for T and arbitrary bounded and uniformly pseudo-continuous boundary values $f(\gamma)$. Then a necessary and sufficient condition that every such function $U(P)$ approach with pseudo-continuity at γ_1 the corresponding boundary value $f(\gamma_1)$ is that the element γ_1 be pseudo-regular.*

We shall first establish the necessity of the given condition. For this purpose let $U(P)$ be the sequence solution of the extended Dirichlet problem for T and the boundary values $\psi(\gamma)$ determined on t by the function $\Psi(\gamma_1, P)$ representing the pseudo-distance from γ_1 to the point P of T. We shall prove that the given condition is necessary by showing that if, in this particular case, the function $U(P)$ approaches pseudo-continuously the value $\psi(\gamma_1)$ at the boundary element γ_1, then $U(P)$ is a pseudo-barrier for T at γ_1.

Let $\{T_i\}$ be any pseudo-normal sequence of domains in T. The function $U(P)$ may be obtained as the limit of the sequence of functions $\{U_i(P)\}$ associated with T, $\{T_i\}$, and $\Psi(\gamma_1, P)$. Now $\Psi(\gamma_1, P)$ is subharmonic in T, and so $\{U_i(P)\}$ is a monotone increasing sequence, and

$$U(P) \geqq \Psi(\gamma_1, P),$$

throughout T. That $U(P)$ is a pseudo-barrier for T at γ_1 is now obvious from the properties of pseudo-distance given in Theorem 20.

In order to establish the sufficiency of the given condition, we now let $U(P)$ be the sequence solution of the extended Dirichlet problem for T and arbitrarily given bounded and uniformly pseudo-continuous boundary values, $f(\gamma)$. We shall show that if there exists a barrier, $V(\gamma_1, P)$, for T at a given element γ_1 of t, then to any preassigned positive ϵ there corresponds a positive δ such that

$$|U(P) - f(\gamma_1)| < \epsilon,$$

throughout $\mathfrak{S}(\gamma_1, \delta)$. We may assume, without loss of generality, that $V(\gamma_1, P)$ is bounded* in T.

Let $F(P)$ be any function which is bounded and continuous in T, and approaches the boundary values $f(\gamma)$ with uniform pseudo-continuity.† There exist positive constants $\bar{\delta}$, B, and b such that

* The existence of a pseudo-barrier for T at γ_1 implies the existence of a bounded pseudo-barrier for T at γ_1, as may be readily seen from the discussion of Theorem 22.

† The continuity of $F(P)$ in T is necessarily pseudo-uniform, by Theorem 8.

$$|F(P) - f(\gamma_1)| < \frac{\epsilon}{2}, \qquad \text{in } \mathfrak{S}(\gamma_1, \bar{\delta}),$$

$$b < V(\gamma_1, P), \qquad \text{in } T - \mathfrak{S}(\gamma_1, \bar{\delta}),$$

and

$$|F(P) - f(\gamma_1)| < Bb, \qquad \text{in } T - \mathfrak{S}(\gamma_1, \bar{\delta}).$$

From these inequalities we infer that

$$|F(P) - f(\gamma_1)| < BV(\gamma_1, P), \qquad \text{in } T - \mathfrak{S}(\gamma_1, \bar{\delta}),$$

and

$$|F(P) - f(\gamma_1)| < BV(\gamma_1, P) + \frac{\epsilon}{2}, \qquad \text{in } T.$$

Let $\{T_i\}$ be any pseudo-normal sequence of domains in T, and let $\{U_i(P)\}$ be the sequence of functions associated with T, $\{T_i\}$, and $F(P)$. From the Corollary of Theorem 10, we know that given any positive ϵ' and any non-negative integer i, there exists a positive δ_i', independent of P, such that

$$-\epsilon' < U_i(P) - U_{i+1}(P),$$

at each point P of T_{i+1} at a distance less than δ_i' from the boundary of T_{i+1}. Suppose now that for each point P of T we have also, for some such i,

$$0 \leqq \left[f(\gamma_1) + BV(\gamma_1, P) + \frac{\epsilon}{2}\right] - U_i(P),$$

a relation which is already known to be valid when $i=0$. Then for each point P of T_{i+1} at a distance less than δ_i' from the boundary of T_{i+1} we have

$$-\epsilon' < \left[f(\gamma_1) + BV(\gamma_1, P) + \frac{\epsilon}{2}\right] - U_{i+1}(P).$$

From Theorem 17 we infer that

$$0 \leqq \left[f(\gamma_1) + BV(\gamma_1, P) + \frac{\epsilon}{2}\right] - U_{i+1}(P),$$

throughout T_{i+1}, and so throughout T. By mathematical induction we conclude that for every non-negative integer i,

$$0 \leqq \left[f(\gamma_1) + BV(\gamma_1, P) + \frac{\epsilon}{2}\right] - U_i(P),$$

throughout T.

By allowing i to become infinite in this relation we see that

$$U(P) \leqq f(\gamma_1) + BV(\gamma_1, P) + \frac{\epsilon}{2},$$

throughout T. By analogous reasoning we infer that

$$U(P) \geqq f(\gamma_1) - BV(\gamma_1, P) - \frac{\epsilon}{2},$$

throughout T. Now there exists a positive δ such that

$$0 < V(\gamma_1, P) < \frac{\epsilon}{2B},$$

throughout $\mathfrak{S}(\gamma_1, \delta)$. Hence we have

$$| U(P) - f(\gamma_1) | < \epsilon,$$

throughout $\mathfrak{S}(\gamma_1, \delta)$. This relation establishes the sufficiency of the given condition.

COROLLARY. *If T is an ordinary finite domain and if each boundary element of T is pseudo-regular, then T is pseudo-normal.*

The validity of this proposition is readily established by the use of Theorems 12 and 23.

THEOREM 24. *If the point p contained in a given boundary element γ of T is a regular boundary point of some partial domain G of T corresponding to a closed partial region of T' which contains γ, then γ is a pseudo-regular boundary element of T.*

We may choose a positive number ρ_1 such that $\mathfrak{S}(\gamma, \rho_1)$ is a proper subset of G. If $\rho_1 > \rho_2 > \rho_3 > \cdots$, $\rho_i \to 0$, then $\mathfrak{S}'(\gamma, \rho_1)$, $\mathfrak{S}'(\gamma, \rho_2)$, $\mathfrak{S}'(\gamma, \rho_3)$, $\cdots$ determines γ. Let $\mathcal{V}(p, P)$ be a barrier for G at p. For each $i > 1$ let β_i be the greatest lower bound of $\mathcal{V}(p, P)$ in $G - G \cdot S_i$, where S_i is the interior of the sphere of radius ρ_i with center at p. We now define $V_i(\gamma, P)$, $i = 2, 3, 4, \cdots$, by requiring that

(1) if P is in $\mathfrak{S}(\gamma, \rho_i)$ then $V_i(\gamma, P)$ is the smaller of $\mathcal{V}(p, P)$ and β_i, or their common value if $\mathcal{V}(p, P) = \beta_i$;

(2) if P is in $T - \mathfrak{S}(\gamma, \rho_i)$ then $V_i(\gamma, P) = \beta_i$.

For fixed γ each of the functions $V_i(\gamma, P)$ is continuous and superharmonic in T, and $0 < V_i(\gamma, P) \leqq \beta_2$, $i = 2, 3, 4, \cdots$. Hence

$$V(\gamma, P) = \sum_{i=2}^{\infty} 2^{-i} V_i(\gamma, P)$$

is continuous and superharmonic in T, inasmuch as the series is uniformly convergent in T. Now in $\mathfrak{S}(\gamma, \rho_2)$

$$V(\gamma, P) \leqq \mathcal{V}(p, P) \cdot \sum_{i=2}^{\infty} 2^{-i}.$$

Hence (considered as a function of P) $V(\gamma, P)$ approaches the value zero pseudo-continuously at γ. On the other hand, given any positive ρ, we can choose k so that $\mathfrak{S}(\gamma, \rho_k)$ is a subset of $\mathfrak{S}(\gamma, \rho)$. In $T - \mathfrak{S}(\gamma, \rho)$,

$$V(\gamma, P) \geqq \sum_{i=k}^{\infty} 2^{-i}\beta_i > 0.$$

Hence $V(\gamma, P)$ is a pseudo-barrier for T at γ, and the theorem is proved.

We now consider two lemmas*:

Lemma 1. *Let T be a finite normal domain. Then to each point p of t there corresponds a function $W(p, P)$ of a variable point P of T having the following properties:*

(1) *$W(p, P)$ is bounded and harmonic in T.*

(2) *Given any positive ϵ there exists a positive δ, independent of p, such that if $\overline{pP} < \delta$ then $|W(p, P)| < \epsilon$.*

(3) *$W(p, P) \geqq \overline{pP}$, throughout T.*

For fixed p we take $W(p, P)$ as the solution of the Dirichlet problem with boundary values $w(p, q) = \overline{pq}$ on t. Then $W(p, P) - \overline{pP}$ is superharmonic in T, and approaches the boundary value zero continuously on t. Hence items (1) and (3) of the conclusion of the lemma are valid.

Given any point p_1, of t, there exists a positive $\delta(p_1)$ such that $W(p_1, P) < \epsilon/2$ for each point P of T such that $\overline{p_1P} < \delta(p_1)$. Furthermore,

$$W(p, P) \leqq W(p_1, P) + \overline{pp_1}$$

for any point P of T and any pair of points p and p_1 on t. This follows from the fact that $W(p, P) - W(p_1, P)$ is harmonic in T and approaches continuously the boundary value $\overline{pq} - \overline{p_1q}$ at the arbitrary point q of t. Hence for $\overline{pp_1} < \epsilon/2$ and $\overline{p_1P} < \delta(p_1)$ we have

$$0 < W(p, P) < \epsilon.$$

This implies that each point p_1 of the bounded closed set t is the center of a sphere $S(p_1)$ such that if P and p are any points of T and t respectively

* I wish to thank Professor J. J. Gergen for many helpful suggestions in the revision of this paper, particularly in connection with these lemmas and their application to Theorem 25, which was originally stated and proved only for ordinary domains.

which lie in the interior of the same $S(p_1)$ then $|W(p, P)| < \epsilon$. By the Heine-Borel Theorem there exists a finite set of overlapping spheres of this type enclosing t. There exists a positive constant δ, independent of p, such that a sphere of radius δ with center at any point of t lies entirely in the interior of some sphere of this finite set. For this δ the assertion given in item (2) of the conclusion of the lemma is valid.

LEMMA 2. *Let T be a finite normal domain. Then to each element γ of t there corresponds a function $V(\gamma, P)$ of a variable point P of T having the following properties:*

(1) *$V(\gamma, P)$ is bounded, continuous and superharmonic in T.*

(2) *Given any positive ϵ, there exists a positive δ, independent of γ, such that $|V(\gamma, P)| < \epsilon$ for every P in $\mathfrak{S}(\gamma, \delta)$.*

(3) *Given any ρ such that $0 < \rho < \bar{\rho}$ (where $2\bar{\rho}$ is the diameter of T) there exists a positive $b(\rho)$, independent of γ and P, such that $b(\rho) \leqq V(\gamma, P)$ for every point P in the proper set $T - \mathfrak{S}(\gamma, \rho)$.*

Choose $\bar{\rho} > \rho_1 > \rho_2 > \rho_3 > \cdots$, so that ρ_i tends to zero. We now define a set of functions $V_i(\gamma, P)$, $i = 1, 2, 3, \cdots$, for any γ of t and any P of T by making the following requirements:

(1) If P is in $\mathfrak{S}(\gamma, \rho_i)$ then $V_i(\gamma, P)$ is the smaller of $W(p, P)$ and ρ_i (where p is the point contained in γ) or their common value if $W(p, P) = \rho_i$.

(2) If P is in $T - \mathfrak{S}(\gamma, \rho_i)$, then $V_i(\gamma, P) = \rho_i$.

For fixed γ, each $V_i(\gamma, P)$ is bounded, continuous and superharmonic in T. Moreover, $0 < V_i(\gamma, P) \leqq \rho_1$, throughout T. Hence, for each γ of t,

$$V(\gamma, P) = \sum_{i=1}^{\infty} 2^{-i} V_i(\gamma, P)$$

is continuous and superharmonic in T; also,

$$0 < V(\gamma, P) \leqq W(p, P),$$

throughout T. Hence items (1) and (2) of the conclusion of the lemma are valid. Moreover, given any ρ such that $0 < \rho < \bar{\rho}$, we can choose, independently of γ, an integer k such that $\mathfrak{S}(\gamma, \rho_k)$ is contained in the partial domain $\mathfrak{S}(\gamma, \bar{\rho})$ of T. We have then in $T - \mathfrak{S}(\gamma, \rho)$,

$$V(\gamma, P) \geqq \sum_{i=k}^{\infty} 2^{-i} \rho_i.$$

Choosing $b(\rho)$ as the value of the series given in this relation, we see that item (3) of the conclusion of the lemma is valid.

THEOREM 25. *If T is a normal finite domain, then T is pseudo-normal.*

Let $U(P)$ be the sequence solution of the extended Dirichlet problem for the normal finite domain T and arbitrarily assigned bounded and pseudo-uniformly continuous boundary values $f(\gamma)$. We need to show that given any positive ϵ, there exists a positive δ independent of the arbitrarily chosen element γ_1 of t, such that

$$| U(P) - f(\gamma_1) | < \epsilon,$$

throughout $\mathfrak{S}(\gamma_1, P)$.

Let $F(P)$ be any function which is bounded and continuous in T and approaches with uniform pseudo-continuity the boundary values $f(\gamma)$ on t. We may then choose, independently of γ_1, a positive quantity $\bar{\delta}$ less than half the diameter of T' such that in $\mathfrak{S}(\gamma_1, \bar{\delta})$

$$|F(P) - f(\gamma_1)| < \epsilon/2.$$

Let us now consider again the reasoning by which we established the sufficiency of the condition given in Theorem 23. Since T is now assumed to be normal we may use as a pseudo-barrier $V(\gamma_1, P)$ the function introduced in Lemma 2. We know then that there exist constants $b(\bar{\delta})$ and $B(\bar{\delta})$, independent of γ_1, such that

$$b(\bar{\delta}) < V(\gamma_1, P), \qquad \text{in } T - \mathfrak{S}(\gamma_1, \bar{\delta}),$$

and

$$|F(P) - f(\gamma_1)| < B(\bar{\delta})b(\bar{\delta}), \qquad \text{in } T - \mathfrak{S}(\gamma_1, \bar{\delta}).$$

Hence,

$$|F(P) - f(\gamma_1)| < B(\bar{\delta})V(\gamma_1, P) + \epsilon/2, \qquad \text{in } T.$$

By the same reasoning as that used in the proof of Theorem 23, we infer that

$$| U(P) - f(\gamma_1)| < B(\bar{\delta})V(\gamma_1, P) + \epsilon/2,$$

throughout T. Again using Lemma 2, we may choose a positive constant δ, independent of γ_1, such that

$$0 < V(\gamma_1, P) < \frac{\epsilon}{2B(\bar{\delta})},$$

in $\mathfrak{S}(\gamma_1, \delta)$. Hence

$$| U(P) - f(\gamma_1) | < \epsilon,$$

in $\mathfrak{S}(\gamma_1, \delta)$. This completes the proof of the theorem.

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THE EQUIVALENCE OF PAIRS OF HERMITIAN MATRICES*

BY

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Two pairs of n-ary Hermitian forms with $n\times n$ matrices A, B and C, D with elements in the complex field are equivalent if there exists a non-singular matrix T such that $\overline{T}'AT=C$ and $\overline{T}'BT=D$, where $\overline{T}'$ is the conjugate-transpose of T.

As is usual in the study of equivalence of pairs of matrices the work divides itself into the consideration of the non-singular and singular cases. These two cases are taken up in Parts II and I respectively.

In the non-singular case the rank of $\rho A+\sigma B$ is n except for special values of ρ and σ. It has frequently been pointed out that in this case no generality is lost by assuming B is of rank n.

In the singular case the rank r of $\rho A+\sigma B$ is less than n for all values of ρ and σ, but as above no generality is lost in assuming that the rank r of B is the maximum rank of $\rho A+\sigma B$.

By the elementary divisors of a pair of matrices A, B is meant the elementary divisors of $A-\lambda B$ when B is non-singular, and the elementary divisors of $\rho A+\sigma B$ when B is singular but the determinant $|\rho A+\sigma B|$ is not identically zero in ρ and σ. In the non-singular case, the well known necessary and sufficient condition for the equivalence in any field of pairs of bilinear forms, or of their corresponding matrices, and for the equivalence in the field of complex numbers of pairs of symmetric matrices is that the pairs have the same elementary divisors. This condition is known to be not sufficient

* Presented to the Society, April 7, 1934; received by the editors June 27, 1934. Part II is in essence a dissertation of Dr. Wegner, written at the University of Wisconsin. Part I is in general the work of Professor Ingraham.

The subject of this paper has been the source of a considerable amount of current investigation some of which has led to results which in part are equivalent to some of those arrived at here. Dr. Wegner presented a paper containing his results at the April, 1934, meeting of the Society, the abstract appearing in the Bulletin of the American Mathematical Society, vol. 40, No. 1, January, 1934, as abstract No. 103. Simultaneously Dr. G. R. Trott obtained by somewhat analogous methods results which were equivalent to those of Dr. Wegner. These along with the proof of their equivalence are given in the American Journal of Mathematics, vol. 56, July, 1934, pp. 359 ff.: *On the canonical form of a non-singular pencil of Hermitian matrices.* Since preparing this paper it has been brought to the attention of the authors that Professor Turnbull had considered this problem and would soon publish a paper on the subject. The authors immediately submitted a copy of this paper to Professor Turnbull and in reply received word that the treatments were totally different, his treatment following the analogous classical treatment for the case of real quadratic forms.

for the equivalence in the field of real numbers of pairs of real symmetric matrices. This is illustrated by the pairs of one-by-one matrices

$$A = (1),\ B = (1);\ C = (-1),\ D = (-1)$$

which have the same elementary divisor $(\lambda - 1)$, but for which there obviously exists no real $P = (p)$ such that $P'AP = p^2 = -1$. In 1905 Muth* gave the necessary and sufficient conditions for the real equivalence of real symmetric matrices.

It has sometimes been stated that for the non-singular case the coincidence of the elementary divisors of the pairs is also a sufficient condition for the equivalence in the complex field of pairs of Hermitian matrices. That this is not the case is illustrated by the above pair considered as Hermitian matrices, there existing no $P = (p)$ such that $\bar{P}'AP = \bar{p}p = -1$. The present paper gives the necessary and sufficient conditions for the equivalence of pairs of Hermitian matrices. Although the method of proof is much simpler, the conditions for the non-singular case are the same as those arrived at by Muth for the real symmetric case, a result which is entirely reasonable when one considers that Hermitian matrices should be thought of as a generalization of real symmetric matrices. Also, when one remembers that the necessary and sufficient conditions for the real equivalence of two real symmetric matrices or for the equivalence in the complex field of two Hermitian matrices is that they have the same rank and the same index, the results of Part II of this paper seem quite reasonable when stated in the following form:

THEOREM. *Two pairs of Hermitian matrices A, B and C, D, where $|B| \neq 0$ and $|D| \neq 0$, are equivalent if and only if*

(1) *they have the same elementary divisors,*

and

(2) *the matrices $B(B^{-1}A - \lambda I)^n$ and $D(D^{-1}C - \lambda I)^n$ have the same index for all positive integral n and real λ.*

Dickson's† treatment of the singular case is reduced to the above mentioned erroneous treatment of the non-singular case. A direct reduction to the non-singular case, leading to a canonical form and using the Hermitian properties of the matrices involved, has been found. This is given in Part I. Part II, which treats the non-singular case, may be read independently of Part I.

* Muth, P., *Über reelle Äquivalenz von Scharen reeller quadratischer Formen*, in Journal für die reine und angewandte Mathematik, vol. 128 (1905), pp. 302–321.

† Dickson, L. E., *Singular case of pairs of bilinear, quadratic, or Hermitian forms*, these Transactions, vol. 29 (1927), pp. 239–253.

I. Singular case

In this section the finding of a canonical form for a pair A, B of Hermitian matrices is reduced to the treatment of a pair of lower order. By successive reductions the problem is completely solved or is finally reduced to the treatment of the non-singular case.

Consider a pair of $n\times n$ Hermitian matrices A, B such that the rank of $\rho A+\sigma B$ never exceeds r, the rank of B. Without loss of generality we may assume that B is of the form

$$\begin{Vmatrix} B_{11} & 0 \\ 0 & 0 \end{Vmatrix} \quad \text{where } B_{11} = \begin{Vmatrix} I_s & 0 \\ 0 & -I_t \end{Vmatrix}, \tag{1}$$

I_k is the $k\times k$ identity matrix, the 0's stand for 0 matrices, and $r=s+t$.

Let

$$A = \begin{Vmatrix} A_{11} & A_{12} \\ \overline{A}'_{12} & A_{22} \end{Vmatrix},$$

where A_{11} is an $r\times r$ Hermitian matrix, A_{12} is an $r\times(n-r)$ matrix, and A_{22} is an $(n-r)\times(n-r)$ Hermitian matrix. Since the rank of $A+\sigma B$ never exceeds r, $A_{22}=0$, for if that were not the case there would be a minor of order $r+1$ the determinant of which would have $\pm k\sigma^r$ for the leading term in σ, where k is a non-zero element of A_{22}, and hence this determinant is not identically zero.

Thus

$$A + \sigma B = \begin{Vmatrix} A_{11} + \sigma B_{11} & A_{12} \\ \overline{A}'_{12} & 0 \end{Vmatrix},$$

where $A_{11}+\sigma B_{11}$ is non-singular, i.e., of rank r except for a finite number of values of σ, and the rank of $A+\sigma B$ never exceeds r. Clearly,

$$(A_{11} + \sigma B_{11})(A_{11} + \sigma B_{11})^{-1}A_{12} = A_{12}$$

and from above the same relation must hold between the columns of the last $n-r$ rows of $A+\sigma B$ and hence

$$\overline{A}'_{12}(A_{11} + \sigma B_{11})^{-1}A_{12} = 0. \tag{2}$$

Since $B_{11}=B_{11}^{-1}$, for sufficiently large values of σ we have the expansion

$$(A_{11} + \sigma B_{11})^{-1} = \frac{1}{\sigma} B_{11} - \frac{1}{\sigma^2} B_{11}A_{11}B_{11} + \frac{1}{\sigma^3} B_{11}(A_{11}B_{11})^2 + \cdots,$$

and therefore from equation (2) we see that

$$\overline{A}'_{12}B_{11}A_{12} = 0 \tag{3}$$

and in general

$$\bar{A}'_{12}B_{11}(A_{11}B_{11})^k A_{12} = 0 \qquad (k = 1, 2, 3, \cdots). \tag{4}$$

If we let

$$A_{12} = \begin{Vmatrix} A_{121} \\ A_{122} \end{Vmatrix}$$

where A_{121} is an $s\times(n-r)$ matrix and A_{122} is a $t\times(n-r)$ matrix, condition (3) becomes

$$\bar{A}'_{121}A_{121} = \bar{A}'_{122}A_{122}, \tag{5}$$

and hence the ranks of A_{121} and A_{122} can not exceed the smallest of the three numbers s, t, and $n-r$.

We shall specify that in our canonical form for the pair A,B, B as defined in equation (1) be left invariant.

Let T be a non-singular matrix

$$\begin{Vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{Vmatrix}$$

with the same conventions as above for the dimensions of the sub-matrices and satisfying the condition

$$\bar{T}'BT = B. \tag{6}$$

This condition is equivalent to the following three conditions:

$$\bar{T}'_{11}B_{11}T_{11} = B_{11}, \qquad \bar{T}'_{11}B_{11}T_{12} = 0, \qquad \bar{T}'_{12}B_{11}T_{12} = 0. \tag{7}$$

From the first of these conditions it follows that T_{11} must be non-singular and hence from the second condition we see that $T_{12}=0$. Since T is non-singular, T_{22} must be non-singular.

If $F=\bar{T}'AT$, we have, using the same conventions as above,

$$F_{11} = \bar{T}'_{11}A_{11}T_{11} + \bar{T}'_{21}\bar{A}'_{12}T_{11} + \bar{T}'_{11}A_{12}T_{21}, \tag{8}$$

$$F_{12} = \bar{T}'_{11}A_{12}T_{22} = \bar{F}'_{21}, \tag{9}$$

$$F_{22} = 0. \tag{10}$$

As a special case we may take

$$T_{11} = \begin{Vmatrix} S_{11} & 0 \\ 0 & S_{22} \end{Vmatrix}$$

where S_{11} is an $s\times s$ unitary orthogonal matrix and S_{22} is a $t\times t$ unitary orthogonal matrix, and for this case

$$F_{121} = \bar{S}'_{11}A_{121}T_{22}, \quad \text{and } F_{122} = \bar{S}'_{22}A_{122}T_{22}.$$

Let the rank of A_{121} be l_1. It is readily shown that, by a proper choice of S_{11}, $\bar{S}'_{11}A_{121}$ may be taken as a matrix in which all the elements below the l_1th row are zero. This being the case, it is readily seen that T_{22} may be so chosen that

$$F_{121} = \begin{Vmatrix} I_{l_1} & 0 \\ 0 & 0 \end{Vmatrix}.$$

Although T_{22} may have been determined, S_{22} may be so chosen that all but the last l rows of F_{122} are zero, where l is the rank of A_{122}, and such that in these last l rows, if for the ith and $(i+1)$st rows a_{ij} and $a_{i+1,k}$ are the first non-zero elements, then $j<k$. We may also pick S_{22} such that these first non-zero elements are positive real numbers. From these conditions and the fact that according to (5)

$$\bar{F}'_{122}F_{122} = \bar{F}'_{121}F_{121},$$

we see that

$$F_{122} = \begin{Vmatrix} 0 & 0 \\ I_{l_1} & 0 \end{Vmatrix}$$

and that l_1 is the rank of A_{12}. In order not to accumulate notations we will assume from now on that

$$A_{12} = \begin{Vmatrix} I_{l_1} & 0 \\ 0 & 0 \\ I_{l_1} & 0 \end{Vmatrix}, \tag{11}$$

where, of course, if $l_1=n-r$ the second column of zeros is absent. Call l_1 the first invariant sub-rank of A.

We will now use only such transformations T as will leave B and the A_{12} invariant. Let

$$T_{11} = S$$

and

$$S = (S_{ij}) \qquad (i = 1, 2, 3; j = 1, 2, 3),$$

S_{11} and S_{33} being $l_1\times l_1$ matrices and S_{22} an $(r-2l_1)\times(r-2l_1)$ matrix except where $r-2l_1=0$, in which case the second row and column of S are deleted. Let

$$B_{11} = \begin{Vmatrix} I_{l_1} & 0 & 0 \\ 0 & B_{11}^{(1)} & 0 \\ 0 & 0 & -I_{l_1} \end{Vmatrix},$$

where $B_{11}^{(1)}$ is an $(r-2l_1)\times(r-2l_1)$ matrix of structure similar to B_{11} with $s-l_1$ plus ones and $t-l_1$ minus ones on the main diagonal and elsewhere zero. From (9), we see that $\bar{S}'A_{12}T_{22}=A_{12}$, and from this and the fact that T_{22} is non-singular, it follows that

$$\bar{S}'A_{12} = \begin{Vmatrix} \bar{K}' & 0 \\ 0 & 0 \\ \bar{K}' & 0 \end{Vmatrix},$$

where K is an $l_1\times l_1$ non-singular matrix. Hence, remembering the form (11) for A_{12} we see that

(12) $$S_{31} = K - S_{11},$$

(13) $$S_{13} = K - S_{33},$$

(14) $$S_{32} = - S_{12}.$$

Making these substitutions in the form for S, we see that the conditions that $\bar{S}'B_{11}S=B_{11}$ are

(15) $$\bar{S}_{21}'B_{11}^{(1)}S_{21} + \bar{S}_{11}'K - \bar{K}'K + \bar{K}'S_{11} = I_{l_1},$$

(16) $$\bar{S}_{21}'B_{11}^{(1)}S_{22} + \bar{K}'S_{12} = 0,$$

(17) $$\bar{S}_{11}'K + \bar{S}_{21}'B_{11}^{(1)}S_{23} - \bar{K}'S_{33} = 0,$$

(18) $$\bar{S}_{22}'B_{11}^{(1)}S_{22} = B_{11}^{(1)},$$

(19) $$\bar{S}_{12}'K + \bar{S}_{22}'B_{11}^{(1)}S_{23} = 0,$$

(20) $$\bar{K}'K - \bar{K}'S_{33} - \bar{S}_{33}'K + \bar{S}_{23}'B_{11}^{(1)}S_{23} = - I_{l_1}.$$

Subtracting the conjugate-transpose of (16) from (19) we get

$$\bar{S}_{22}'B_{11}^{(1)}(S_{23} - S_{21}) = 0$$

and since $B_{11}^{(1)}$ is non-singular and, by (18), S_{22} is non-singular, we see that

(21) $$S_{21} = S_{23}.$$

Since K is non-singular, from (16) we see that

(22) $$S_{12} = - \bar{K}'^{-1}\bar{S}_{21}'B_{11}^{(1)}S_{22}.$$

Using (21) and subtracting (17) from (15) we see that

$$\bar{K}'S_{11} + \bar{K}'S_{33} - \bar{K}'K = I_{l_1},$$

that is,

$$S_{11} + S_{33} - K = \overline{K}'^{-1}. \tag{23}$$

Since subtracting (17) from (20) also yields (23) we see that conditions (17), (18), (21), (22) and (23) are equivalent to conditions (15) to (20). Moreover, it may be seen that S_{21} may be chosen arbitrarily, that S_{22} need only satisfy (18), and that S_{11} and S_{33} may be determined so as to satisfy (17) and (23).

Let us now turn our attention to equation (8) for F_{11} and, in particular, study the last two terms $\overline{T}'_{21}\overline{A}'_{12}T_{11}+\overline{T}'_{11}A_{12}T_{21}$ which we will call M.

If

$$T_{21} = \begin{Vmatrix} T_{211} & T_{212} & T_{213} \\ T_{214} & T_{215} & T_{216} \end{Vmatrix},$$

we see that

$$M = \begin{Vmatrix} \overline{K}'T_{211} + \overline{T}'_{211}K & \overline{K}'T_{212} & \overline{K}'T_{213} + \overline{T}'_{211}K \\ \overline{T}'_{212}K & 0 & \overline{T}'_{212}K \\ \overline{K}'T_{211} + \overline{T}'_{213}K & \overline{K}'T_{212} & \overline{K}'T_{213} + \overline{T}'_{213}K \end{Vmatrix},$$

and we see, since K is non-singular, that $M_{11}=\overline{K}'T_{211}+\overline{T}'_{211}K$ may be taken as an arbitrary $l_1\times l_1$ Hermitian matrix and that $M_{12}=\overline{K}'T_{212}$ and $M_{13}=\overline{K}'T_{213}+\overline{T}'_{211}K$ may be chosen as arbitrary matrices of the correct dimensions; that $M_{21}=M_{23}=\overline{M}'_{12}=\overline{M}'_{2}$, and that $M_{33}=M_{13}+M_{31}-M_{11}$. If now we write

$$A_{11} = G = \begin{Vmatrix} G_{11} & G_{12} & G_{13} \\ \overline{G}'_{12} & G_{22} & G_{23} \\ \overline{G}'_{13} & \overline{G}'_{23} & G_{33} \end{Vmatrix},$$

we see that condition (4) with $k=1$, namely, that $\overline{A}'_{12}B_{11}GB_{11}A_{22}=0$, reduces to

$$G_{33} = G_{13} + \overline{G}'_{13} - G_{11}.$$

Hence M may be so chosen that

$$F_{11} = \overline{T}'_{11}A_{11}T_{11} + M$$

will be of the form

$$\begin{Vmatrix} 0 & 0 & 0 \\ 0 & A^{(1)}_{11} & A^{(1)}_{12} \\ 0 & \overline{A^{(1)}_{12}}' & 0 \end{Vmatrix}. \tag{24}$$

We may consider A_{11} to be in this form from now on. We must now study transformations T which leave B and B_{11} invariant and which leave F_{11} in form (24). Letting $H=\overline{S}'A_{11}S$, we see that

$$
\begin{aligned}
H_{12} &= \overline{S}'_{21}A^{(1)}_{11}S_{22} + \overline{K}'\overline{A^{(1)}_{12}}{}'S_{22} - \overline{S}'_{11}\overline{A^{(1)}_{12}}{}'S_{22} - \overline{S}'_{21}A^{(1)}_{12}S_{12},\\
H_{22} &= \overline{S}'_{22}A^{(1)}_{11}S_{22} - \overline{S}'_{12}\overline{A^{(1)}_{12}}{}'S_{22} - \overline{S}'_{22}A^{(1)}_{12}S_{12},\\
H_{23} &= \overline{S}'_{22}A^{(1)}_{11}S_{21} - \overline{S}'_{12}\overline{A^{(1)}_{12}}{}'S_{21} + \overline{S}'_{22}A^{(1)}_{12}S_{33}.
\end{aligned}
\tag{25}
$$

If M is so chosen that

$$
H + M = \left\| \begin{matrix} 0 & 0 & 0 \\ 0 & F^{(1)}_{11} & F^{(1)}_{12} \\ 0 & \overline{F^{(1)}_{12}}{}' & F^{(1)}_{22} \end{matrix} \right\|,
$$

then

$$
F^{(1)}_{22} = 0, \qquad F^{(1)}_{11} = H_{22}, \tag{26}
$$

and

$$
F^{(1)}_{12} = H_{23} - \overline{H}'_{12} = \overline{S}'_{22}A^{(1)}_{12}(S_{33} + S_{11} - K),
$$

and by (23),

$$
F^{(1)}_{12} = \overline{S}'_{22}A^{(1)}_{12}\overline{K}'^{-1}. \tag{27}
$$

If in (26) and (27) we replace

$$
F^{(1)}_{11}, F^{(1)}_{12}, F^{(1)}_{22}, A^{(1)}_{11}, \text{ and } A^{(1)}_{12} \text{ by } F_{11}, F_{12}, F_{22}, A_{11}, \text{ and } A_{12},
$$

and replace

$$
S_{22} \text{ by } T_{11}, S_{12} \text{ by } -T_{21}, \text{ and } \overline{K}'^{-1} \text{ by } T_{22},
$$

we arrive at conditions (8), (9) and (10), and if in addition $B^{(1)}_{11}$ be replaced by B_{11}, (18) becomes (7). $A^{(1)}+\sigma B$, which is equal to

$$
\left\| \begin{matrix} \sigma I_{t_1} & 0 & 0 & I_{t_1} & 0 \\ 0 & A^{(1)}_{11} + \sigma B^{(1)}_{11} & A^{(1)}_{12} & 0 & 0 \\ 0 & \overline{A^{(1)}_{12}}{}' & -\sigma I_{t_1} & I_{t_1} & 0 \\ I_{t_1} & 0 & I_{t_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \right\|,
$$

has the same rank as

$$
\left\| \begin{matrix} \sigma I_{t_1} & 0 & 0 & 0 & 0 \\ 0 & A^{(1)}_{11} + \sigma B^{(1)}_{11} & A^{(1)}_{12} & 0 & 0 \\ 0 & \overline{A^{(1)}_{12}}{}' & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{t_1}/\sigma & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \right\|,
$$

and hence, since $A^{(1)}+\sigma B$ must have the same rank as B except for a finite number of values of σ, the matrix

$$\begin{Vmatrix} A_{11}^{(1)} + \sigma B_{11}^{(1)} & A_{12}^{(1)} \\ \overline{A_{11}^{(1)\prime}} & 0 \end{Vmatrix}$$

must have the same rank as $B_{11}^{(1)}$. This completes the reduction of the problem to the consideration of a pair of matrices of order r. Let l_2 be the first invariant sub-rank for $A^{(1)}$, $B^{(1)}$, etc. Finally at some stage l_k is zero and then $A_{12}^{(k-1)}$ is zero and (8) reduces to the consideration of the non-singular case of the reduction of $A^{(k)}$, $B^{(k)}$. This completes the proof of the sufficiency of the conditions of Theorem 1. The necessity may be readily checked from the above considerations.

THEOREM 1. *Two pairs of Hermitian matrices A, B and C, D where the rank of B is maximal for $\rho A+\sigma B$ are equivalent if and only if*

(1) *B is equivalent to D;*

(2) *the invariant sub-ranks are equal;*

(3) *the non-singular pair $A^{(k)}$, $B^{(k)}$ is equivalent to the pair $C^{(k)}$, $D^{(k)}$.*

II. NON-SINGULAR CASE

1. **Preliminary reduction of pair.** Consider any pair of Hermitian matrices A, B with complex elements and such that the determinant $|\rho A+\sigma B|$ is not identically zero in ρ and σ. As stated above, no generality is lost in assuming that $|B|\neq 0$. Calling G the classical canonical form (described below) of $B^{-1}A$, we know there exists a matrix T such that

$$G = T^{-1}B^{-1}AT = T^{-1}B^{-1}\overline{T}'^{-1}\overline{T}'AT = B_1^{-1}A_1,$$

where $B_1=\overline{T}'BT$ and $A_1=\overline{T}'AT$. Therefore we lose no generality in allowing the pair A, B to be such that $B^{-1}A=G$ is in canonical form.

We shall call a matrix whose elements are all zero except for square blocks along the main diagonal a diagonal block matrix. It shall be shown that there exists a diagonal block matrix E determined by G such that $E=\overline{E}'=E^{-1}$ and such that B must be of the form ES_1, where S_1 is a matrix commutative with the canonical form G. A canonical pair A_c, B_c for A, B will then be obtained by showing that it is always possible to find a non-singular matrix S commutative with G such that

$$B_c = \overline{S}'BS \text{ and } A_c = \overline{S}'AS = B_cG.$$

Let the elementary divisors of $A-\lambda B$ be $(\lambda-\lambda_i)^{e_i}$. Call J_i the square matrix of order e_i having ones in the diagonal above the main diagonal and other-

wise zeros. Call J_i^0 the identity matrix of order e_i. We may then describe the canonical form G as a diagonal block matrix having a block $(\lambda_i J_i^0 + J_i)$ corresponding to each elementary divisor $(\lambda - \lambda_i)^{e_i}$. We may assume that blocks of G which correspond to conjugate imaginary pairs of elementary divisors are adjacent blocks. Call E_i the square matrix of order e_i with ones along its secondary diagonal and otherwise zeros, i.e., having elements (c_{jk}) where $c_{jk}=1$ for $j+k=e_i+1$, and $c_{jk}=0$ for $j+k\neq e_i+1$. Define E as a diagonal block matrix such that a block $(\lambda_i J_i^0 + J_i)$ of C corresponds to a block E_i of E when λ_i is real, and two blocks $(\lambda_i J_i^0 + J_i)$ and $(\bar{\lambda}_i J_i^0 + J_i)$ of G correspond to one block

$$\begin{Vmatrix} 0 & E_i \\ E_i & 0 \end{Vmatrix}$$

of E when λ_i is not real. (In this paper the symbol 0 used in this way represents a matrix all of whose elements are zero.) This E is such that $E=\bar{E}'=E^{-1}$ and $EGE=\bar{G}'$, whence, since we are assuming $B^{-1}A=G$,

$$A = \bar{A}' = BG = \bar{G}'B = EGEB,$$

and therefore

$$EBG = GEB.$$

Hence B must be of the form ES_1, where S_1 is a matrix commutative with G.

The form of any matrix S commutative with the canonical form G will now be described. To facilitate this description, we may assume that the blocks of G are arranged so that those corresponding to elementary divisors involving the same root appear in non-increasing order with respect to size.

Call I_{ik}, $e_i \geqq e_k$, the $e_i \times e_k$ matrix made up of J_k^0 augmented below by $e_i - e_k$ rows of zeros. S is a block matrix of the following form: To the block $(\lambda_i J_i^0 + J_i)$ of G corresponds a block $S_{ii}(J_i)$ of S, where S_{ii} is a polynomial with complex coefficients. When $\lambda_i = \lambda_k$, $e_i \geqq e_k$, there is also an $e_i \times e_k$ block of the form $I_{ik}S_{ik}(J_k)$ in the columns of $S_{kk}(J_k)$ and the rows of $S_{ii}(J_i)$, and an $e_k \times e_i$ block of the form $S_{ki}(J_k)E_k I'_{ik} E_i$ in the rows of $S_{kk}(J_k)$ and the columns of $S_{ii}(J_i)$.

$B=ES$ will then be a block matrix of the following form: Considering first the blocks related to elementary divisors involving real roots we find that to the block $S_{ii}(J_i)$ of S corresponds a block $E_i B_{ii}(J_i)$ of B, to the block $I_{ik}S_{ik}(J_k)$ of S the block $E_i I_{ik} B_{ik}(J_k)$ of B, and to the block $S_{ki}(J_k)E_k I'_{ik} E_i$ of S the block $\bar{B}_{ik}(J_k')I'_{ik}E_i$ of B. Considering then double blocks related to conjugate imaginary pairs of elementary divisors we find that to

$$\begin{Vmatrix} S_{ii1}(J_i) & 0 \\ 0 & S_{ii2}(J_i) \end{Vmatrix}$$

of S corresponds

$$\begin{Vmatrix} 0 & E_iB_{ii1}(J_i) \\ \bar{B}_{ii1}(J_i')E_i & 0 \end{Vmatrix}$$

of B; to

$$\begin{Vmatrix} I_{ik}S_{ik1}(J_k) & 0 \\ 0 & I_{ik}S_{ik2}(J_k) \end{Vmatrix}$$

of S corresponds

$$\begin{Vmatrix} 0 & E_iI_{ik}B_{ik2}(J_k) \\ E_iI_{ik}B_{ik1}(J_k) & 0 \end{Vmatrix}$$

of B; and to

$$\begin{Vmatrix} S_{ki1}(J_k)E_kI'_{ik}E_i & 0 \\ 0 & S_{ki2}(J_k)E_kI'_{ik}E_i \end{Vmatrix}$$

of S corresponds

$$\begin{Vmatrix} 0 & \bar{B}_{ik1}(J_k')I'_{ik}E_i \\ \bar{B}_{ik2}(J_k')I'_{ik}E_i & 0 \end{Vmatrix}$$

of B.

2. **Reduction of pair to canonical form.** The canonical pair A_c, B_c that we shall obtain has the following form: B_c is a diagonal block matrix with blocks of the same dimensions as those of E, a block E_i of E corresponding to a block ϵ_iE_i of B_c, where $\epsilon_i = \pm 1$, and a block

$$\begin{Vmatrix} 0 & E_i \\ E_i & 0 \end{Vmatrix}$$

of E corresponding to a block

$$\begin{Vmatrix} 0 & E_i \\ E_i & 0 \end{Vmatrix}$$

of B_c. $A_c = B_cG$ is also a diagonal block matrix with blocks of the same dimensions as those of E, a block E_i of E corresponding to a block $\epsilon_iE_i(\lambda_iJ_i^0 + J_i)$ of A_c, and a block

$$\begin{Vmatrix} 0 & E_i \\ E_i & 0 \end{Vmatrix}$$

of E corresponding to a block

$$\begin{Vmatrix} 0 & E_i(\bar{\lambda}_iJ_i^0 + J_i) \\ E_i(\lambda_iJ_i^0 + J_i) & 0 \end{Vmatrix}$$

of A_c.

Suppose the real elementary divisors $(\lambda-\lambda_i)^{e_i}$ of A,B divide into m classes of equal elementary divisors (i.e., involving the same root and the same exponent). For each class define a σ as the sum of the ϵ's corresponding to the elementary divisors of that class. It will be shown in §3 that these m σ_i are invariants of the pair A,B.

To reduce to this canonical pair we shall show that it is always possible to choose a non-singular matrix S commutative with the canonical form $G=B^{-1}A$ so that $\bar{S}'BS=B_c$. It follows that

$$A_c = \bar{S}'AS = \bar{S}'BSS^{-1}GS = \bar{S}'BSG = B_cG. \tag{1}$$

Because of the block form of B and S it is evidently necessary to consider elementary divisors involving but a single real root, or a pair of conjugate imaginary roots. The reduction is divided into eight cases. In Case I the canonical form is obtained for a pair of matrices having but one real elementary divisor, and in Case II for a pair of matrices having but one pair of conjugate imaginary elementary divisors. Every other situation is shown to depend essentially on Cases I and II. In Cases III and IV induction is used to obtain the canonical forms where there are any number of distinct (i.e., with distinct exponents, but involving the same characteristic root) and no repeated, elementary divisors (III) and any number of distinct, but no repeated, pairs of conjugate imaginary elementary divisors (IV). In Cases V and VI the situations of a cluster of equal real elementary divisors and a cluster of equal pairs of conjugate imaginary elementary divisors are reduced so as to be handled by the methods of Cases III and IV. Finally, in Cases VII and VIII, induction on Cases V and VI is used to cover the situation of any number of clusters of equal real elementary divisors and of any number of clusters of equal pairs of conjugate imaginary elementary divisors.

Before taking up these cases, it will be well to list some relations that shall be used repeatedly in the reductions:

$$J_i^j = 0 \qquad \text{when } j \geqq e_i; \tag{2}$$

$$P(J_i')E_i = E_iP(J_i), \qquad P \text{ a polynomial}; \tag{3}$$

$$P_1(J_j)I_{jk} = I_{jk}P_2(J_k), \qquad P_1 \text{ and } P_2 \text{ polynomials}; \tag{4}$$

$$E_kI_{jk}'E_jI_{jk} = J_k^{e_j-e_k}; \tag{5}$$

$$I_{ij}I_{jk} = I_{ik}. \tag{6}$$

Also, it will be found convenient in some of the reductions to use for references the following multiplications, in which $\mathcal{S}_{ij}$, $\mathcal{B}_{ij}$, and $\mathcal{R}_{ij}$ are themselves matrices, square if $i=j$.

If

$$S = \begin{Vmatrix} \mathfrak{S}_{11} & \mathfrak{S}_{12} \\ 0 & I \end{Vmatrix} \text{ and } B = \begin{Vmatrix} \mathfrak{B}_{11} & \mathfrak{B}_{12} \\ \overline{\mathfrak{B}}_{12}' & \mathfrak{B}_{22} \end{Vmatrix},$$

then

$$(7) \quad \overline{S}'BS = \begin{Vmatrix} \overline{\mathfrak{S}}_{11}'\mathfrak{B}_{11}\mathfrak{S}_{11} & \overline{\mathfrak{S}}_{11}'(\mathfrak{B}_1\mathfrak{S}_{12}+\mathfrak{B}_{12}) \\ (\overline{\mathfrak{S}}_{12}'\overline{\mathfrak{B}}_{11}' + \overline{\mathfrak{B}}_{12}')\mathfrak{S}_{11} & \overline{\mathfrak{S}}_{12}'(\mathfrak{B}_{11}\mathfrak{S}_{12} + \mathfrak{B}_{12}) + \overline{\mathfrak{B}}_{12}'\mathfrak{S}_{12} + \mathfrak{B}_{22} \end{Vmatrix}.$$

If

$$R = \begin{Vmatrix} I & 0 \\ 0 & \mathfrak{R}_{22} \end{Vmatrix} \text{ and } B = \begin{Vmatrix} \mathfrak{B}_{11} & \mathfrak{B}_{12} \\ \overline{\mathfrak{B}}_{12}' & \mathfrak{B}_{22} \end{Vmatrix},$$

then

$$(8) \quad \overline{R}'BR = \begin{Vmatrix} \mathfrak{B}_{11} & \mathfrak{B}_{12} \\ \overline{\mathfrak{B}}_{12}' & \overline{\mathfrak{R}}_{22}'\mathfrak{B}_{22}\mathfrak{R}_{22} \end{Vmatrix}.$$

In some of the cases below, manipulative details have been omitted. Those interested may refer to the doctor's thesis, University of Wisconsin, 1934, by K. W. Wegner.

Case I. A single elementary divisor $(\lambda-\lambda_1)^{e_1}$.

$$S = S_1(J_1) = \sum_{i=1}^{e_1} s_i J_1^{i-1}, \qquad B = E_1B_1(J_1) = E_1 \cdot \sum_{i=1}^{e_1} b_i J_1^{i-1};$$

$$\begin{aligned} \overline{S}'BS &= \overline{S}_1(J_1')E_1B_1(J_1)S_1(J_1) = E_1\overline{S}_1(J_1)B_1(J_1)S_1(J_1) \quad \text{by (3)} \\ &= E_1[s_1^2b_1J_1^0 + (2s_1s_2b_1 + s_1^2b_2)J_1 + (2s_1s_3b_1 + s_2^2b_1 + 2s_1s_2b_2 + s_1^2b_3)J_1^2 \\ &\quad + (2s_1s_4b_1 + 2s_2s_3b_1 + 2s_1s_3b_2 + s_2^2b_2 + 2s_1s_2b_3 + s_1^2b_4)J_1^3 + \cdots \\ &\quad + (2s_1s_{e_1}b_1 + 2s_2s_{e_1-1}b_1 + \cdots)J_1^{e_1-1}], \end{aligned}$$

using (2) and choosing the s_i to be real. The element b_1 is real since B is Hermitian and $b_1 \neq 0$ since $|B| \neq 0$. Hence we may choose $s_1 = (\pm b_1)^{-1/2}$ and s_j, $j=2, 3, \cdots, e_1$, so that the coefficient of J_1^{j-1} in the last expression for $\overline{S}'BS$ above is zero. Then $\overline{S}'BS = S'BS = \pm E_1J_1^0 = \pm E_1 = \epsilon_1E_1 = B_c$.

Case II. Elementary divisors: $(\lambda-\lambda_1)^{e_1}$, $(\lambda-\overline{\lambda}_1)^{e_1}$, $\lambda_1 \neq \overline{\lambda}_1$.

$$S = \begin{Vmatrix} S_1(J_1) & 0 \\ 0 & S_2(J_1) \end{Vmatrix}, \qquad B = \begin{Vmatrix} 0 & E_1B_1(J_1) \\ \overline{B}_1(J_1')E_1 & 0 \end{Vmatrix},$$

$$\overline{S}'BS = \begin{Vmatrix} 0 & \overline{S}_1(J_1')E_1B_1(J_1)S_2(J_1) \\ \overline{S}_2(J_1')\overline{B}_1(J_1')E_1S_1(J_1) & 0 \end{Vmatrix}.$$

Choose $S_1(J_1) = J_1^0$ and call

$$B_1(J_1) = \sum_{i=1}^{e_1} b_i J_1^{i-1}, \qquad S_2(J_1) = \sum_{i=1}^{e_1} s_i J_1^{i-1}.$$

Then

$$\bar{S}_1(J_1')E_1B_1(J_1)S_2(J_1) = E_1[s_1b_1J_1^0 + (s_2b_1 + s_1b_2)J_1 + (s_3b_1 + s_2b_2 + s_1b_3)J_1^2 + \cdots + (s_{e_1}b_1 + s_{e_1-1}b_2 + \cdots + s_2b_{e_1-1} + s_1b_{e_1})J_1^{e_1-1}].$$

Since $|B| \neq 0$, we know $b_1 \neq 0$. Hence we may choose $s_1 = 1/b_1$ and s_j, $j=2, 3, \cdots, e_1$, so that the coefficient of J_1^{j-1} in the above expression is zero. Then

$$\bar{S}'BS = \begin{Vmatrix} 0 & E_1 \\ E_1 & 0 \end{Vmatrix} = B_c.$$

Case III. Elementary divisors:

$$(\lambda - \lambda_1)^{e_1},\ (\lambda - \lambda_1)^{e_2},\ (\lambda - \lambda_1)^{e_3},\ \cdots,\ (\lambda - \lambda_1)^{e_k},\quad e_1 > e_2 > \cdots > e_k.$$

In (7) we may take

$\mathfrak{S}_{11}$ of the form described for S in §1 with $k-1$ diagonal blocks and $\frac{1}{2}(k-1)(k-2)$ blocks above the diagonal. The $\frac{1}{2}(k-1)(k-2)$ blocks below the diagonal are taken to be zero;

$\mathfrak{S}_{12}$ a matrix of dimensions $(e_1+e_2+\cdots+e_{k-1})\times e_k$ made up of matrices, each above the next, of the form $I_{ik}S_{ik}(J_k)$, $i=1, 2, \cdots, (k-1)$;

$\mathfrak{B}_{11}$ of the form described for B in §1 with $k-1$ diagonal blocks and $\frac{1}{2}(k-1)(k-2)$ blocks above and also below the diagonal;

$\mathfrak{B}_{12}$ a matrix of dimensions $(e_1+e_2+\cdots+e_{k-1})\times e_k$ made up of matrices, each above the next, of the form $E_iI_{ik}B_{ik}(J_k)$, $i=1, 2, \cdots, (k-1)$;

$\mathfrak{B}_{22} = E_kB_{kk}(J_k)$.

We know that $|\mathfrak{B}_{11}| \neq 0$ and $|\mathfrak{B}_{22}| \neq 0$ since $|B| \neq 0$ and $e_1 > e_k$. Let us assume that $\mathfrak{S}_{11}$ can be chosen so that $\bar{\mathfrak{S}}_{11}'\mathfrak{B}_{11}\mathfrak{S}_{11}$ is of the desired form, i.e., a matrix of $k-1$ diagonal blocks of the form ϵ_iE_i. (See Case I for start of induction.) Since $\mathfrak{B}_{11}\mathfrak{S}_{12}$ is of the same form as $\mathfrak{B}_{12}$, we may choose $\mathfrak{S}_{12}$ so that $\mathfrak{B}_{11}\mathfrak{S}_{12}+\mathfrak{B}_{12}=0$.

$$\bar{\mathfrak{B}}_{12}'\mathfrak{S}_{12} = \sum_{i=1}^{k-1} \bar{B}_{ik}(J_k')I_{ik}'E_iI_{ik}S_{ik}(J_k) = \sum_{i=1}^{k-1} E_k\bar{B}_{ik}(J_k)J_k^{e_i-e_k}S_{ik}(J_k) = E_kP_1(J_k),$$

where P_1 has no constant term. Since $\mathfrak{B}_{22}+E_kP_1(J_k)$ is therefore non-singular, we employ a further transformation R of form (8) in which we may, according to Case I, choose $\mathfrak{R}_{22}$ so that

$$\bar{\mathfrak{R}}_{22}'\,[\mathfrak{B}_{22} + E_kP_1(J_k)]\mathfrak{R}_{22} = \epsilon_kE_k.$$

Calling $S_1 = SR$, we have $\bar{S}_1'BS_1 = B_c$.

Evidently this method would cover the situation with $e_1 \geqq e_2 \geqq \cdots \geqq e_k$ if it were known that $|\mathfrak{B}_{11}| \neq 0$, $|\mathfrak{B}_{22}| \neq 0$, and that $\mathfrak{B}_{12}$ involved only polynomials without constant terms.

Case IV. Elementary divisors:

$$(\lambda-\lambda_1)^{e_1},\ (\lambda-\bar{\lambda}_1)^{e_1},\ (\lambda-\lambda_1)^{e_2},\ (\lambda-\bar{\lambda}_1)^{e_2},\ \cdots,\ (\lambda-\lambda_1)^{e_k},\ (\lambda-\bar{\lambda}_1)^{e_k},$$
$$e_1 > e_2 \cdots > e_k,\ \lambda_1 \neq \bar{\lambda}_1.$$

This case may be handled by exactly the same method as that used in Case III, double blocks being dealt with in place of single ones. The note at the end of Case III is also valid here.

Case V. Elementary divisors:

$$(\lambda-\lambda_1)^{e_1},\ (\lambda-\lambda_1)^{e_2},\ \cdots,\ (\lambda-\lambda_1)^{e_k}, \qquad e_1 = e_2 = \cdots = e_k.$$

S contains k^2 blocks of form

$$S_{ij}(J_1) = \sum_{n=1}^{e_1} s_{ijn}J_1^{n-1} \qquad (i, j = 1, 2, \cdots, k).$$

B contains k^2 blocks of form

$$E_1B_{ij}(J_1) = E_1\sum_{n=1}^{e_1} s_{ijn}J_1^{n-1} \qquad (i, j = 1, 2, \cdots, k).$$

Hence $\bar{S}'BS$ contains k^2 blocks of form $E_1P_{ij}(J_1)$ in which the coefficients of J_1^0 in the polynomials P_{ij} involve only s_{ijn} and b_{ijn} with $n=1$. Since

$$\pm\begin{vmatrix} b_{111} & b_{121} \cdots b_{1k1} \\ \bar{b}_{121} & b_{221} \cdots b_{2k1} \\ \cdot & \cdot \ \cdots \ \cdot \\ \cdot & \cdot \ \cdots \ \cdot \\ \bar{b}_{1k1} & \bar{b}_{2k1} \cdots b_{kk1} \end{vmatrix}^{e_1} = |\,B\,| \neq 0,$$

we may choose s_{ij1} so that

$$\begin{Vmatrix} \bar{s}_{111} & \bar{s}_{211} \cdots \bar{s}_{k11} \\ \bar{s}_{121} & \bar{s}_{221} \cdots \bar{s}_{k21} \\ \cdot & \cdot \ \cdots \ \cdot \\ \bar{s}_{1k1} & \bar{s}_{2k1} \cdots \bar{s}_{kk1} \end{Vmatrix} \cdot \begin{Vmatrix} b_{111} & b_{121} \cdots b_{1k1} \\ \bar{b}_{121} & b_{221} \cdots b_{2k1} \\ \cdot & \cdot \ \cdots \ \cdot \\ \bar{b}_{1k1} & \bar{b}_{2k1} \cdots b_{kk1} \end{Vmatrix} \cdot \begin{Vmatrix} s_{111} & s_{121} \cdots s_{1k1} \\ s_{211} & s_{221} \cdots s_{2k1} \\ \cdot & \cdot \ \cdots \ \cdot \\ s_{k11} & s_{k21} \cdots s_{kk1} \end{Vmatrix} = \begin{Vmatrix} \delta_1 & & 0 \\ & \delta_2 & \\ & & \ddots \\ 0 & & \delta_k \end{Vmatrix},$$

where $\delta_i = \pm 1$. Choosing $s_{ijn}=0$ for $n \neq 1$, we then apply the method of Case III to $\bar{S}'BS = B_1$, the latter being such that $|\mathcal{B}_{11}| \neq 0$, $|\mathcal{B}_{22}| \neq 0$, and $\mathcal{B}_{12}$ involves polynomials in J_1 without constant terms. (See the remark at the end of Case III.)

Case VI. Elementary divisors:

$$(\lambda-\lambda_1)^{e_1},\ (\lambda-\bar{\lambda}_1)^{e_1},\ (\lambda-\lambda_1)^{e_2},\ (\lambda-\bar{\lambda}_1)^{e_2},\ \cdots,\ (\lambda-\lambda_1)^{e_k},\ (\lambda-\bar{\lambda}_1)^{e_k};$$
$$e_1 = e_2 = \cdots = e_k;\ \lambda_1 \neq \bar{\lambda}_1.$$

S contains k^2 blocks of form

$$\begin{vmatrix} S_{ij1}(J_1) & 0 \\ 0 & S_{ij2}(J_1) \end{vmatrix} \qquad (i, j = 1, 2, \cdots, k),$$

B contains k^2 blocks of form

$$\begin{vmatrix} 0 & E_1B_{ij2}(J_1) \\ E_1B_{ij1}(J_1) & 0 \end{vmatrix} \qquad (i, j = 1, 2, \cdots, k),$$

where

$$S_{ijm}(J_1) = \sum_{n=1}^{e_1} s_{ijmn}J_1^{n-1} \qquad (m = 1, 2),$$

and

$$B_{ij2}(J_1) = \overline{B}_{ji1}(J_1') = \sum_{n=1}^{e_1} b_{ij2n}J_1^{n-1}.$$

Then $\overline{S}'BS$ contains k^2 blocks of form

$$\begin{Vmatrix} 0 & E_1P_{ij2}(J_1) \\ E_1P_{ij1}(J_1) & 0 \end{Vmatrix} \qquad (i, j = 1, 2, \cdots, k),$$

in which the coefficients of J_1^0 in the polynomials P_{ijm} involve only s_{ijmn} and b_{ijmn} with $n=1$. Choose $s_{ijmn}=0$, $n\neq 1$. Using the fact that $|B|\neq 0$, it can be shown by a method similar to that used in Case V that we may choose $s_{ij11}=s_{ij21}$ in such a way that $\overline{S}'BS=S_1$ can be handled by the method of Case IV. (See remark at end of Case IV.)

Case VII. Elementary divisors: $(\lambda-\lambda_1)^j$, and

Case VIII. Elementary divisors: $(\lambda-\lambda_1)^j$, $(\lambda-\overline{\lambda}_1)^j$, $\lambda_1\neq\overline{\lambda}_1$,
in which, for both cases, j takes on the values $e_1, e_2, \cdots, e_{\beta_k}$, where

$$e_1 = e_2 = e_3 = \cdots = e_{\beta_1} > e_{\beta_1+1} = e_{\beta_1+2} = \cdots = e_{\beta_2} > e_{\beta_2+1}$$
$$= \cdots = e_{\beta_3} > e_{\beta_3+1} = \cdots > e_{\beta_{k-1}+1} = \cdots = e_{\beta_k}.$$

These cases may be handled as were Cases III and IV, they being made to depend on V and VI as III and IV depended on I and II. Clusters of blocks are dealt with in place of single blocks.

3. **Conditions for equivalence.** Consider the pair of Hermitian matrices A, B, where $|B|\neq 0$. The matrix $B(B^{-1}A-\lambda I)^n$ is Hermitian for any real λ and positive integral n since it is a sum of matrices of the form $B(B^{-1}A)^j$, which is easily shown to be Hermitian for any positive integral j.

Referring to the σ_i defined at the beginning of §2, the following theorem may be stated:

THEOREM 2. *In the non-singular case, two pairs of Hermitian matrices are equivalent if and only if they have the same elementary divisors and also the same* σ_i.

We shall prove this theorem in a more illuminating form already stated above:

THEOREM 2a. *Two pairs of Hermitian matrices* A, B *and* C, D, *where* $|B| \neq 0$ *and* $|D| \neq 0$, *are equivalent if and only if*

(1) *they have the same elementary divisors,*

and

(2) *the matrices* $B(B^{-1}A-\lambda I)^n$ *and* $D(D^{-1}C-\lambda I)^n$ *have the same index for all positive integral* n *and real* λ.

Necessity. The necessity of (1) is known from classical theory.

Suppose there exists a non-singular P so that $\bar{P}'AP=C$ and $\bar{P}'BP=D$. Then

$$\begin{aligned} D(D^{-1}C - \lambda I)^n &= \bar{P}'BP(P^{-1}B^{-1}\bar{P}'^{-1}\bar{P}'AP - \lambda I)^n \\ &= \bar{P}'BP[P^{-1}(B^{-1}A - \lambda I)P]^n \\ &= \bar{P}'BP[P^{-1}(B^{-1}A - \lambda I)^n P] \\ &= \bar{P}'[B(B^{-1}A - \lambda I)^n]P, \end{aligned}$$

whence the necessity of (2) follows.

Sufficiency. We may assume the two pairs are in their canonical forms. Because of (1), these canonical pairs are the same except possibly in their ϵ_i. The coincidence of the σ_i is a sufficient condition for equivalence since equal ϵ_i could be made to correspond by a proper interchange of blocks. We have then merely to prove that for any variation in the σ_i of the canonical pairs there will exist a real λ and a positive integral n such that the indices of $B(B^{-1}A-\lambda I)^n$ and $D(D^{-1}C-\lambda I)^n$ are not the same.

Since A and B are in canonical form, $B(B^{-1}A-\lambda I)^n$ is a diagonal block matrix like A and B, the real block $\epsilon_i E_i(\lambda_i J_i^0+J_i)$ of A corresponding to the block $\epsilon_i E_i[(\lambda_i-\lambda)J_i^0+J_i]^n$ of $B(B^{-1}A-\lambda I)^n$. Obviously, the index of $B(B^{-1}A-\lambda I)^n$ will be the sum of the indices of its blocks.

Consider first the case in which all the elementary divisors involve the same real root λ_1, the exponents being

$$\begin{aligned} e_1 = e_2 = \cdots = e_{\beta_1} > e_{\beta_1+1} = e_{\beta_1+2} = \cdots = e_{\beta_2} > e_{\beta_2+1} \\ = \cdots = e_{\beta_3} > e_{\beta_3+1} = \cdots > \cdots = e_{\beta_u}. \end{aligned}$$

Let e_{β_s} be the largest exponent such that

$$\sigma_s = \sum_{j=\beta_{s-1}+1}^{\beta_s} \epsilon_j$$

of one pair is different from

$$\sigma_s' = \sum_{j=\beta_{s-1}+1}^{\beta_s} \epsilon_j'$$

of the other pair. Choose $\lambda = \lambda_1$ and $n = e_{\beta_s} - 1$. Blocks $\epsilon_i E_i J_i^{e_{\beta_s}-1}$ will have the same indices in each pair when $i < \beta_s$, since we are assuming $\sigma_j = \sigma_j'$, $j < s$. Also blocks $\epsilon_i E_i J_i^{e_{\beta_s}-1}$ will have the same indices in each pair when $i > \beta_s$, for then the blocks are entirely zeros. However block $\epsilon_\beta E_{\beta_s} J_{\beta_s}^{e_{\beta_s}-1}$ has index σ_s in one pair and $\sigma_s' \neq \sigma_s$ in the other.

Since the index of block $\epsilon_i E_i[(\lambda_i - \lambda)J_i^0 + J_i]^n$ for $\lambda_i \neq \lambda$ depends on the sign of $\epsilon_i(\lambda_i - \lambda)^n$, it is the same for any even n, and the same for any odd n. Consider any general set of elementary divisors, and call λ_1 a root such that the σ_i connected with the set of elementary divisors involving λ_1 are different in the two pairs. Choose $\lambda = \lambda_1$ and choose n as above so that the total indices of the corresponding blocks of the two pairs are different. If the remaining blocks have the same total indices for each pair for this λ and n, we are done. If not, increase n by 2, whence the difference of the indices of blocks involving λ_1 disappears since the blocks causing the difference become entirely zeros, but the difference of the indices of blocks not involving λ_1 remains unchanged.

This proof shows that condition (2) in the above theorem could be replaced by the following condition (2′) which is more easily applied but less easily stated:

(2′) *the matrices $B(B^{-1}A - \lambda I)^n$ and $D(D^{-1}C - \lambda I)^n$ have the same index for all positive integral n which are less than or equal to the order of the matrices involved and of the form $e_i \pm 1$, where the e_i are the exponents of the real elementary divisors of the pairs, and for all λ which are real roots λ_i involved in these elementary divisors.*

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ON THE EQUIVALENCE OF QUADRICS IN m-AFFINE n-SPACE AND ITS RELATION TO THE EQUIVALENCE OF $2m$-POLE NETWORKS*

BY

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1. **Introduction.** The recent work of Cauer† and others† in the study of equivalent $2m$-pole networks has given considerable importance to the matric study of quadratic forms under the real m-affine non-singular group of linear transformations T.

It is the purpose of this paper to exhibit a system of integer, matric, and algebraic invariants of the matrix A of the n-ary quadratic form F, under the m-affine non-singular group of linear transformations T, by means of which necessary and sufficient conditions for the m-affine congruence with respect to T of two matrices A and B as well as the equivalence of the two corresponding forms F and G may be given, where the elements of A and T belong to a field D.

The reduction of A (and F) to canonical forms is indicated, the case where $m=2$ being exhibited in detail, because of its interest in connection with the 4-pole equivalence problem in network theory. The application of these results to the geometry of the locus $F=0$ is also indicated. If $m=0$, T is projective. In case $m=1$, and the field is real, the results of a previous paper‡ appear, in which the matrix A of the real quadric F was shown to have 4 integer invariants (arithmetic invariants) under the real 1-affine group, which are sufficient to give a complete separation of quadrics into types.

In the closing paragraph, the relation of the present paper to the theory of $2m$-pole linear electrical networks is discussed. Each invariant of the network matrix A may be given a physical interpretation and the various theorems of the present paper become theorems relating to the network. So that this paper deals essentially with the mathematical structure underlying the theory relating to electrical networks.

2. **Invariants.** Consider§ the symmetric matrix

$$A \equiv (a_{ij}) \qquad (i, j = 1, \cdots, n), \tag{2.1}$$

of the quadric

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† See references at end of paper, under (1) and (2).

‡ See references at end of paper, under (3).

§ See references, under (3).

$$F \equiv \sum_{i,j=1}^{n} a_{ij}x_ix_j, \tag{2.2}$$

under the non-singular m-affine transformations

$$\begin{aligned} x_i &= x_i' && (i = 1, \cdots, m), \\ x_j &= \sum_{k=1}^{n} b_{jk}x_k' && (j = m+1, \cdots, n), \end{aligned} \tag{2.3}$$

of matrix

$$T \equiv \left[\begin{array}{ccc|ccc} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hline b_{m+1,1} & \cdots & b_{m+1,m} & b_{m+1,m+1} & \cdots & b_{m+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} & b_{n,m+1} & \cdots & b_{nn} \end{array}\right], \quad d(T) \neq 0, \tag{2.4}$$

where $d(T)$ is the determinant of T, and where the elements of A and T belong to a field D.

Under T, A becomes

$$\bar{A} = T' \cdot A \cdot T, \tag{2.5}$$

where T' is the transpose of T.

If $\bar{A}_{r_1 \cdots r_s}$ is $\bar{A}$ with the $r_1, \cdots, r_s$ rows and columns deleted, the r_i's being all distinct and less than or equal to m, then

$$\bar{A}_{r_1 \cdots r_s} = T'_{r_1 \cdots r_s} \cdot A_{r_1 \cdots r_s} \cdot T_{r_1 \cdots r_s}, \tag{2.6}$$

where $T_{r_1 \cdots r_s}$ and $A_{r_1 \cdots r_s}$ are T and A, respectively, with the $r_1, \cdots, r_s$ rows and columns deleted. Thus $A_{r_1 \cdots r_s}$ is an *invariant matrix* of A under T in the sense that $\bar{A}_{r_1 \cdots r_s}$ can be found either (i) by transforming and then deleting the $r_1, \cdots, r_s$ rows and columns, or (ii) by deleting the $r_1, \cdots, r_s$ rows and columns of A and T and then transforming.

Let the *ranks* and *signatures* of $A_{r_1 \cdots r_s}$ be denoted by $\rho_{r_1 \cdots r_s}$, $\sigma_{r_1 \cdots r_s}$, respectively; the $\sigma_{r_1 \cdots r_s}$ being meaningless if $A_{r_1 \cdots r_s}$ cannot be reduced to a diagonal matrix, or if the field is not ordered.

As is well known,*

* See list of references, under (4).

THEOREM 2.1. *The $\rho, \rho_1, \cdots, \rho_{r_1 \cdots r_s}$ are integer invariants* of $A, A_1, \cdots, A_{r_1 \cdots r_s}$, respectively, and hence of A and F. If the field D is ordered, the $\sigma_{r_1 \cdots r_s}$ are integer invariants.*

By taking determinants of (2.6), it follows that

$$d(\overline{A}_{r_1 \cdots r_s}) = [d(T_{r_1 \cdots r_s})]^2 \cdot d(A_{r_1 \cdots r_s}), \tag{2.7}$$

whence

THEOREM 2.2. *The $d(A)$, $d(A_1)$, $\cdots$, $d(A_{r_1 \cdots r_s})$ are relative invariants of A and F under T.*

Likewise, it is easy to show

THEOREM 2.3. *If rows $r_1 \cdots r_t$ and columns $s_1 \cdots s_t$ be deleted from A and yield $A_{r_1 \cdots r_t}^{s_1 \cdots s_t}$; and if $T_{s_1 \cdots s_t} \equiv T_{s_1 \cdots s_t}^{s_1 \cdots s_t}$ be T with the $s_1 \cdots s_t$ rows and columns deleted, and $T'_{r_1 \cdots r_t}$ be T' with the $r_1 \cdots r_t$ rows and columns deleted, where all the s_i and r_i are less than or equal to m, then*

$$\overline{A}_{r_1 \cdots r_t}^{s_1 \cdots s_t} = T'_{r_1 \cdots r_t} \cdot A_{r_1 \cdots r_t}^{s_1 \cdots s_t} \cdot T_{s_1 \cdots s_t} \tag{2.8}$$

is an invariant matrix; and $d(A_{r_1 \cdots r_t}^{s_1 \cdots s_t})$ is a relative invariant of A and F under T.

Since $d(T_{r_1 \cdots r_t}) = d(T_{s_1 \cdots s_t}) = d(T)$,

THEOREM 2.4. *If R_1 and R_2 be any two of the above relative invariants, then*

$$I_{1,2} \equiv R_1/R_2 \tag{2.9}$$

is an absolute invariant of A and F under T.

(In case $I_{1,2}$ is indeterminate, recourse may be had to a limiting process to define and determine $I_{1,2}$.)

Thus, with each form F, of matrix A, there is associated a set of matric, integer, relative and absolute invariants.

If the transformation $T_{1, \cdots, q-1, q+1, \cdots, m}$ is insufficient to reduce $A_{1, \cdots, q-1, q+1, \cdots, m}$ to a diagonal form, then $A_{1, \cdots, q-1, q+1, \cdots, m}$ is *parabolic* and A is *q-parabolic.*

As in the paper cited under (3) in list of references,

THEOREM 2.5. *A necessary and sufficient condition that A be q-parabolic is that $\rho_{1 \cdots m} - \rho_{1, \cdots, q-1, q+1, \cdots, m} = 2$.*

Two matrices A and B in D are said to be *m-affine congruent* if and only if there exists a non-singular matrix T of type (2.4) in D such that

$$A = T' \cdot B \cdot T.$$

---

* Integer invariants are known as arithmetic invariants in paper (3) in list of references. The term *integer invariant* was adopted at the suggestion of Professor Arthur B. Coble.

3. **Reduction to canonical forms.** It is known* that there exists a transformation T which reduces F to a form for which $A_{1\cdots m-1}$ becomes (see paper (3) in list of references for proof)

$$\left\{\begin{array}{c|ccccc} a_{mm} & 0 & 0 & \cdots 0 & & 0 \\ \hline 0 & \delta_{m+1} & 0 & \cdots 0 & & 0 \\ 0 & 0 & \delta_{m+2} & \cdots & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \delta_{n-1} & 0 \\ 0 & 0 & 0 & \cdots 0 & & \delta_n \end{array}\right\}, \text{ if } \nu \equiv \rho_{1\cdots m-1} - \rho_{1\cdots m} \neq 2, \tag{3.1}$$

and

$$\left\{\begin{array}{c|ccccc} 0 & 0 & 0 & \cdots 0 & & 1 \\ \hline 0 & \delta_{m+1} & 0 & \cdots 0 & & 0 \\ 0 & 0 & \delta_{m+2} & \cdots & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \delta_{n-1} & 0 \\ 1 & 0 & 0 & \cdots 0 & & 0 \end{array}\right\}, \text{ if } \nu = 2. \tag{3.2}$$

If the field is real, each positive δ can be reduced to 1, and each negative δ to -1. The number of positive δ's in $A_{1\cdots m}$ is $(\rho_{1\cdots m}+\sigma_{1\cdots m})/2$ and the number of negative δ's is $(\rho_{1\cdots m}-\sigma_{1\cdots m})/2$, the remaining δ's being zero. If the field is algebraically closed, each non-zero δ can be reduced to 1. No further reduction of $A_{1\cdots m-1}$ is possible. The *parameter* a_{mm} is an absolute invariant (by Theorem 2.4).

Thus,

THEOREM 3.1. *Every symmetric matrix $A_{1\cdots m-1}$ of A with elements in a field D not of characteristic 2 is m-affine congruent in D with a diagonal matrix* (3.1) *if $\nu\neq 2$, and with a parabolic matrix* (3.2) *if $\nu=2$, the number of non-zero δ's in $A_{1\cdots m}$ being equal to the rank $\rho_{1\cdots m}$ of $A_{1\cdots m}$.*

THEOREM 3.2. *Every symmetric matrix $A_{1\cdots m-1}$ of A with elements in a real field R is m-affine congruent in R with a diagonal matrix* (3.1) *if $\nu\neq 2$, and with a parabolic matrix* (3.2) *if $\nu=2$; the number of positive δ's being $(\rho_{1\cdots m}+\sigma_{1\cdots m})/2$ and the number of negative δ's being $(\rho_{1\cdots m}-\sigma_{1\cdots m})/2$.*

---

* See (4) in list of references.

THEOREM 3.3. *A necessary and sufficient condition for the m-affine congruence of two matrices* $A^{(1)}_{1\cdots m-1}$ *and* $A^{(2)}_{1\cdots m-1}$ *of the symmetric matrices* $A^{(1)}$ *and* $A^{(2)}$, *whose elements belong to any algebraically closed field, is that these matrices have the same ranks* $\rho^{(1)}_{1\cdots m-1}, \rho^{(1)}_{1\cdots m}$, *and* $\rho^{(2)}_{1\cdots m-1}, \rho^{(2)}_{1\cdots m}$, *respectively, with* $\nu^{(1)}=\nu^{(2)}=2$, *and if* $\nu^{(1)}, \nu^{(2)}\neq 2$, *that in addition to the above, their parameters* $a^{(1)}_{mm}$ *and* $a^{(2)}_{mm}$ *be identical.*

THEOREM 3.4. *Two matrices* $A^{(1)}_{1\cdots m-1}$ *and* $A^{(2)}_{1\cdots m-1}$ *of the symmetric matrices* $A^{(1)}$ *and* $A^{(2)}$, *whose elements belong to a real field R, are m-affine congruent in R if and only if these matrices have the same ranks and signatures,* $\rho^{(1)}_{1\cdots m-1}, \rho^{(1)}_{1\cdots m}, \sigma^{(1)}_{1\cdots m-1}, \sigma^{(1)}_{1\cdots m}$ *and* $\rho^{(2)}_{1\cdots m-1}, \rho^{(2)}_{1\cdots m}, \sigma^{(2)}_{1\cdots m-1}, \sigma^{(2)}_{1\cdots m}$, *respectively, with* $\nu^{(1)}=\nu^{(2)}=2$; *and if* $\nu^{(1)}, \nu^{(2)}\neq 2$, *that in addition to the above, their parameters* $a^{(1)}_{mm}$ *and* $a^{(2)}_{mm}$ *be identical.*

THEOREM 3.5. *The quadratic form* $F_{1\cdots m-1}$ *of F can be reduced by a non-singular m-affine transformation to the form*

$$a_{mm}x_m^2 + \sum_{j=m+1}^{n} \delta_j x_j^2, \qquad \text{if } \nu \neq 2, \tag{3.3}$$

and to the form

$$2x_1x_n + \sum_{j=m+1}^{n} \delta_j x_j^2, \qquad \text{if } \nu = 2. \tag{3.4}$$

THEOREM 3.6. *A necessary and sufficient condition for the m-affine equivalence of two quadratic forms* $F^{(1)}_{1\cdots m-1}, F^{(2)}_{1\cdots m-1}$ *of forms* $F^{(1)}$ *and* $F^{(2)}$, *with elements in a field D, is that their matrices* $A^{(1)}_{1\cdots m-1}$ *and* $A^{(2)}_{1\cdots m-1}$ *be m-affine congruent.*

The classification of quadratics $F_{1\cdots m-1}$ can now be made as in paper (3)* in terms of the parameter a_{mm}, and the ranks (and signatures) of $A_{1\cdots m-1}$.

The above theorems hold, in a like manner, for $A_{1,\cdots,q-1,q+1,\cdots,m}$.

As an aid to the reduction, in case D is real, it is agreed that the δ's will be so ordered that all the positive δ's are followed by all the negative δ's and then by the zero δ's. No loss of generality will result.

Case $\rho_{1\cdots m}=r-m-1$. Suppose that $\delta_r=\cdots=\delta_n=0$ for $r\geqq m+1$, that is, that $A_{1\cdots m}$ is of rank $(r-m-1)$. Transformation T with $b_{jj}=1$, $j=1, \cdots, n$; $a_{sr}+b_{sr}\cdot\delta_s=0$, $r=1,\cdots, m-1$, $s=m+1,\cdots, n$; $b_{sr}=0$ for $s=1,\cdots, m$ and $r=1,\cdots, n$; $b_{sr}=0$ for $s=m+1,\cdots, n$, $r=m,\cdots, n$, $r\neq s$; reduces A to the forms

* See list of references at end of paper.

$$
(3.5)\qquad \left(\begin{array}{cccc|ccc|ccc}
a_{11} & \cdots & a_{1,m-1} & a_{1m} & 0 & \cdots & 0 & a_{1r} & \cdots & a_{1n} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\cdot & & \cdot & a_{m-1,m} & 0 & \cdots & 0 & a_{m-1,r} & \cdots & a_{m-1,n} \\
a_{m1} & \cdots & a_{m,m-1} & a_{mm} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hline
0 & \cdots & 0 & 0 & \delta_{m+1} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & \delta_{r-1} & 0 & \cdots & 0 \\
\hline
a_{r1} & \cdots & a_{r,m-1} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
a_{n1} & \cdots & a_{n,m-1} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right), \quad \text{if } \nu \neq 2,
$$

and

$$
(3.6)\qquad \left(\begin{array}{cccc|ccc|ccc}
a_{11} & \cdots & a_{1,m-1} & a_{1m} & 0 & \cdots & 0 & a_{1r} & \cdots & a_{1n} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\cdot & & \cdot & a_{m-1,m} & 0 & \cdots & 0 & a_{m-1,r} & & a_{m-1,n} \\
a_{m1} & \cdots & a_{m,m-1} & 0 & 0 & \cdots & 0 & 0 & \cdots 0 & 1 \\
\hline
0 & \cdots & 0 & 0 & \delta_{m+1} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & \delta_{r-1} & 0 & \cdots & 0 \\
\hline
a_{r1} & \cdots & a_{r,m-1} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\cdot & & \cdot & 0 & & & & & & \\
a_{n1} & \cdots & a_{n,m-1} & 1 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right), \quad \text{if } \nu = 2.
$$

The case where $m=2$ will now be considered in detail. (This case is of interest in the 4-pole equivalence problem.)

Case $m=2$; $\nu=\rho_1-\rho_{12}\neq 2$. If $a_{1n}\neq 0$, transformation T with

$$
a_{1n}\cdot x_n = \left(-\frac{a_{11}}{2}\bar{x}_1 - a_{12}\bar{x}_2 - a_{1r}\bar{x}_r - a_{1,r+1}\cdot\bar{x}_{r+1} - \cdots - a_{1,n-1}\bar{x}_{n-1} + \bar{x}_n\right),
$$

$$
x_j = \bar{x}_j \qquad (j = 1, \cdots, n-1),
$$

reduces (3.5), $m=2$, to the form

$$
(3.7)\qquad
\left(\begin{array}{cc|ccc|cccc}
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1\\
0 & a_{22} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
\hline
0 & 0 & \delta_3 & \cdots & 0 & 0 & \cdots & & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & & \vdots\\
0 & 0 & 0 & \cdots & \delta_{r-1} & 0 & \cdots & & 0\\
\hline
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & & 0\\
\vdots & \vdots & \vdots & & \vdots & \vdots & & & \vdots\\
0 & 0 & & & & & & & \\
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & & 0
\end{array}\right)
\equiv
\left(\begin{array}{cc|c|cccc}
0 & 0 & \multirow{2}{*}{0} & 0 & \cdots & 0 & 1\\
0 & a_{22} & & 0 & \cdots & 0 & 0\\
\hline
\multicolumn{2}{c|}{0} & \delta & \multicolumn{4}{c}{0}\\
\hline
0 & 0 & & & & & \\
\vdots & \vdots & 0 & \multicolumn{4}{c}{0}\\
0 & 0 & & & & & \\
1 & 0 & & & & &
\end{array}\right).
$$

It is easy to show that no further reduction is possible which preserves this form.

If $a_{1n}=a_{1,n-1}=\cdots=a_{1,n-k-1}=0$, $a_{1,n-k}\neq 0$, $(n-k)\geqq r$, (3.5) with $m=2$ may be reduced to a form similar to (3.7), whence by a simple transformation to form (3.7).

If $a_{1n}=\cdots=a_{1r}=0$, then A becomes

$$
(3.8)\qquad
\left(\begin{array}{cc|c|c}
a_{11} & a_{12} & \multirow{2}{*}{0} & \multirow{2}{*}{0}\\
a_{21} & a_{22} & & \\
\hline
\multicolumn{2}{c|}{0} & \delta & 0\\
\hline
\multicolumn{2}{c|}{0} & 0 & 0
\end{array}\right).
$$

No further reduction is possible.

Case $m=2$, $\nu=2$. If $a_{1,n-1}\neq 0$, the transformation

$$
\begin{aligned}
a_{1,n-1}x_{n-1} &= (-\ a_{11}/2)\ \bar{x}_1 - a_{12}\bar{x}_2 - a_{1r}\bar{x}_r - a_{1,r+1}\bar{x}_{r+1}\\
&\qquad - \cdots - a_{1,n-2}\bar{x}_{n-2} + \bar{x}_{n-1} - a_{1n}\bar{x}_n),\\
x_j &= \bar{x}_j \qquad\qquad (j = 1, \cdots, n-2, n),
\end{aligned}
$$

reduces (3.6), with $m=2$, to the form

$$
(3.9)\qquad
\left(\begin{array}{cc|c|ccccc}
\multicolumn{2}{c|}{\multirow{2}{*}{0}} & \multirow{2}{*}{0} & 0 & \cdots & 0 & 1 & 0\\
 & & & 0 & \cdots & 0 & 0 & 1\\
\hline
\multicolumn{2}{c|}{0} & \delta & \multicolumn{5}{c}{0}\\
\hline
0 & 0 & & & & & & \\
\vdots & \vdots & & & & & & \\
0 & 0 & 0 & \multicolumn{5}{c}{0}\\
1 & 0 & & & & & & \\
0 & 1 & & & & & &
\end{array}\right),
$$

and no further reduction is possible.

If $a_{1,n-1}= \cdots =a_{1,k-1}=0$, $a_{1k}\neq 0$, $k\geqq r$, (3.6) may be reduced to a form similar to (3.9), whence to form (3.9).

If $a_{1,n-1}= \cdots =a_{1r}=0$, then (3.6), with $m=2$, becomes

$$
\left(\begin{array}{cc|c|cccc}
a_{11} & a_{12} & & 0 & \cdots & 0 & a_{1n} \\
a_{21} & 0 & 0 & 0 & \cdots & 0 & 1 \\
\hline
\multicolumn{2}{c|}{0} & \delta & \multicolumn{4}{c}{0} \\
\hline
0 & 0 & & & & & \\
\vdots & \vdots & & & & & \\
\vdots & \vdots & 0 & \multicolumn{4}{c}{0} \\
0 & 0 & & & & & \\
a_{n1} & 1 & & & & &
\end{array}\right). \tag{3.10}
$$

The transformation $x_n=-a_{21}\bar{x}_1+\bar{x}_n$ reduces (3.10) to the form

$$
\left(\begin{array}{cc|c|cccc}
a_{11} & 0 & & 0 & \cdots & 0 & a_{1n} \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
\hline
\multicolumn{2}{c|}{0} & \delta & \multicolumn{4}{c}{0} \\
\hline
0 & 0 & & & & & \\
\vdots & \vdots & & & & & \\
\vdots & \vdots & 0 & \multicolumn{4}{c}{0} \\
0 & 0 & & & & & \\
a_{n1} & 1 & & & & &
\end{array}\right), \quad \text{if } a_{1n} \neq 0, \tag{3.11}
$$

and to the form

$$
\left(\begin{array}{cc|c|cccc}
a_{11} & 0 & & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
\hline
\multicolumn{2}{c|}{0} & \delta & \multicolumn{4}{c}{0} \\
\hline
0 & 0 & & & & & \\
\vdots & \vdots & & & & & \\
\vdots & \vdots & 0 & \multicolumn{4}{c}{0} \\
0 & 0 & & & & & \\
0 & 1 & & & & &
\end{array}\right), \quad \text{if } a_{1n} = 0. \tag{3.12}
$$

No further reduction is possible.

Each of the forms (3.7), (3.8), (3.9), (3.11), (3.12) may be subdivided, in case of the ordered (and real) field, according to the signature of A_{12} (which is of rank $(r-3)$). For each value of $r=3, 4, \cdots, (n+1)$, there corresponds a set of forms (3.7), $\cdots$, (3.12). By Theorem 2.4, the parameters a_{11}, a_{12}, a_{22},

a_{1n} are absolute invariants.* It can be shown that the various forms thus obtained cannot be reduced any further with preservation of the invariance of their forms. The forms thus obtained are called *canonical forms*. The various parameters $a_{11}, \cdots, a_{1n}$ may or may not vanish, the conditions for this being indicated in the following table. The separation into forms can be made according to the ranks (and signatures) of A as indicated.

CLASSIFICATION OF MATRIX A (AND FORM F) FOR CASE $m=2$

TABLE I

$\rho_{12}=r-3,\ r=3, 4, \cdots, n+1$

$\rho_1-\rho_{12}$	$\rho_2-\rho_{12}$	ρ	ρ_1	ρ_2	ρ_1^2	a_{11}	a_{12}	a_{22}	a_{1n}	Form
$\neq 2$	2		$r-2$					$\neq 0$		3.7
$\neq 2$	2		$r-3$					$=0$		3.7
$\neq 2$	$\neq 2$		$r-2$	$r-2$	$r-2$	$\neq 0$	$\neq 0$	$\neq 0$		3.8
$\neq 2$	$\neq 2$		$r-3$	$r-2$	$r-2$	$\neq 0$	$\neq 0$	$=0$		3.8
$\neq 2$	$\neq 2$		$r-3$	$r-2$	$r-3$	$\neq 0$	$=0$	$=0$		3.8
$\neq 2$	$\neq 2$		$r-2$	$r-2$	$r-3$	$\neq 0$	$=0$	$\neq 0$		3.8
$\neq 2$	$\neq 2$		$r-2$	$r-3$	$r-2$	$=0$	$\neq 0$	$\neq 0$		3.8
$\neq 2$	$\neq 2$		$r-3$	$r-3$	$r-2$	$=0$	$\neq 0$	$=0$		3.8
$\neq 2$	$\neq 2$		$r-3$	$r-3$	$r-3$	$=0$	$=0$	$=0$		3.8
$\neq 2$	$\neq 2$		$r-2$	$r-3$	$r-3$	$=0$	$=0$	$\neq 0$		3.8
2	2	$r+1$								3.9
2	2	r				$\neq 0$			$(\neq 0)$	3.11
2	2	$r-1$				$=0$			$(\neq 0)$	3.11
2	$\neq 2$	r		$(r-2)$		$\neq 0$				3.12
2	$\neq 2$	$r-1$		$(r-3)$		$=0$				3.12

Each form is subdivided according to signature of A_{12}, if the field is real.

Case $m=m$. The reduction of A to canonical form for the m-affine case is done in a manner similar to that used above in the case where $m=2$. Reduce A to the forms for which $A_{1\cdots m-2}$ assume the canonical forms exhibited above for $m=2$. Continue the reduction by T, as was done above, until no further reduction of $A_{1\cdots m-3}$ can be made. Next, repeat the operations on A, by means of T, with respect to $A_{1\cdots m-4}$, etc., until no further reduction is possible. In this manner certain canonical forms $C_1, C_2, \cdots$ for matrix A are found, with corresponding forms $g_1, g_2, \cdots$ for the form F. The following theorems are evident:

* Suppose $r=n+1$, and F belongs to form (3.8). Then

$$I \equiv d(A_2)/d(A_{12}) = (a_{11}\delta_3 \cdots \delta_{r-1}\delta_r \cdots \delta_n)/(\delta_3 \cdots \delta_{r-1}\delta_r \cdots \delta_n).$$

Note that $\lim_{\delta\to 0} I = a_{11}$. If $d(A_2)/d(A_{12})$ is indeterminate, as is the case when $r<n+1, \cdots$, define I to be the $\lim_{\delta\to 0}[d(A_2)/d(A_{12})]$. Similarly, the other parameters $a_{11}, \cdots, a_{1n}$ may be handled. (See Theorem 2.4.)

THEOREM 3.7. *The matrix A of F can be reduced by a non-singular m-affine transformation to one of the forms C_1, C_2, $\cdots$, according to the ranks (and signatures in real field) of the invariant matrices of A, with corresponding canonical forms g_1, g_2, $\cdots$ for F, as indicated in Table* I. *In case $m=2$ these forms are*

$$f_1 = a_{22}x_2^2 + 2x_1x_n + \sum_{j=3}^{r-1} \delta_j x_j^2 ,$$

$$f_2 = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + \sum_{j=3}^{r-1} \delta_j x_j^2 ,$$

$$f_3 = 2x_1x_{n-1} + 2x_2x_n + \sum_{j=3}^{r-1} \delta_j x_j^2 ,$$

$$f_4 = a_{11}x_1^2 + 2a_{1n}x_1x_n + 2x_2x_n + \sum_{j=3}^{r-1} \delta_j x_j^2 ,$$

$$f_5 = a_{11}x_1^2 + 2x_2x_n + \sum_{j=3}^{r-1} \delta_j x_j^2 .$$

THEOREM 3.8. *A necessary and sufficient condition for the m-affine congruence of two matrices $A^{(1)}$ and $A^{(2)}$ whose elements belong to the real field is that their invariant matrices have the same ranks and signatures, $\rho^{(1)}$, $\rho_1^{(1)}$, $\cdots$, $\sigma_{12}^{(1)}$, $\cdots$ and $\rho^{(2)}$, $\rho_1^{(2)}$, $\cdots$, $\sigma_{12}^{(2)}$, $\cdots$, respectively; and that their parameters (in case they appear in the canonical form dictated by the ranks named above) $a_{11}^{(1)}$, $a_{12}^{(1)}$, $a_{22}^{(1)}$, $a_{1n}^{(1)}$, $\cdots$ and $a_{11}^{(2)}$, $a_{12}^{(2)}$, $a_{22}^{(2)}$, $a_{1n}^{(2)}$, $\cdots$ be identical. If the elements of $A^{(1)}$ and $A^{(2)}$ belong to an algebraically closed field, the above holds without the signatures.*

THEOREM 3.9. *A necessary and sufficient condition for the m-affine equivalence of quadratic forms $F^{(1)}$ and $F^{(2)}$, with elements in a field, is that their matrices $A^{(1)}$ and $A^{(2)}$ be m-affine congruent in that field.*

Theorems similar to Theorems 3.7, 3.8, 3.9 hold for any invariant matrix of A; e.g., Theorems 3.1 to 3.6.

4. **Application to the locus $F=0$.** In a manner similar to that given in paper (3),* a classification of locus $F=0$ can be made, the numerical value of the signature being used instead of the signature in case the field is real. With the various interpretations that can be placed upon the transformation T, a geometric study of the quadrics can be made.

5. **Relation to the theory of linear networks.** Consider a linear network of a finite number n of meshes. Let R_{st}, L_{st}, D_{st}, $s \neq t$ (real numbers) be the mutual circuit parameters (the resistance, inductance and *elastance*, respec-

* See list of references at end of paper.

tively), between mesh s and mesh t; and R_{ss}, L_{ss}, D_{ss} the total circuit parameters of mesh s. If $I_1, \cdots, I_m$ be the (complex numbers) currents through the m-terminal pairs of $2m$-poles and $E_1, \cdots, E_m$ (complex numbers) be the corresponding electromotive forces, subject to the restriction that the currents through the terminals be linearly independent, and if I_j be the current in jth mesh, then the Kirchoff equations of the network may be written

$$A(I) = (E), \tag{5.1}$$

where $(I)=(I_1, \cdots, I_n)$ and $(E)=(E_1, \cdots E_m, 0, \cdots, 0)$ are column matrices and $A=(a_{st})$ is the *network matrix*, with

$$a_{st} = R_{st} + L_{st}\lambda + D_{st}\lambda^{-1}, \tag{5.2}$$

and $\lambda = i\omega$, $i^2=-1$, the "imaginary frequency parameter."

The total power loss, instantaneous magnetic energy and electrostatic energy for the complete network are given by the (symmetric) quadratic forms

$$\mathcal{R} = \sum_{j,k}^{n} R_{jk}\cdot I_j\cdot I_k, \quad \mathcal{L} = \tfrac{1}{2}\sum_{j,k}^{n} L_{jk}\cdot I_j\cdot I_k, \quad \mathcal{D} = \tfrac{1}{2}\sum_{j,k}^{n} D_{jk}\cdot Q_j\cdot Q_k, \tag{5.3}$$

where Q_j is the instantaneous charge in mesh j, I_j is the corresponding current, and $L_{jk}=L_{kj}$, $D_{jk}=D_{kj}$, $R_{jk}=R_{kj}$.

The pencil of forms

$$\mathcal{A} = \mathcal{R} + 2\mathcal{L}\lambda + 2\mathcal{D}\lambda^{-1} \tag{5.4}$$

has the (energy) matrix

$$A = R + L\lambda + D\lambda^{-1}. \tag{5.5}$$

Thus the energy matrix is identical with the network matrix A of system (5.1).

If $d(A)\neq 0$, that is, A is of rank n, (5.1) may be solved for the currents

$$(I) = A^{-1}(E). \tag{5.6}$$

Let $(I)_m$ and $(E)_m$ denote (I) and (E), respectively, with all but the first m rows and columns deleted. If $Y\equiv(Y_{st})$ be A^{-1} with all but the first m rows and columns deleted, then

$$(I)_m = Y(E)_m. \tag{5.7}$$

Cauer has called Y a *characteristic coefficient matrix* of the network A. Two $2m$-pole linear networks are *equivalent* if, for all frequencies $(\omega=-\lambda i)$, they have equal characteristic coefficient matrices $Y(\lambda)$ (or $Z(\lambda)$); i.e., for all ω, they have equal electrical characteristics.

To each $2m$-pole linear network of matrix A, there corresponds a set of equivalent networks (1)* whose matrices (3)* may be obtained one from the other by a non-singular linear transformation (5)* of matrix T. If the driving-point currents (and charges) across the terminal pairs in meshes $1, \cdots, m$ be left invariant, T is m-affine.

It is evident that the methods and results given in the earlier parts of this paper are available for use in the study of linear networks (3).*

By Theorem 2.4, the elements of Y

$$Y_{st} = d(A_t^s)/d(A) \qquad (s, t = 1, \cdots, m) \tag{5.8}$$

are absolute invariants of A (and F) under non-singular m-affine linear transformations of matrix T. Y_{st}, $s \neq t$, is the short circuit transfer admittance between terminal pairs s and t and Y_{ss} is the short circuit driving point admittance at terminal pairs s. In fact, Y is an *absolutely invariant matrix* of A (and F) under T. The ranks of A, A_t^s, Y are integer invariants. The rank of A^{-1} is the number of linearly independent mesh currents; the rank of Y, the number of linearly independent driving-point currents.

In view of the assumption made upon the independence of currents $I_1, \cdots, I_m$, the rank of Y must equal m. Whence (5.7) gives

$$(E)_m = Z(I)_m \tag{5.9}$$

where

$$Z \equiv (z_{st}) = Y^{-1} \qquad (s, t = 1, \cdots, m). \tag{5.10}$$

Z is also known as a *characteristic coefficient matrix* of the network.

Equation (5.9) may also be obtained from (5.1) by eliminating the inner currents $I_{m+1}, \cdots, I_n$, provided the rank of $A_{1 \cdots m}$ is $(n-m)$, in which case $I_{m+1}, \cdots, I_n$ are linear functions of $I_1, \cdots, I_m$. The number of linearly independent inner mesh currents is equal to the rank of $A_{1 \cdots m}$.

Evidently, each z_{st} is an absolute invariant of A, whence Z is an absolutely invariant matrix of A. In fact, z_{st}, $s \neq t$, is the open circuit transfer impedance between terminal pairs s and t; and z_{ss} the open circuit driving point impedance at terminal pairs s. The rank of Z is equal to the number of linearly independent e.m.f.'s imposed across the terminal pairs $1, \cdots, m$.

The various invariant matrices and their several invariants may be given physical interpretations. For example, A_k^k $(k \leq m)$ is the matrix of the network derived from a network of matrix A by removing the imposed e.m.f. in mesh k and leaving the circuit open across terminal pairs k; i.e., the network matrix corresponding to the original network with mesh k removed.

* Numbers refer to list of references at end of paper.

A_k^k $(k>m)$ is the network matrix derived from A upon removing mesh k from the original network.

If the terminal pairs in mesh k are shorted, $k\leqq m$, the network becomes a $2(m-1)$-pole network and the corresponding mathematical theory is that of $(m-1)$-affine n-space, provided I_k is no longer held invariant. If mesh $k(k>m)$ be opened and an e.m.f. inserted, this increases the number of pole pairs by one and the mathematical theory becomes that of $(m+1)$-affine n-space.

If in addition to the invariance of the currents through terminal pairs $1, \cdots, m$, it is required to preserve the invariance of the current in an inner mesh $k(k>m)$, a restriction on T is imposed which dictates the theory used for $(m+1)$-affine n-space.

The various theorems given in the earlier part of this paper may be interpreted physically. For example, Theorem 3.6 may be interpreted as a theorem on the equivalence of two networks with one type of circuit parameter. The various canonical forms may be interpreted in terms of *canonical network forms* and the parameters appearing therein interpreted in terms of circuit parameters. The detailed treatment of the 2-affine case given above should be of particular interest because of the importance of 4-pole networks.

Two networks may be "equivalent" and yet one or both of them may not be physically realizable. Necessary and sufficient conditions for the physical realizability of a network corresponding to forms (5.3), in the case of networks containing but two types of circuit parameters, have been discussed by Cauer. The forms (5.3) used in existing theory have been positive definite because the networks considered have been passive. Other restrictions such as $2|R_{11}|\geqq\sum_{j=1}^{n}|R_{1j}|$ are necessitated by the nature of the physical problem. Should some future development occur that would make the study of non-passive circuits desirable, the generalizations of this paper are applicable.

It is evident that two $2m$-pole networks with a different number of meshes, of numbers p and q respectively, may be equivalent. The theory corresponding to this situation is really that of one quadric in p-space embedded in a q-space. The theorems of this paper include this possibility.

In conclusion, it should be noted that this paper deals essentially with the mathematical structure underlying the theory relating to electrical networks.

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THE POTENTIAL FUNCTION METHOD FOR THE SOLUTION OF TWO-DIMENSIONAL STRESS PROBLEMS*

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I. INTRODUCTION

At different intervals during the development of the theory of elasticity various methods of solution for two-dimensional stress problems have been proposed. Among these may be mentioned methods based on the use of the Airy stress function†; the strain energy function‡; the recently developed so-called "displacement function"§; and the potential function.|| The latter method was originally suggested independently by S. D. Carothers|| and by A. Nádai||; and although a useful and convenient one it does not as yet seem to have found broader application.

In his derivation, S. D. Carothers obtained the expressions for the stress components from solutions of the stress equations of equilibrium and the identical relations between strain components, while A. Nádai derived them for two important special cases by a considerably shorter method which will be developed further in the present investigation. More recently, the potential method has also been discussed and applied to various problems by L. Föppl¶, E. Kohl,** and H. Neuber.†† Both Föppl and Kohl derived the general expressions for the stress components from solutions of the fundamental

* Presented to the Society, April 20, 1935; received by the editors November 1, 1934.

† For details cf. Love, A. E. H., *Mathematical Theory of Elasticity*, London, Cambridge University Press, 4th edition, 1927, p. 88 ff.

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** Kohl, E., *Beitrag zur Lösung des ebenen Spannungsproblems*, Zeitschrift für Angewandte Mathematik und Mechanik, vol. 10 (1930), p. 141.

†† Neuber, H., *Elastisch-strenge Lösungen zur Kerbwirkung bei Scheiben und Umdrehungskörpern*, Zeitschrift für Angewandte Mathematik und Mechanik, vol. 13 (1933), pp. 439–443.

elastic equations in terms of displacements. It will be shown later that these expressions for the general stress components may also be derived by following a somewhat different procedure.

It is the intention here in particular to deal with the potential method more completely than has been done heretofore and in a somewhat different manner, and to apply it to a number of examples.

Notation

p_0, τ_1: distributed normal and shearing forces per unit area.

$\sigma_x, \sigma_y, \tau_{xy}$: normal and shearing stresses in the $[x, y]$ plane.

$\epsilon_x, \epsilon_y, \gamma_{xy}$: strain components in the $[x, y]$ plane.

ξ, η: displacements in the x and y directions.

E, G, ν: moduli of elasticity and rigidity, Poisson's ratio.

F: Airy's stress function.

c: a unit of distance.

$z, \bar{z}$: complex variables $x+iy$ and $x-iy$ respectively.

Φ, Ψ, χ, ψ: potential functions where $W(z)=\Psi-i\Phi$, $K(z)=\chi+i\psi$.

Re: real part of.

$Z(z), H(z)$: functions of a complex variable where $Z(z)=\Theta+i\Omega$ and $H(z)=\Theta_0+i\Omega_0$.

II. General expressions for stresses in terms of potential functions or functions of a complex variable

Following Airy's stress function method for the solution of stress problems in the plane, the stresses will be completely determined if a function $F[x, y]$ can be found which is a solution of the biharmonic equation

$$\nabla^2\nabla^2 F = 0, \tag{1}$$

where ∇^2 represents the Laplacean operator in two dimensions, and which when substituted in the expressions for the stresses

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \tag{2}$$

satisfies the boundary conditions of the problem. The general solution of equation (1) may be expressed in any one of the following equivalent forms*:

$$F = y\Theta + \Theta_0 = x\Theta_2 + \Theta_1 = (x^2 + y^2)\Theta_4 + \Theta_3, \tag{3}$$

where the functions Θ_i are various logarithmic potential functions. Hence any

* Cf. *Selected Problems in the Theories of Flat Plates and Plane Stress*, Dissertation, University of Pittsburgh, 1934.

one of these forms may be used to express the most general biharmonic stress function in two dimensions. Selecting the first form of equations (3) or $y\Theta+\Theta_0$ and substituting in equations (2) making use of the relations

$$\Phi = \frac{\partial\Theta}{\partial y}, \quad \Psi = \frac{\partial\Theta}{\partial x}, \quad \chi = -\frac{\partial\Theta_0}{\partial x}, \quad \psi = \frac{\partial\Theta_0}{\partial y}, \quad W(z) = \Psi - i\Phi,$$

we obtain

$$\begin{aligned} \sigma_x &= 2\Phi + y\frac{\partial\Phi}{\partial y} + \frac{\partial\chi}{\partial x}, \\ \sigma_y &= -y\frac{\partial\Phi}{\partial y} - \frac{\partial\chi}{\partial x}, \\ \tau_{xy} &= -\Psi - y\frac{\partial\Phi}{\partial x} + \frac{\partial\chi}{\partial y}, \end{aligned} \tag{4}$$

which are the general expressions for the stresses in two dimensions in terms of logarithmic potential functions only. These equations hold for a body of any shape stressed in its plane. Considering the special case of the semi-plane, equations (4) may be put in a simpler form treating the cases of normal and shear loading separately. For normal loading only along the line $y=0$ of the semi-plane, it follows that $\Psi=\partial\chi/\partial y$ and from the Cauchy-Riemann equations we find that $\Phi=-\partial\chi/\partial x$. Hence for this case equations (4) become

$$\sigma_x = \Phi + y\frac{\partial\Phi}{\partial y}, \quad \sigma_y = \Phi - y\frac{\partial\Phi}{\partial y}, \tag{5}$$

$$\tau_{xy} = -y\frac{\partial\Phi}{\partial x},$$

which show that along $y=0$, $\sigma_y=\Phi$, $\tau_{xy}=0$. In a similar manner for shear loading only along the line $y=0$ of the semi-plane, placing $\chi=0$ equations (4) reduce to

$$\begin{aligned} \sigma_x &= 2\Phi + y\frac{\partial\Phi}{\partial y}, \\ \sigma_y &= -y\frac{\partial\Phi}{\partial y}, \\ \tau_{xy} &= -\Psi - y\frac{\partial\Phi}{\partial x}, \end{aligned} \tag{6}$$

indicating that for $y=0$, $\sigma_y=0$, $\tau_{xy}=-\Psi$. The stress problem for these two cases has thus been reduced to the first boundary value problem of the potential theory. By computing the dilatation and rotation for the stresses given

in equations (5) and (6) it can be seen that the physical meaning of the potential functions $\Phi(x, y)$ and $\Psi(x, y)$ is that they represent to a constant factor the dilatation and rotation respectively.

There are several inherent advantages in having the stress components given in terms of functions of a complex variable and its derivatives. The advantages are (a) less labor involved in the computation of a given case; (b) it being unnecessary to compute the conjugate function which in most cases is tedious and often difficult; and (c) greater ease in recognizing the necessary function $W(z)$ for a given load distribution on the semi-plane than its real or imaginary parts.

The stress components in complex form may be derived either by making use of the stress equations already obtained, and of certain relations between the potential functions, or from the beginning in complex form. The latter method will be followed here. Following a suggestion made by Busemann* we may express the biharmonic stress function as

$$(7) \qquad F(x, y) = F\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) = F_1(z, \bar{z})$$

where F_1 is real. Substituting this stress function in equations (2) and differentiating, using the relations

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \qquad \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right),$$

we get, if

$$(8) \qquad F(x, y) = F_1(z, \bar{z}) = \mathrm{Re}\left(H(z) + \frac{i(\bar{z} - z)}{2} Z(z)\right),$$

where

$$H(z) = \Theta_0 + i\Omega_0, \qquad Z(z) = \Theta + i\Omega,$$

and

$$\frac{dH(z)}{dz} = -K(z) = -\chi - i\psi,$$

$$\frac{dZ(z)}{dz} = W(z) = \Psi - i\Phi,$$

the general expressions for the stress components in complex form as

* Busemann, A., *Schematischer Übergang von Vectorgleichungen auf komplexe Gleichungen bei ebenen Problemen*, Zeitschrift für Angewandte Mathematik und Mechanik, vol. 11 (1931), pp. 71–72.

$$
\begin{aligned}
\sigma_x &= \mathrm{Re}\left[2iW(z) - y\frac{dW(z)}{dz} + \frac{dK(z)}{dz}\right], \\
\sigma_y &= \mathrm{Re}\left[\qquad + y\frac{dW(z)}{dz} - \frac{dK(z)}{dz}\right], \\
\tau_{xy} &= \mathrm{Re}\left[-W(z) - iy\frac{dW(z)}{dz} + i\frac{dK(z)}{dz}\right].
\end{aligned} \tag{9}
$$

Using certain obvious relations between the complex functions, we get for the special case of normal loading on the semi-plane

$$
\begin{aligned}
\sigma_x &= \mathrm{Re}\left[iW(z) - y\frac{dW(z)}{dz}\right], \\
\sigma_y &= \mathrm{Re}\left[iW(z) + y\frac{dW(z)}{dz}\right], \\
\tau_{xy} &= \mathrm{Re}\left[\qquad - iy\frac{dW(z)}{dz}\right],
\end{aligned} \tag{10}
$$

and for the special case of shear loading on the semi-plane,

$$
\begin{aligned}
\sigma_x &= \mathrm{Re}\left[2iW(z) - y\frac{dW(z)}{dz}\right], \\
\sigma_y &= \mathrm{Re}\left[\qquad + y\frac{dW(z)}{dz}\right], \\
\tau_{xy} &= \mathrm{Re}\left[-W(z) - iy\frac{dW(z)}{dz}\right].
\end{aligned} \tag{11}
$$

In equations (10) and (11), the boundary $y=0$ is loaded by normal pressure $\sigma_y = \mathrm{Re}[iW(z)]_{y=0}$ and shear pressure $\tau_{xy} = \mathrm{Re}[-W(z)]_{y=0}$ respectively.

III. General expressions for displacements in terms of potential functions

The expressions for the displacements corresponding to the states of stress described in the preceding section may now easily be calculated. Consider a function $Z(z) = \Theta + i\Omega$ which is analogous to the so-called complex stream function of hydrodynamics. Then

$$
\frac{dZ(z)}{dz} = \frac{\partial\Theta}{\partial x} + i\frac{\partial\Omega}{\partial x} = \frac{\partial\Theta}{\partial x} - i\frac{\partial\Theta}{\partial y} = \Psi - i\Phi = W(z)
$$

which follows from relations used in the previous section. In the latter equa-

tions $W(z)$ is analogous to the complex velocity function in fluid dynamics. Hence it follows that

$$\Phi = \frac{\partial\Theta}{\partial y} = -\frac{\partial\Omega}{\partial x}; \qquad \Psi = \frac{\partial\Theta}{\partial x} = \frac{\partial\Omega}{\partial y}. \tag{12}$$

The general expressions for the stress components given in equations (4) may now be substituted in

$$\begin{aligned} \epsilon_x &= \frac{\partial\xi}{\partial x} = \frac{1}{2G}[\sigma_x(1-\nu) - \nu\sigma_y], \\ \epsilon_y &= \frac{\partial\eta}{\partial y} = \frac{1}{2G}[\sigma_y(1-\nu) - \nu\sigma_x], \\ \gamma_{xy} &= \frac{\partial\xi}{\partial y} + \frac{\partial\eta}{\partial x} = \frac{\tau_{xy}}{G}, \end{aligned} \tag{13}$$

which hold for plane strain ($\epsilon_z = 0$). If the resulting equations are then integrated for the displacements (ξ, η) making use of (12), the general expressions for the displacements in plane strain become

$$\begin{aligned} \xi &= \frac{1}{2G}\left[-2(1-\nu)\Omega - y\frac{\partial\Omega}{\partial y} + \chi\right] + c_1y + c_2, \\ \eta &= \frac{1}{2G}\left[(1-2\nu)\Theta - y\frac{\partial\Theta}{\partial y} - \psi\right] - c_1x + c_3. \end{aligned} \tag{14}$$

By introducing the relations between the potential functions and the biharmonic stress function, and through further relations between the potential functions themselves, equations (14) may be put in the forms

$$\begin{aligned} \xi &= \frac{1}{2G}\left[-2(1-\nu)\Omega - \frac{\partial F}{\partial x}\right] + c_1y + c_2, \\ \eta &= \frac{1}{2G}\left[2(1-\nu)\Theta - \frac{\partial F}{\partial y}\right] - c_1x + c_3, \end{aligned} \tag{15}$$

which hold for the general case of plane strain. The general displacement components in plane stress may then be obtained from equations (15) merely by replacing $(1-\nu)$ by $1/(1+\nu)$.

In a similar manner, the displacements in plane strain may be obtained for the special cases of normal and shear loadings on the boundary of the semi-plane. These become

$$
(16) \qquad
\begin{aligned}
\xi &= \frac{1}{2G}\left[-(1-2\nu)\Omega - y\frac{\partial\Omega}{\partial y}\right] - c_1 y + c_3,\\
\eta &= \frac{1}{2G}\left[2(1-\nu)\Theta - y\frac{\partial\Theta}{\partial y}\right] + c_1 x + c_2
\end{aligned}
$$

for the case of normal pressure, and

$$
(17) \qquad
\begin{aligned}
\xi &= \frac{1}{2G}\left[-2(1-\nu)\Omega - y\frac{\partial\Omega}{\partial y}\right] - c_1 y + c_3,\\
\eta &= \frac{1}{2G}\left[(1-2\nu)\Theta - y\frac{\partial\Theta}{\partial y}\right] + c_1 x + c_2
\end{aligned}
$$

for the case of shear loading on the straight boundary of the semi-plane.

IV. Application of the method to special problems in the semi-plane

The expressions for the stress components derived in complex form are very useful in the solution of a large group of problems. In the case of the semi-plane, for example, the complex function $W(z)$ can be selected for many pressure distributions immediately from the form of the given loading. It can be shown in the case of a large number of rational or transcendental functions of the complex variable z that the

$$\mathrm{Re}[W(z)]_{y=0} = \Psi(x, y)\big|_{y=0} = W(x).$$

Hence, for various load distributions $W(x)$ applied along the straight boundary, it is only necessary to replace x by z in $W(x)$ in order to obtain the complex function necessary to derive the corresponding stresses.

A particular group of problems in the semi-plane are of considerable practical importance; namely those in which only one half of the straight boundary is loaded by either normal or shear forces. For many such distributions $f(x)$ of either shear or normal pressure which can be expressed by certain rational or transcendental functions of x, it will be found that considerable use may be made of the function $\log(z/c)$ where c is a real constant. In such cases it will be found that the corresponding complex function $W(z)$ from which the stresses are derived will be represented by

$$(18) \qquad W(z) = f(z)\cdot\log\frac{z}{c}.$$

The function $\log(z/c)$ in this case provides that the pressure will be applied along one half of the boundary only. Equation (18) is quite general for functions restricted to the type mentioned, and with it various problems may be

solved for different functions $f(z)$. In some of the examples to be discussed in this section, a simple type of function will be chosen, namely a function proportional to z^n. It is shown that with this function a considerable number of problems may be solved.

It will however be mentioned here that one should perhaps distinguish between two types of loading along the semi-plane boundary; namely those cases in which the external forces are applied along a small finite strip of the boundary or tend to zero as r increases, and those cases in which the applied forces increase uniformly from the origin along $y=0$. In the former, convergence of the loads towards zero makes possible the requirement that all the stresses converge toward zero values as the distances from the origin become large. In the other case this requirement is not possible.

A. **Normal pressure varying as r^n.** The expressions for the stresses will be derived from two different complex functions $W(z)$ depending on whether n is integral or fractional. For n a positive or a negative integer, let

$$W(z) = \frac{p_0}{\pi} i^{2n} \frac{z^n}{a^n} \log \frac{z}{c},$$

where

$$f(z) = \frac{p_0}{\pi} i^{2n} \frac{z^n}{a^n}$$

in equation (18). Substituting this expression for $W(z)$ in equations (10) we get

$$\begin{aligned} \sigma_x &= \frac{p_0}{\pi a^n} \operatorname{Re}\left[i^{2n}\left(iz^n \log \frac{z}{c} - yz^{n-1}\left[1 + n \log \frac{z}{c}\right]\right)\right], \\ \sigma_y &= \frac{p_0}{\pi a^n} \operatorname{Re}\left[i^{2n}\left(iz^n \log \frac{z}{c} + yz^{n-1}\left[1 + n \log \frac{z}{c}\right]\right)\right], \qquad (19) \\ \tau_{xy} &= -\frac{p_0}{\pi a^n} \operatorname{Re}\left[i^{2n+1} yz^{n-1}\left(1 + n \log \frac{z}{c}\right)\right], \end{aligned}$$

which are the general stress components for the values of n mentioned above. In equations (19) are included such special cases as uniform pressure ($n=0$), linearly increasing pressure ($n=1$), parabolic pressure ($n=2$), hyperbolic pressure ($n=-1$), and various others, the explicit expressions for which may be easily obtained by replacing n in equations (19) by its appropriate value.*

For n a positive or a negative fraction, let

$$W(z) = \frac{p_0}{\sin n\pi} \frac{z^n}{a^n},$$

* For these explicit expressions in the special cases cf. footnote on p. 178.

and by substituting these expressions in equations (10) we get after some reduction

$$
\begin{aligned}
\sigma_x &= \frac{p_0}{a^n \sin n\pi} \operatorname{Re}\left[r^n(\cos (n-1)\phi + i \sin (n-1)\phi)(i \cos \phi - (1+n) \sin \phi)\right],\\
(20) \qquad \sigma_y &= \frac{p_0}{a^n \sin n\pi} \operatorname{Re}\left[r^n(\cos (n-1)\phi + i \sin (n-1)\phi)(i \cos \phi - (1-n) \sin \phi)\right],\\
\tau_{xy} &= \frac{p_0}{a^n \sin n\pi} \operatorname{Re}\left[r^n(\cos (n-1)\phi + i \sin (n-1)\phi)(-in \sin \phi)\right],
\end{aligned}
$$

which are the general stress components for normal loading on one side of the semi-plane boundary with n restricted to fractional values. In equations (20) are included such special cases as parabolic pressure ($n=1/2$), hyperbolic pressure ($n=-1/2$), hyperbolic pressure where $n=-5/16$, and many others. The explicit expressions for the stresses in these cases may also be obtained in the same manner as discussed previously.

B. Shearing forces varying as r^n. For the cases of shearing forces which vary in the same manner as the normal pressures treated in the last section, the stress components will be derived from two different complex functions $W(z)$ depending on whether n is integral or fractional as before.

In case n is a positive or a negative integer, consider the complex function

$$W(z) = \frac{\tau_1}{\pi} i \frac{z^n}{(-a)^n} \log \frac{z}{c},$$

where

$$f(z) = \frac{\tau_1}{\pi} i \frac{z^n}{(-a)^n}$$

in equation (18). Substituting this value of $W(z)$ in equations (11) we obtain

$$
\begin{aligned}
\sigma_x &= \frac{\tau_1}{\pi(-a)^n} \operatorname{Re}\left[z^{n-1}\left([-2z - iyn] \log \frac{z}{c} - iy\right)\right],\\
(21) \qquad \sigma_y &= \frac{\tau_1}{\pi(-a)^n} \operatorname{Re}\left[iyz^{n-1}\left(1 + n \log \frac{z}{c}\right)\right],\\
\tau_{xy} &= \frac{\tau_1}{\pi(-a)^n} \operatorname{Re}\left[z^{n-1}\left([-iz + ny] \log \frac{z}{c} + y\right)\right].
\end{aligned}
$$

These are the general expressions for the stress components for shear loading on one half of the boundary of the semi-plane where n is restricted to integral values. As for the normal pressure cases, equations (21) include such special cases as uniform shear ($n=0$), linearly increasing shearing forces ($n=1$), parabolic shear ($n=2$), hyperbolic shear where $n=-1$ and others.

With n a positive or a negative fraction, the complex function $W(z)$ becomes

$$W(z) = \frac{\tau_1 i}{\sin n\pi} \frac{z^n}{a^n},$$

which when substituted in equations (11) yields the stress components

$$\sigma_x = \frac{\tau_1 r^n}{a^n \sin n\pi}[(n+2)\sin\phi\cdot\sin(n-1)\phi - 2\cos\phi\cdot\cos(n-1)\phi],$$

$$(22) \quad \sigma_y = \frac{\tau_1 r^n}{a^n \sin n\pi}[-n\sin\phi\cdot\sin(n-1)\phi],$$

$$\tau_{xy} = \frac{\tau_1 r^n}{a^n \sin n\pi}[\cos\phi\cdot\sin(n-1)\phi + (n+1)\sin\phi\cdot\cos(n-1)\phi].$$

Such special cases as parabolic shear where $n=1/2$, parabolic shear with $n=5/16$, hyperbolic shear where $n=-1/2$, and various others are contained in these expressions.

V. Conclusion

In this paper the potential method for determining stresses in a body loaded by forces in its plane was developed further than heretofore and in a somewhat different manner. The relations between the general stress components in terms of potentials for plane problems and those for the special cases of normal and shear loading on the boundary of the semi-plane were brought out. The expressions for the stress components in both the general and special cases were developed in complex form, and the displacement components were determined for both the general case and the special cases of normal and shear loading on the boundary of the semi-plane. The application of the method to various simple cases was discussed, and solutions were given for a number of more important examples of both shear and normal loading on one side of the straight semi-plane boundary.

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CYCLOTOMY WHEN e IS COMPOSITE*

BY

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1. **Introduction.** This paper is a sequel to two earlier ones.† Let p be a prime and e a divisor of $p-1=ef$. We seek the e^2 cyclotomic constants (k, h). The difficulties increase roughly as e increases, but more exactly with Euler's function $\phi(e)$. We have $\phi(e)\leqq 4$ only when $e=1$-6, 8, 10, 12; for each of these e's a simple complete theory was given in D. It is known that $\phi(e)$ is even if $e>2$. We have $\phi(e)=6$ only when $e=7, 9, 14, 18$; $\phi(e)=8$ only when $e=15, 16, 20, 24, 30$; $\phi(e)=10$ only when $e=11, 22$. The case in which e is a prime or a double of a prime was treated in T.

Here we give a simple complete theory for $e=9$ and the further facts sufficient for a complete theory for $e=18$. We have overcome difficulties which did not arise in the earlier papers. We treat briefly the five cases having $\phi(e)=8$; there is now trouble in the determination of a unit factor.

2. **Subdivision of periods.** Let d be any divisor of e and write $E=e/d$. In the definition of the e periods η_k, replace e by E and f by df; we get the E periods

$$Y_k = \sum_{j=0}^{d-1} \eta_{k+jE} \qquad (k = 0, \cdots, E-1). \tag{1}$$

By T, (3),

$$Y_0 Y_k = \sum_{h=0}^{E-1} (k, h)_E Y_h + \text{const.}, \quad Y_h = \eta_h + \eta_{h+E} + \cdots .$$

The general term of the product is

$$\eta_{tE}\eta_{k+jE} = \sum_{n=0}^{e-1} (k + jE - tE, n)\eta_{n+tE} + \text{const.}$$

Let $0\leqq k<E$, $0\leqq h<E$. By the terms in η_h,

$$(k, h)_E = \sum_{t,j=0}^{d-1} (k + jE - tE, h - tE).$$

Since the two arguments may be reduced modulo e,

* Presented to the Society, April 20, 1935; received by the editors March 18, 1935.

† These Transactions, vol. 37 (1935), pp. 363–380, cited as T. American Journal of Mathematics, vol. 57 (1935), cited as D.

(2) $$(k, h)_E = \sum_{r,s=0}^{d-1} (k + rE, h + sE).$$

This proof is much simpler than that in D, §14, for the case $d=2$.

The primitive eth roots of unity satisfy an equation of degree $\phi(e)$ with integral coefficients. Its roots are β^k, where $0<k<e$ and k is prime to e. For the field of rational numbers, the general substitution of its Galois group G is induced by the replacement of β by β^k. Hence the latter yields a true relation when applied to a known one. But this may not be the case when k is not prime to e.

In T, (7), take $e=dE$, $m=dM$. In the terms with $j=tE, \cdots, tE+E-1$, take $j=J+tE$ and apply $\beta^{dE}=1$ and (1). We get

$$\sum_{J=0}^{E-1} \beta^{dMJ} \sum_{t=0}^{d-1} \eta_{J+tE} = \sum_{J=0}^{E-1} B^{MJ} Y_J = \phi(B^M),$$

where $B=\beta^d$ is a primitive Eth root of unity. Evidently $\phi(B^M)$ is derived from $F(\beta^M)$ in T, (7), by replacing e by E, β by B, η by Y. Hence $F(\beta^{dM}) =\phi(B^M)$. Then T, (8), gives

(3) $$R(dr, ds, \beta)_e = R(r, s, \beta^d)_E.$$

Part I. Theory for $e=9$

3. The functions $R(m, n)$. If $p=9f+1=$prime, t is even. When β is replaced by β^j, where j is prime to 9, it is known that $R(m,n)$ becomes $R(jm, jn)$, which is called a *conjugate* to $R(m, n)$. If m is prime to 3, we can choose j so that $jm\equiv 1 \pmod 9$. Hence unless m and n are both multiples of 3, $R(m, n)$ is conjugate to a certain $R(1, -)$. But $R(1, 1)=R(1, 7)$ is conjugate to $R(4, 28)=R(1, 4)$. Also $R(1, 6)=R(1, 2)$ is conjugate to $R(5, 10)=R(1, 5) =R(1, 3)$. Hence *every $R(m, n)$ is conjugate to one of* $R(1, 1)$, $R(1, 2)$, $R(3, 3)$.

We readily find $R(3, 3)$. By (32)-(34) of D,

(4) $$2R(1, 1)_3 = L + 3M + 6\beta M, \quad M = (0, 1)_3 - (0, 2)_3,$$

(5) $$4p = L^2 + 27M^2, \qquad L = 9(0, 0)_3 - p + 8 \equiv 7 \pmod 9.$$

By (3), $R(3, 3)_9=R(1, 1, \beta^3)_3$, whence

(6) $$2R(3, 3) = L + 3M + 6\beta^3 M.$$

Jacobi* noted that if $\alpha^{p-1}=1$, $\alpha\neq 1$, and if γ is an imaginary cube root of unity, then

(7) $$F(\alpha)F(\gamma\alpha)F(\gamma^2\alpha) = \alpha^{-3m'}pF(\alpha^3), \qquad 3 \equiv g^{m'} \pmod p.$$

* Journal für Mathematik, vol. 30 (1846), p. 167.

We may take $\gamma=\beta^3, \alpha=\beta^4, p=F(\beta^4)F(\beta^5)$. We get

$$R(1,7) = \beta^{-3m'}R(3,5); \qquad R(1,1) = R(1,7),$$
$$R(3,5,\beta^2) = R(6,1) = R(1,2),$$
$$R(1,2) = \beta^{6m'}R(1,1,\beta^2). \tag{8}$$

4. Determination of the 81 cyclotomic constants $(k, h)=kh$. The equalities T, (4), between the (k, h) become for $e=9$

$$11 = 08,\ 18 = 12,\ 22 = 07,\ 23 = 17,\ 27 = 24,\ 28 = 13,\ 33 = 06,$$
$$34 = 16,\ 35 = 26,\ 37 = 25,\ 38 = 14,\ 44 = 05,\ 45 = 15,\ 46 = 25,$$
$$47 = 26,\ 48 = 15,\ 55 = 04,\ 56 = 14,\ 57 = 24,\ 58 = 16,\ 66 = 03,$$
$$67 = 13,\ 68 = 17,\ 77 = 02,\ 78 = 12,\ 88 = 01,\ kh = hk.$$

The linear relations T, (5), now become

$$\sum_{h=0}^{8}(0,h) = f-1, \qquad 01+08+2(12)+\sum_{h=3}^{7}(1,h) = f,$$
$$02+07+12+13+17+2(24)+25+26 = f, \tag{9}$$
$$03+06+13+14+16+17+25+26+36 = f,$$
$$04+05+14+2(15)+16+24+25+26 = f.$$

The sum of the last four less the first is

$$\begin{aligned} &3(12+13+14+15+16+17+24+25+26)\\ &\quad = 3f+1+(00)-(36). \end{aligned} \tag{10}$$

In (4) and (5) we have by (2),

$$(0,0)_3 = (0,0)+3(0,3)+3(0,6)+2(3,6), \tag{11}$$
$$M = 01-02+04-05+07-08+2\{13-14+16-17+25-26\}. \tag{12}$$

Using T, (8), and checking by T, (16), we get after using

$$\beta^6+\beta^3+1 = 0, \tag{13}$$

$$R(1,1) = \sum_{i=0}^{5} c_i\beta^i, \qquad R(1,2) = \sum_{i=0}^{5} b_i\beta^i, \tag{14}$$

$c_0 = (00) - 3(06) + 2(36)$,

$c_1 = 01 + 04 - 2(07) + 2(13) - 4(16) + 2(25)$,

$c_2 = 2(02) - 05 - 08 + 4(14) - 2(17) - 2(26)$, $\quad c_3 = 3(03) - 3(06)$,

$c_4 = -01 + 2(04) - 07 + 4(13) - 2(16) - 2(25)$,

$c_5 = 02 + 05 - 2(08) + 2(14) + 2(17) - 4(26)$;

$$b_0 = 00 - 01 - 04 - 07 + 13 + 16 + 25 - 36,$$
$$b_1 = 01 + 05 - 07 - 08 + 12 - 2(15) + 16 - 17 + 24 - 25 + 26,$$
$$b_2 = 01 + 02 - 04 - 08 - 12 + 13 - 14 + 2(15) - 24 - 25 + 26,$$
$$b_3 = -01 + 02 - 04 + 05 - 07 + 08 + 13 - 14 + 16 - 17 + 25 - 26,$$
$$b_4 = 02 + 04 - 07 - 08 + 2(12) - 13 - 14 - 15 + 16 - 24 + 26,$$
$$b_5 = -04 + 05 + 07 - 08 + 12 + 13 + 15 - 16 - 17 - 2(24) + 26.$$

These twelve equations with the five in (9), and (11), (12), uniquely determine the nineteen "reduced" (k, h) involved in them and hence all the 81 cyclotomic constants.

We first give combinations which involve 01 and 08, 02 and 07, 04 and 05 only in their sums, which we eliminate by (9). Then $2b_0 - b_3$ is seen to involve the left member of (10), whence

$$2b_0 - b_3 = 1 + 3(0, 0) - 3(3, 6). \tag{15}$$

From this, (11), c_0 and c_3 we get

$$\begin{aligned} 9(0, 0) &= 2(2b_0 - b_3 - 1) + (0, 0)_3 - c_3 + 2c_0, \\ 9(0, 6) &= (0, 0)_3 - c_0 - c_3, \end{aligned} \tag{16}$$

$$(0, 3) = (0, 6) + \tfrac{1}{3}c_3, \qquad (3, 6) = (0, 0) - \tfrac{1}{3}(2b_0 - b_3 - 1). \tag{17}$$

These known $(0, 3t)$ and $(3, 6)$ are allowed in later answers. The new combinations are

$$b_1 - b_2 = 3(1,2) - 6(1,5) + 3(2,4), \quad b_4 + b_5 - b_1 = 3(1,2) + 3(1,5) - 6(2,4),$$
$$\begin{aligned} c_1 - c_2 = {} & 3(1, 3) - 6(1, 4) - 3(1, 5) - 6(1, 6) + 3(1, 7) + 3(2, 4) \\ & + 3(2, 5) + 3(2, 6), \end{aligned}$$
$$b_4 - c_1 = 3(1, 2) - 3(1, 3) + 6(1, 6) - 3(2, 4) - 3(2, 5),$$
$$c_5 - b_1 = 3(1, 4) + 3(1, 5) + 3(1, 7) - 3(2, 4) - 6(2, 6),$$
$$c_4 + b_5 = 3(1, 2) + 6(1, 3) - 3(1, 6) - 3(2, 4) - 3(2, 5),$$
$$\begin{aligned} A &\equiv \tfrac{1}{2}\{M + 2b_0 + f - 1 - 3(0, 0) - 03 - 06 + 2(3, 6)\} \\ &= 2(13 + 16 + 25) - 14 - 17 - 26. \end{aligned}$$

From these and the fourth in (9), we get

$$9(2, 6) = 2B - C, \quad 9(1, 6) = B + C + 3D, \tag{18}$$

$$9(1, 3) = B + C + 3D + G, \qquad 9(2, 5) = B + C - 6D - G, \tag{19}$$
$$B = H - \tfrac{1}{3}(C_5 - b_1) + f - 03 - 06 - 36,$$
$$C = A - H + \tfrac{1}{3}(c_5 - b_1), \quad D = \tfrac{1}{3}(b_4 - c_1 - b_1 + b_2) - 2H,$$
$$G = c_4 + b_5 - b_4 + c_1, \; H = \tfrac{1}{9}\{b_4 + b_5 - b_1 - (b_1 - b_2)\} = 15 - 24.$$

Next, (10) yields

$$3(15 + 16 + 26) = f + \tfrac{1}{3}(1 + 00 - 36 + b_2 - b_1 - c_5 + b_1) + H + D, \tag{20}$$

which gives (1, 5). Then H gives (2, 4). Then $b_1 - b_2$ gives (1, 2). We get (1, 4) from

$$2H + \tfrac{1}{3}(c_1 - c_2 - c_5 + b_1) = 13 - 3(1, 4) - 2(1, 6) + 25 + 3(2, 6). \tag{21}$$

Then $c_5 - b_1$ gives (1, 7). Finally, 01 and 08, 02 and 07, 04 and 05, whose sums are known by (9), are determined by them and c_1, c_4, b_0.

5. **Congruences.** After reductions by $\beta^9 = 1$, but not by (13), let

$$R(1, n) = \sum_{i=0}^{8} B_i \beta^i.$$

By* T, (17) and (18),

$$\sum_{i=0}^{8} B_i \equiv -1, \qquad \sum_{i=0}^{8} iB_i \equiv 0, \qquad \sum_{i=0}^{8} i^2 B_i \equiv 0 \qquad (\text{mod } 3). \tag{22}$$

We now reduce by (13) and get

$$R(1, n) = \sum_{i=0}^{5} C_i \beta^i, \qquad C_0 = B_0 - B_6,\ C_1 = B_1 - B_7,\ C_2 = B_2 - B_8,$$
$$C_3 = B_3 - B_6,\ C_4 = B_4 - B_7,\ C_5 = B_5 - B_8.$$

Hence (22) give

$$\sum_{i=0}^{5} C_i \equiv -1, \qquad \sum_{i=0}^{5} iC_i \equiv 0, \qquad \sum_{i=0}^{5} i^2 C_i \equiv 0 \qquad (\text{mod } 3). \tag{23}$$

These are equivalent to†

$$C_0 + C_3 \equiv -1, \qquad C_1 + C_4 \equiv 0, \qquad C_2 + C_5 \equiv 0 \qquad (\text{mod } 3). \tag{24}$$

For $R(1, 1)$, $c_3 \equiv 0 \pmod 3$. By the fourth and first of (9),

$$c_1 - c_2 \equiv \sum_{h=1}^{8} (0, h) + (3, 6) - f = (3, 6) - (0, 0) - 1 \equiv 0 \qquad (\text{mod } 3)$$

by (10). Using also (24) in small letters, we see that for $R(1, 1)$

$$c_0 \equiv -1,\ c_2 \equiv c_1,\ c_3 \equiv 0,\ c_4 \equiv -c_1,\ c_5 \equiv -c_1 \qquad (\text{mod } 3). \tag{25}$$

* Our conclusion is not altered by the fact that if r, s is 3, 6 or 6, 3, the six numbers in T, (20), now coincide in sets of three. The last two in (22) are multiples of 9.

† For $R(1, 2)$ every linear congruence modulo 3 is a combination of (24).

In Lemmas 1, 2, and their proofs, the summation index takes the values 0, 1, $\cdots$, 5.

LEMMA 1. *Let* $\sum D_i \not\equiv 0 \pmod 3$ *in* $P = \sum D_i\beta^i$. *Then* $\pm\beta^n P = \sum C_i\beta^i$ *satisfies the first two congruences* (23) *for a single choice of the sign and for a single determination of* n *modulo* 3.

We have $\beta P = \sum S_i\beta^i$ where

$$S_0 = -D_5,\ S_1 = D_0,\ S_2 = D_1,\ S_3 = D_2 - D_5,\ S_4 = D_3,\ S_5 = D_4,$$

$$\sum S_i \equiv \sum D_i,\ \sum iS_i \equiv D_0 - D_1 + D_3 - D_4 \equiv \sum iD_i + \sum D_i \pmod 3.$$

Hence in $\beta^n P = \sum \sigma_i\beta^i$,

$$\sum \sigma_i \equiv \sum D_i,\quad \sum i\sigma_i \equiv \sum iD_i + n\sum D_i \equiv 0 \pmod 3$$

by choice of n, uniquely modulo 3.

LEMMA 2. *Let* $C = \sum C_i\beta^i$ *satisfy the first two congruences* (23). *By Lemma* 1, *also* $\beta^3 C$ *and* $\beta^6 C$ *satisfy the same congruences. At most one of* $C, \beta^3 C, \beta^6 C$ *satisfy also* $C_3 \equiv 0,\ C_1 \not\equiv 0$.

In $\beta^3 C = \sum T_i\beta^i$,

$$T_0 = -C_3,\ T_1 = -C_4,\ T_2 = -C_5,\ T_3 = C_0 - C_3,\ T_4 = C_1 - C_4,\ T_5 = C_2 - C_5.$$

Hence in $\beta^6 C = \sum U_i\beta^i$, $U_0 = C_3 - C_0$, $U_3 = -C_0$. If two of C_3, T_3, U_3 are multiples of 3, then $C_0 \equiv C_3 \equiv 0 \pmod 3$ and the coefficients of both β^0 and β^3 in $C, \beta^3 C, \beta^6 C$ are all multiples of 3.

Lemmas 1 and 2 yield

THEOREM 1. *At most one of* $\pm\beta^s P$ *satisfies congruences* (25).

6. **Class number.** If q is any prime, the field defined by $\exp 2\pi i/q^h$ has the discriminant* $D = \pm q^m$, where $m = q^{h-1}(hq - h - 1)$ and the sign is plus except when $q^h = 4$ or $q \equiv 3 \pmod 4$. But Minkowski proved that every ideal class contains an ideal whose norm is $< (\pm D)^{1/2}$. For $q^h = 9$, the latter is $3^{9/2} < 140.3$. Tables† show that every prime < 1000 is a product of actual complex primes, whence every integer < 1000 is a product of principal ideals. Thus every ideal is a principal ideal.

THEOREM 2. *The field of the ninth roots of unity has the class number* 1.

7. **Complex factors of primes** $p = 9f + 1$. To p corresponds a polynomial $L(\beta)$ with integral coefficients which is a complex prime such that p is the

* Kummer. See Hilbert's Report on algebraic numbers, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 4 (1894–95), p. 332.

† C. G. Reuschle, *Tafeln Complexer Primzahlen*, 1875, pp. 173–75.

product of a unit and the $L(\beta^i)$ for $i=1, 2, 4, 5, 7, 8$. Evidently the only possible factorizations

$$(26) \qquad p = uf(\beta)f(\beta^{-1}), \quad u = \text{unit}, \quad f(\beta) \text{ with factor } L(\beta),$$

have the following four forms of $f(\beta)$:

$$(\text{I}) \quad L(\beta)L(\beta^2)L(\beta^4); \qquad (\text{II}) \quad L(\beta)L(\beta^2)L(\beta^5);$$

$$(\text{III}) \quad L(\beta)L(\beta^5)L(\beta^7); \qquad (\text{IV}) \quad L(\beta)L(\beta^4)L(\beta^7).$$

Since (IV) is unaltered when β is replaced by β^4 or β^7, it corresponds to $R(3, 3)$. When β is replaced by β^2, (III) becomes (II), (II) becomes (I), and (I) becomes the complement $L(\beta^8)\ L(\beta^4)\ L(\beta^2)$ to (III).

8. **Diophantine equations determining $R(1, 1)$.** As in T, §13, $u=1$ in (26), while $v = \pm\beta^s$ is the only unit such that

$$(27) \qquad p = F(\beta)F(\beta^{-1}), \qquad F(\beta) = vf(\beta) = \sum_{i=0}^{5} c_i\beta^i.$$

By (13) this product is the sum of (28) and

$$(\beta + \beta^{-1})A + (\beta^2 + \beta^{-2})B + (\beta^4 + \beta^{-4} = -\beta - \beta^{-1} - \beta^2 - \beta^{-2})C,$$

where

$$(28) \qquad p = \sum_{i=0}^{5} c_i^2 - c_0c_3 - c_1c_4 - c_2c_5, \qquad A = C, \quad B = C,$$

$$(29) \qquad A = c_0c_1 + c_1c_2 + c_2c_3 + c_3c_4 + c_4c_5,$$

$$(30) \qquad B = c_0c_2 + c_1c_3 + c_2c_4 + c_3c_5, \qquad C = c_0c_4 + c_1c_5 + c_0c_5.$$

By Theorem 1 there is at most one choice of v in (27) such that the c_i satisfy congruences (25), which must hold if $F(\beta)$ serves as $R(1, 1)$. Replace β by β^2 and write $F(\beta^2) = \sum b_i\beta^i$. Then

$$(31) \quad b_0 = c_0 - c_3,\ b_1 = c_5,\ b_2 = c_1 - c_4,\ b_3 = -c_3,\ b_4 = c_2,\ b_5 = -c_4.$$

Evidently (28)–(30) hold when the c_i are replaced by these b_i. If the c_i satisfy congruences (25), also the b_i satisfy them.

We saw that $R(1, 1)$ is the product of a unit by (I), (II), (III), or one of their complements, the six being permuted when β is replaced by β^2.

THEOREM 3. *Equations* (28) *have exactly six sets of integral solutions satisfying congruences* (25). *These sets are derived from any one set by applying the powers of substitution* (31) *of period* 6. *Any of the six sets may be chosen as the coefficients of* $R(1, 1) = \sum c_i\beta^i$. *Then* (8) *gives* $R(1, 2)$. *Except for the double sign of* M, $R(3, 3)$ *is defined by* (5) *and* (6). *Then all the cyclotomic constants are determined as in* §4.

The ambiguity in $R(3, 3)$ may be removed by using*

$$R(3, 3) = \beta^{-6m'}R(1, 1)R(1, 2, \beta^2)/R(1, 2), \tag{32}$$

viz.,

$$\beta^{6m'}F(\beta^2)F(\beta^3) = F(\beta)F(\beta^4),$$

which follows from (8).

Theorem 3 permits a six-fold choice for $R(1, 1)$. This is in accord with the fact that β may be chosen as any of the six roots of (13). The cyclotomic constants themselves have a six-fold ambiguity involved in the choice of the primitive root g of p. When g is replaced by a new primitive root g^r, $R(1, 1)$ becomes $R(t, t) = R(1, 1, \beta^t)$, where $tr \equiv 1 \pmod 9$. But t ranges with r over the six integers <9 and prime to 9. By (28),

$$\begin{aligned} &4p = C_0^2 + C_1^2 + C_2^2 + 3(c_3^2 + c_4^2 + c_5^2), \\ &C_0 = 2c_0 - c_3, \qquad C_1 = 2c_1 - c_3, \qquad C_2 = 2c_2 - c_5. \end{aligned} \tag{33}$$

By (25), $C_1 = 3y$, $C_2 = 3z$, $c_3 = 3w$, where y, z, w are integers. Thus

$$4p = C_0^2 + 9(y^2 + z^2) + 27w^2 + 3c_4^2 + 3c_5^2, \quad C_0 \equiv 1, \quad c_4 \equiv c_5 \pmod 3, \tag{34}$$

so that the five congruences (25) reduce to two after choosing our new variables.

PART II. THEORY FOR $e = 18$

9. Unless m and n are both even or both multiples of 3, $R(m, n)$ is conjugate to some $R(1, -)$ and hence to a single one of

$$R(1, 1),\ R(1, 2),\ R(1, 3),\ R(1, 4),\ R(1, 5),\ R(1, 9). \tag{35}$$

The $R(3x, 3y)$ are conjugate to $R(3, 3)$, $R(3, 6)$, or $R(6, 6)$. The $R(2r, 2s)$ are given by (3). By T, §10,

$$\begin{aligned} &R(1, 9) = \beta^{2m}R(1, 1), \qquad R(1, 4) = (-1)^f\beta^{-6m}R(1, 1), \\ &R(2, 8) = (-1)^f\beta^{4m}R(1, 1). \end{aligned} \tag{36}$$

We regard $R(1, 1)_9$ as known by Theorem 3. Then our $R(2, 2)$ is known. Replacing β by β^{13}, we get $R(8, 8) = R(2, 8)$. Thus (36) give $R(1, 1)$, $R(1, 9)$, $R(1, 4)$. In (7) we may take $\gamma = \beta^6$, $\alpha = \beta^7$, $p = F(\beta^7)F(\beta^{11})$, and get

$$R(1, 13) = \beta^{-3m'}R(3, 11), \qquad R(1, 2) = \beta^{-3m'}R(1, 4, \beta^5), \tag{37}$$

since the latter is derived from the former by replacing β by β^5. By the value of R in terms of F, we get

* In case M is not divisible by 9, the change of the sign of M subtracts M from A above (18) and hence from C, whence by (18) the solution (26) is an integer for a single choice of $\pm M$.

(38) $$R(m, t)R(n, m + t) = R(m, n)R(m + n, t),$$

(39) $$R(1, 4)R(1, 5) = R(1, 1)R(2, 4), \quad R(2, 3)R(1, 5) = R(1, 3)R(2, 4).$$

By the first and (36_2), and then by the second,

(40) $$R(1, 5) = (-1)^f\beta^{6m}R(2, 4), \qquad R(1, 3) = (-1)^f\beta^{6m}R(2, 3).$$

We now know all functions (35) except $R(1, 3)$. While the case $R(1, 2)\ R(1, 3) = R(1, 1)\ R(2, 2)$ of (38) gives $R(1, 3)$, it is not found linearly.

10. We prove the following theorem:

THEOREM 4. *If $[x]$ denotes the least positive residue of x modulo e, we have the following decomposition into prime ideals*:

(41) $$R(h, t) = \pm\, \beta^s \Pi\, f(\beta^Z), \qquad zZ \equiv 1 \pmod{e},$$

where z ranges over those positive integers $<e$ and prime to e such that

(42) $$[hz] + [tz] > e.$$

Let r and g be primitive roots of

$$r^{p-1} = 1, \qquad g^{p-1} \equiv 1 \pmod{p}, \quad p = ef + 1.$$

Write $\psi(r) = F(r^{-m})\ F(r^{-n})/F(r^{-m-n})$, where m and n are positive and $<p-1$. Jacobi noted that

$$\psi(g) \equiv 0 \pmod{p} \quad \text{if } m + n > p - 1.$$

Write

$$r^{-f} = \beta, \quad g^{-f} \equiv u \pmod{p}, \qquad m \equiv hzf, \quad n \equiv tzf \pmod{(p - 1)}.$$

Then $r^{-m} = \beta^{hz}$, $r^{-n} = \beta^{tz}$, and β is a primitive eth root of unity. Thus $\psi(r)$ becomes $R(h, t, \beta^z)$. Since m/f and n/f are positive integers $<e$ and are congruent modulo e to hz and tz, respectively, $m+n>p-1$ is equivalent to (42). Then $R(h, t, u^z) \equiv 0 \pmod{p}$. This implies* (41).

Since $R\ R(h, t) = p$ if $R = R(h, t, \beta^{-1})$, the solutions z of

(43) $$[hz] + [tz] < e$$

yield the factors of R. We pass to the factors of $R(h, t)$ itself if we replace $f(\beta)$ by $f(\beta^{-1})$.

11. For $e = 18$, we use (43) and see that $R(1, 3)$ and $R(6, 6)$ are both prod-

* For e a prime, Kummer, Journal für Mathematik, vol. 35 (1847), p. 362, where there are two misprints of m for μ in the second line. Since we are taking his $f = 1$, the periods η are the powers of α, and the symbolic "$f(\alpha) \equiv 0 \pmod{q}$ for $\eta = u_r$" on p. 339 now means $f(u_r) \equiv 0 \pmod{q}$ in the ordinary sense. For e composite, Kummer, Mathematische Abhandlungen, Akademie der Wissenschaften, Berlin (for 1856), 1857, p. 45, where he used (43).

ucts of $f(\beta)\, f(\beta^7)\, f(\beta^{13})$ by units $\pm\beta^s$. Hence $R(1, 3)=\beta^k R(6, 6)$. Replacing β by β^{13}, we get

$$R(13, 3) = (-1)^f R(2, 3) = \beta^{13} R(6, 6).$$

Then (40_2) gives $12k+6m\equiv 0 \pmod{18}$, $k=m+3t$. We omit the indirect determination of t, yielding

$$R(1, 3) = \pm\, \beta^{m+3m'} R(6, 6). \tag{44}$$

Since we know all the $R(m, n)$, we can find the cyclotomic constants as in D or T.

Part III. Theory for $\phi(e)=8$, $e=15, 16, 20, 24, 30$

12. Let a, b, c, d denote the positive integers $<e/2$ and prime to e. Then $a'=e-a, \cdots, d'=e-d$ give the integers $>e/2$ and prime to e. Then p is the product of eight prime ideals $f(\beta^Z)$, denoted by Z, for $Z=a, \cdots, d'$. The following give $F(\beta)$ in the only decompositions $p=F(\beta)\, F(\beta^{-1})$, where $F(\beta)$ is a product of four of the prime ideals, one of which is $f(\beta^a)$:

(45)
I, II: a, b, c, d or d'; III, IV: a, b, c', d or d';
V, VI: a, b', c, d or d'; VII, VIII: a, b', c', d or d'.

If $F=F(\beta)$ is such a product of four, the product $F(\beta^{-1})$ of the complementary set of four is denoted by F'.

13. **Case** $e=16$. Every* ideal is a principal ideal (or the class number is 1). In §12, $a=1$, $b=3$, $c=5$, $d=7$. For the equation having the eight roots β^k, k odd and <16, the Galois group G for the domain of rational numbers is generated by

$$(\beta\beta^3\beta^9\beta^{11})(\beta^5\beta^{15}\beta^{13}\beta^7), \qquad (\beta\beta^5\beta^9\beta^{13})(\beta^3\beta^{15}\beta^{11}\beta^7). \tag{46}$$

These induce the respective substitutions

(47)
(II V′ VIII III)(I VII′ to I′)(IV)(VI to VI′),
(II III′ VIII V)(I VII′ to I′)(IV to IV′)(VI).

Each $R(m, n)$ is conjugate to one and only one of $R(1, j)$, $j=1, 2, 3, 6, 7$, $R(2, 2)$, $R(2, 4)$, $R(2, 6)$, $R(4, 4)$. By T, (48)–(51),

$$(-1)^f R(1, 6) = R(1, 9) = \beta^{2m} R(2, 2), \qquad R(1, 7) = (-1)^f \beta^{2m} R(1, 1), \tag{48}$$

where $g^m\equiv 2 \pmod p$. Applying Theorem 4 with (42) replaced by (43), we see that, apart from unit factors $\pm\beta^i$,

$$R(1, 1) = \text{VII},\ R(1, 2) = \text{VIII},\ R(1, 3) = \text{III},\ R(1, 6) = \text{IV},$$
$$R(2, 2) = R(2, 6) = R(1, 6),\ R(4, 4) = R(2, 2) = \text{VI},$$

* Weber, *Algebra*, 2d edition, vol. 2, 1899, p. 808, foot-note.

after a proper choice of $f(\beta)$ among the eight prime factors.

Consider the Diophantine equations found as in §8. A set of integral solutions which gives rise to 8 distinct sets under the group G generated by (46) may be taken as the coefficients of $R(1, 3)$. After choice of β among the eight roots of the octic satisfied by β^k, k odd and <16, we may assume that $R(1, 3)$ is the product of a unit $\pm\beta^i$ and III, rather than another of II, III, V, VII or the complements II$'$, $\cdots$ in the cycles of four in (47).

This unit is partially determined as follows. Write

$$R = (-1)^{nf}R(1, n) = \sum_{i=0}^{15} B_i\beta^i, \tag{49}$$

without reduction by $\beta^8 = -1$. As in T, §3,

$$\sum B_i = p - 2, \quad \sum iB_i \equiv 0 \pmod{16}, \quad \sum i^2 B_i \equiv 0 \pmod{8}. \tag{50}$$

After reduction by $\beta^8 = -1$, we get

$$R = \sum_{i=0}^{7} C_i\beta^i, \quad C_i = B_i - B_{i+8}, \tag{51}$$

$$\sum C_i \equiv 1, \qquad \sum iC_i \equiv C_1 + C_3 + C_5 + C_7 \equiv 0 \pmod{2}. \tag{52}$$

By the difference of the last two in (50) taken modulo 4, we get

$$C_2 + C_3 + C_6 + C_7 \equiv 0 \pmod{2}. \tag{53}$$

Consider any polynomial (51_1) with $\sum C_i$ odd. Then

$$\beta R = \sum_{i=0}^{7} D_i\beta^i, \quad D_0 = -C_7, \quad D_i = C_{i-1} \quad (i = 1, \cdots, 7),$$

$$\Delta = D_1 + D_3 + D_5 + D_7 = C_0 + C_2 + C_4 + C_6 \equiv 1 + s \pmod{2},$$

where $s = C_1 + C_3 + C_5 + C_7$. Hence if $s \equiv 1 \pmod 2$, βR has $\Delta \equiv 0 \pmod 2$. Hence by choice between R and βR we may assume that $s \equiv 0 \pmod 2$ in R. Then

$$\beta^2 R = \sum_{i=0}^{7} H_i\beta^i, \quad H_0 = -C_6, \quad H_1 = -C_7, \quad H_i = C_{i-2} \quad (i = 2, \cdots, 7),$$

$$H_1 + H_3 + H_5 + H_7 = C_1 + C_3 + C_5 - C_7 \equiv 0 \pmod{2},$$

$$\delta = H_2 + H_3 + H_6 + H_7 = C_0 + C_1 + C_4 + C_5 \equiv 1 + t \pmod{2},$$

where $t = C_2 + C_3 + C_6 + C_7$. Hence if t is odd, δ is even. Hence just one of $R, \beta R, \beta^2 R, \beta^3 R$ is a polynomial $\sum C_i\beta^i$ for which the three congruences (52) and (53) hold, viz.,

$$C_1 + C_5 \equiv C_3 + C_7 \equiv C_2 + C_6 \equiv 1 + C_0 + C_4 \pmod{2}. \tag{54}$$

These four sums remain unaltered modulo 2 when we replace R by $\beta^4 R$. Thus $R(1, 3)$ is determined* up to a factor β^{4i}. We have not undertaken the investigation similar to that in §5, but much longer, to find further linear congruences which determine j. For a given p, j is probably determined by the formulas expressing $(0, 0)$ or other cyclotomic constants (k, h) in terms of the coefficients of the $R(m, n)$.

The $R(2x, 2y)$ are known by the theory for $e=8$ in D. Then (48) gives $R(1, 6)$. We get $R(2, 3)$ from

$$R(1, 6) = R(33, 22) = R(2, 3, \beta^{11}). \tag{55}$$

The above discussion yielded $R(1, 3)$ and hence its conjugate $R(11, 33) = R(1, 11) = \pm R(1, 4)$. We get $R(1, 1)$ and $R(1, 2)$ from

$$R(1, 3)R(1, 4) = R(1, 1)R(2, 3), \qquad R(1, 2)R(1, 3) = R(1, 1)R(2, 2). \tag{56}$$

Also $R(1, 7)$ is known by (48). We now have a conjugate to every $R(m, n)$ and can find the (k, h) by linear equations as in T.

14. **Case** $e=15$. Let d_1, d_2, D be the discriminants of the fields defined by a primitive nth root of unity for $n=3, 5, 15$, respectively. Then $D=d_1^4 d_2^2$ by Hilbert's Report, loc. cit., p. 267. By §6, $d_1=-3$, $d_2=5^3$. Thus $D^{1/2}=1125$. By Reuschle's Tables, every integer <1000 is a product of principal ideals. If this were verified on to 1125, Minkowski's theorem (§6) would show that, in the field of the fifteenth roots of unity, every ideal is a principal ideal. A complete proof may be made by use of the real subfield of degree 4 as by Weber, loc. cit.

The $R(m, n)$ are conjugate to a single one of $R(1, j)$, $j=1, \cdots, 5$, $R(3, 3)$, $R(5, 5)$. The last two may be regarded as known by (3).

Consider (7) with $\gamma=\beta^5$, $\alpha=\beta^6$ or β^{14}, $p=F(\alpha)\,F(\alpha^{-1})$. We get

$$R(1, 3) = \beta^{-3m'}R(3, 3), \qquad R(1, 6, \beta^4) = R(4, 9) = \beta^{3m'}R(1, 2). \tag{57}$$

Expressing the former in terms of F's we get

$$R(1, 6) = \beta^{-3m'}R(3, 4).$$

By the case $24\cdot 16=12\cdot 34$ of (38), we get

$$R(2, 4) = R(1, 2, \beta^2) = \beta^{3m'}R(1, 2). \tag{58}$$

In (41) denote $f(\beta^z)$ by Z and use (43). Then, apart from factors $\pm\beta^s$,

$$R(1, 1) = 1\cdot 4\cdot 8\cdot 13, \qquad R(1, 2) = 1\cdot 2\cdot 4\cdot 8, \qquad R(1, 3) = 1\cdot 8\cdot 11\cdot 13,$$

while $R(1, 4)$ has the same factors as $R(1, 2)$, and $R(1, 5)$ the same as $R(1, 1)$.

* Likewise $R(1,5)$. While the factor for $R(1,4)$ is β^{-4i}, $R(1,1)$ is uniquely determined. See the later formulas.

Hence

$$R(1, 5) = \pm \beta^x R(1, 1). \tag{59}$$

Expressing the R's in terms of F's, we see that $R(2, 5) = \pm\beta^x R(1, 8)$. Replacing β by β^2, we get $R(1, 4) = \pm\beta^{2x}R(1, 2)$. The case $R(1, 4)\ R(1, 5) = R(1, 1) \cdot R(2, 4)$ of (38), and (58) give $3x \equiv 3m'$ (mod 15), whence $x = m' + 5y$.

In a formula due to Jacobi, loc. cit., p. 168, take $\lambda = 5$ and replace β by β^3, a primitive fifth root of unity. Then

$$F(\alpha)F(\beta^3\alpha)F(\beta^6\alpha)F(\beta^9\alpha)F(\beta^{12}\alpha) = \alpha^{-5M}p^2F(\alpha^5) \tag{60}$$

if $5 \equiv g^M$ (mod p), $\alpha^{p-1} = 1$. We take

$$p = F(\alpha)F(\alpha^{-1}), \qquad p = F(\beta^3\alpha)F(\beta^{-3}\alpha^{-1}),$$

and have two equal products of three F's. Take $\alpha = \beta^{-1}$ and divide by $F(\beta^{11}) \cdot F(\beta^{13})$. We get

$$R(5, 8) = \beta^{5M}R(1, 10), \quad \text{or} \quad R(2, 8) = \beta^{5M}R(1, 4) = \pm\ \beta^{2x}\beta^{5M}R(1, 2).$$

Replacing β by β^2 in the earlier $R(1, 4)$, we get

$$R(2, 8) = \pm\ \beta^{4x}R(2, 4) = \pm\ \beta^{4x}\beta^{3m'}R(1, 2).$$

Hence $5M \equiv 2x + 3m'$ (mod 15). Thus $y \equiv m' - M$ (mod 3),

$$x = 6m' - 5M.$$

In §12, $a=1$, $b=2$, $c=4$. By the replacement of β by either β^2 or β^7, I, IV, VI, VII (or their complements) are permuted in a cycle of four, while III and VIII are interchanged, and V is unaltered (or goes to V′). After a choice of β among the eight β^k, k prime to 15 and $k<15$, we may take $R(1, 1)$ to be a product of a unit $\pm\beta^i$ by VI (rather than I, IV or VII). An equivalent choice for the Diophantine equations found as in §8 is that a set of integral solutions, which give rise to 8 distinct sets under the transformations induced by the Galois group generated by the replacements of β by β^2 and β^7, may be taken as the coefficients of $R(1, 1)$. But the unit factor cannot be determined as heretofore since there exists no linear congruence modulo 3 or 5 between the coefficients of $R(1, 1)$, after* reduction to a polynomial of degree 7 in β.

If we waive this difficulty and regard $R(1, 1)$ as known, we have $R(1, 5)$ by (59), $R(1, 3)$ by (57), and find $R(1, 2)$ by (56_2). Then $R(1, 4) = \pm\beta^{2x}R(1, 2)$. We now know a conjugate to every $R(m, n)$.

15. **Case** $e=20$. The $R(m, n)$ are conjugate to $R(1, j)$, $j=1$-5, 8, 9,

* Before that reduction by the octic in β, we have the congruences T, (18), and see that T, (69)–(71), apply also here.

$R(2, 2)$, $R(2, 4)$, $R(2, 8)$, $R(4, 4)$, $R(5, 5)$. The last five are found by (3). By T, (48)–(51),

$$R(1, 8) = \pm \beta^{-2m}R(2, 8), \qquad R(1, 9) = \pm \beta^{2m}R(1, 1).$$

By (60) with β^3 replaced by β^4, we get $R(2, 7) = rR(1, 2)$, $r = \beta^{6M}$. Then (38) gives $R(3, 6) = rR(1, 6)$. The factorizations into prime ideals yield only the facts that $R(1, 3)/R(1, 1)$, $R(1, 8)/R(1, 2)$, $R(2, 2)/R(1, 2)$ are units. The latter are found from one by use of (56_2), $18 \cdot 19 = 12 \cdot 28$, and $R(2, 8) = \beta^{4M}R(2, 2)$.

16. Case $e = 24$. The $R(m, n)$ are conjugate to $R(1, j)$, $j = 1$-11; $R(2, j)$, $j = 2, 4, 6, 8, 10$; $R(3, 3)$, $R(3, 6)$, $R(3, 9)$; $R(4, 4)$, $R(4, 8)$, $R(6, 6)$, $R(8, 8)$. By (7) with $\alpha = \beta^{17}$, $\gamma = \beta^8$, $R(1, 6) = \beta^{-3m'}R(3, 6)$. Expressed in F's, the latter gives $R(1, 9) = \beta^{-3m'} R(1, 2, \beta^7)$.

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POTENTIALS OF POSITIVE MASS. PART II*

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IV. The sweeping-out process

11. **Decreasing sequences of potentials.** As we have seen in §2.1, the limit of an increasing bounded sequence of potential functions of positive mass distributions on a bounded set F is itself a potential of positive mass. The limit of a decreasing sequence of such functions is however not necessarily superharmonic. Nevertheless, de la Vallée Poussin, in the memoir on the Poincaré sweeping-out process already cited in §1, is able to associate a positive mass distribution with a particular type of decreasing sequence, and the ideas underlying this association do not lose in force in a wider application.† Accordingly we shall consider an arbitrary monotone-decreasing sequence of potential functions of positive mass distributions on a bounded set F, which set without loss of generality may be assumed to be closed.

Let then $U_1, U_2, \cdots$ be a monotone-decreasing sequence of potentials

$$U_{i+1}(M) \leqq U_i(M), \qquad M \text{ in } W,$$

of positive mass distributions $f_1(e), f_2(e), \cdots$, respectively, on F. Denote the limiting function by $U_0(M)$. It is everywhere $\geqq 0$, but not necessarily superharmonic. It is harmonic in T.

The distributions $f_i(e)$ are bounded in their set, since, by §2, $f_i(F) \leqq f_1(F)$, $i>1$, and accordingly the sequence contains a subsequence $\{f_{i_n}(e)\}$ which converges in the weak sense to a positive mass function $f(e)$ on F or on a subset of F, that is, converges so that

$$\lim_{i_n=\infty} \int_W \phi(M)df_{i_n}(e) = \int_W \phi(M)df(e)$$

for every continuous function $\phi(M)$. Let $U(M)$ be the potential of $f(e)$.

In particular,

$$(1) \qquad \int_W h^{1/p}(M, P)df(e_P) = \lim_{i_n=\infty} \int_W h^{1/p}(M, P)df_{i_n}(e_P).$$

* See these Transactions, vol. 37 (1935), pp. 226–253. Presented to the Society, December 29, 1932, and September 6, 1934; received by the editors January 14, 1935.

† These methods and ideas are closely related to those of N. Wiener and G. Bouligand. See G. Bouligand, *Fonctions Harmoniques. Principes de Picard et de Dirichlet*, Mémorial des Sciences Mathématiques, fascicule 11, Paris, 1926.

If M is distant δ from F, the equation (1) takes the form

$$U(M) = \lim_{i_n=\infty} U_{i_n}(M),$$

for an arbitrary value of ρ, $\rho<\delta$. Hence $U(M)=U_0(M)$, for M not on F.

If M is on F, we have from (1)

$$\int_W h^{1/\rho}(M,P)df(e_P) \leqq \liminf_{i_n=\infty} \int_W \frac{1}{MP}\, df_{i_n}(e_P) = \liminf_{i_n=\infty} U_{i_n}(M),$$

and since this relation is true for all ρ, we may let ρ approach zero and obtain the equation $U(M)\leqq U_0(M)$, M in F.

Finally, equation (1) is a statement of the fact that

$$A_U(\rho, M) = \lim_{i_n=\infty} A_{U_{i_n}}(\rho, M) = \lim_{i_n=\infty} \frac{1}{4\pi\rho^2}\int_{C(\rho,M)} U_{i_n}(P)dP,$$

the last quantity being $A_{U_0}(\rho, M)$ since the $U_{i_n}(P)$ form a monotone-decreasing sequence with limit $U_0(P)$, for all P. Hence

$$A_U(\rho, M) = A_{U_0}(\rho, M).$$

From this equation and (17), §4, follows a similar result for the operation $a_U(\rho, M)$.

We may speak of the process just described in terms of a monotone-decreasing sequence as a *general sweeping-out process*, and summarize the results in the following theorem.

THEOREM. *For the general sweeping-out process, in which* $U_1(M)\geqq U_2(M) \geqq \cdots \geqq U_0(M)=\lim\ (i=\infty)\ U_i(M)$, *and* $U(M)$ *is the potential of a distribution* $f(e)$ *defined by the weak convergence of a subsequence of the* $f_i(e)$ *on* F, *we have*

$$(2) \qquad U(M) = U_0(M), \qquad M \textit{ not on } F,$$

$$(3) \qquad U(M) \leqq U_0(M), \qquad M \textit{ on } F,$$

$$(4) \qquad A_U(\rho, M) = A_{U_0}(\rho, M), \qquad a_U(\rho, M) = a_{U_0}(\rho, M), \qquad M \textit{ in } W.$$

The potential $U(M)$ *and the distribution* $f(e)$, *for sets e measurable Borel, are uniquely determined, independently of the subsequence on which there is weak convergence.*

In fact, by (9), §2.2, and this equation (4), letting ρ approach zero, it follows from the uniqueness of $U_0(M)$ that there is only one possible function

$U(M)$. But given the potential $U(M)$ its mass distribution $f(e)$ is uniquely determined on all sets measurable Borel.*

In particular, as a further consequence of (4), letting ρ approach zero, $U(M)=U_0(M)$ wherever the latter is the point set derivative of its spatial integral. Hence if E is any set of positive spatial Lebesgue measure, we shall have

$$\int_E U(M)dM = \int_E U_0(M)dM. \tag{4'}$$

Corollary. *If $U_1(M)=U_1'(M)+U_1''(M)$, $U_1'(M)$ and $U_1''(M)$ being potentials of distributions of positive masses on F, and the generalized sweeping-out process is carried out separately on $U_1'(M)$ and $U_1''(M)$, then $U_i(M)=U_i'(M)+U_i''(M)$ determines a sweeping-out process for $U_1(M)$, so that $U(M)=U'(M)+U''(M)$.*

12. **Poincaré sweeping-out process for continuous potentials.** As a first case, we consider that discussed in the main by de la Vallée Poussin, in which a given potential $U(M)$ of a distribution of positive mass on F is assumed to be continuous on the set $\Sigma+s$ of §1, with respect to $\Sigma+s$ itself. We define a decreasing sequence $V_n(M)$ in terms of the sequence solution for the domain Σ and the boundary values $U(P)$, P on s, and describe the process of removing the mass from Σ as the *Poincaré sweeping-out process*.

More precisely, let Σ_n be a sequence of nested regular domains approximating to Σ,† and choose $V_n(M)$ as the following uniquely defined function:

$V_n(M)$ is continuous in W,

$V_n(M)$ is the solution of Laplace's equation in Σ_n which takes on the given values $U(P)$ on s_n, regular at ∞ if Σ is unbounded,

$V_n(M)=U(M)$ for M in $C\Sigma_n$, the complement of Σ_n.

Then for M in Σ, $V_0(M)=\lim\ (n=\infty)\ V_n(M)$ is the desired sequence solution, and is independent of the choice of the set of nested domains.‡

Since $V_n(M)$ is $\leqq U(M)$ and is harmonic wherever $V_n(M)<U(M)$, it is, being continuous, superharmonic, and, by §2, a potential of a positive mass

* F. Riesz, Memoir (2), cited in §2. See also G. C. Evans, *Fundamental points of potential theory*, Rice Institute Pamphlet, vol. 7 (1920), pp. 252–329, p. 271 and p. 285, where the determination of the additive function of point sets is given in terms of a uniquely determined function of curves, with regular discontinuities.

† That is, Σ contains Σ_n with its boundary, and Σ_n contains Σ_{n-1} with its boundary; and the boundary of Σ_n is regular. (Hence there is one and only one solution of the Dirichlet problem for Σ_n which takes on continuously assigned boundary values which are continuous.) Every point of Σ is to lie ultimately in some Σ_n.

‡ O. D. Kellogg, *Foundations of Potential Theory*, Berlin, 1929, p. 317 ff.

distribution $f_n(e)$. Moreover $V_{n+1}(M)$ is everywhere $\leqq V_n(M)$, since the two functions are identical except in Σ_{n+1}, while in that region $V_n(M)$ is superharmonic and $V_{n+1}(M)$ harmonic. The sequence is therefore a special case of the sequence of §11, and the mass functions converge in the weak sense for a subsequence $\{n_i\}$ to a mass distribution $\mu(e)$ whose potential $V(M)$ is dominated by $V_0(M)$.

If Σ is a bounded domain, the total mass $f_n(W)$ or $f_n(C\Sigma_n)$ is $f(W)=f(F)$, for we have $V_n(M)=U(M)$ outside of a properly chosen sphere; if Σ is an infinite domain the $f_n(W)$ is $\leqq f(F)$, and some of the mass may be described as lost to infinity. In the limiting distribution there is no mass in Σ, and $\mu(s+B)=f(F)$ or is $\leqq f(F)$ according as Σ is bounded or unbounded; in the two cases the Poincaré sweeping-out process may be described as a transfer of the mass from Σ to its boundary s, or in part to s and in part to infinity.* In fact, from what is given above, it follows evidently that if $\nu_n(e)$ denotes the distribution $f_n(e\cdot s_n)$ and $\nu(e)$ a limiting distribution in weak convergence for the subsequence $\{n_i\}$, then

$$\begin{aligned} \nu(s) &= f(\Sigma), &&\text{if } \Sigma \text{ is bounded,} \\ &\leqq f(\Sigma), &&\text{if } \Sigma \text{ is unbounded.} \end{aligned}$$

Under this first case we may include that where $U(M)$ is continuous in the part of $\Sigma+s$ within a distance δ of s, for after a certain n the boundaries s_n will all lie within that neighborhood of s.

The following statement is immediate, as a property of the sequence solution. *Given two potentials $U'(M)$, $U''(M)$ of the kind just specified such that $U'(M)\geqq U''(M)$ for all M; then the corresponding functions $V_0'(M)$, $V_0''(M)$ satisfy the relation*

$$V_0'(M) \geqq V_0''(M), \qquad \textit{for all } M. \tag{5}$$

12.1. **Sweeping out of discontinuous potentials.** Turn now to the general case, $U(M)$ being the potential of an arbitrary distribution of positive mass on F, and take a sequence of nested domains Σ_n as in §12. Since $U(M)$ is lower semicontinuous and positive there exists a sequence $U^{(p)}(M)$ of not negative functions, defined and continuous on s_n and tending to $U(M)$ at every point of s_n. Let $U_n^{(p)}(M)$ be the function which is defined and continuous in Σ_n+s_n, harmonic in Σ_n, regular at ∞ if Σ is unbounded, and takes on the values $U^{(p)}(M)$ continuously on s_n. Define

* In de la Vallée Poussin, loc. cit., the process is described in terms of an actual transfer of mass for a domain of sufficiently smooth boundary, based on its approximation by a domain consisting of a finite number of spheres.

$$
(6) \qquad \begin{aligned} V_n(M) &= \lim_{p=\infty} U_n^{(p)}(M), \qquad && M \text{ in } \Sigma_n, \\ &= U(M), && M \text{ in } C\Sigma_n. \end{aligned}
$$

The function $V_n(M)$, in Σ_n, is independent of the choice of the monotone-increasing sequence $U^{(p)}(M)$; this is in fact well known. We note that if we define $U^{(p)}(M)$, for all M, as the average $U(\rho, M)$, $\rho=1/p$, it follows by §2 that $U^{(p)}(M)$ is a (continuous) potential of a distribution of positive mass on a set bounded independently of p, the total mass being $f(F)$. The function $V_n^{(p)}(M)$, equal to $U_n^{(p)}(M)$ for M in Σ_n and equal to $U^{(p)}(M)$ for M in $C\Sigma_n$, is therefore a potential of positive mass on a set which is bounded independently of n, p. But the functions $V_n^{(p)}(M)$ form a monotone-increasing sequence with respect to p, and $V_n(M)=\lim\ (p=\infty)\ V_n^{(p)}(M)$ for all M. Hence by §2.1, $V_n(M)$ is a potential of a distribution of positive mass on a bounded portion of $C\Sigma_n$, and the total mass is in value $f(F)$ or $\leqq f(F)$, according as Σ is bounded or unbounded. The same remark applies to a mass distribution to which these converge weakly as n tends to infinity.

The functions $U, V_1, V_2, \cdots$ constitute the decreasing sequence of §11. In fact, $V_{n+1}^{(p)}(M) \leqq V_n^{(p)}(M) \leqq U^{(p)}(M)$ for every p.

THEOREM. *The functions $V_0(M)$, $V(M)$ and the mass distribution $\mu(e)$, for sets e measurable Borel, are uniquely determined, independently of the choice of the sequence of nested regular domains for Σ, and of the subsequence over which the weak convergence is established.*

Consider first two regular domains Σ_1, Σ_2 such that Σ contains Σ_2, which contains Σ_1. We note that the corresponding potentials V_1, V_2 satisfy the relation

$$
V_1(M) \geqq V_2(M), \qquad M \text{ in } W.
$$

In fact, given M, and choosing $U^{(p)}(M)=U(\rho, M)$, $\rho=1/p$, as above,

$$
V_1^{(p)}(M) \geqq V_2^{(p)}(M), \qquad \text{for all } p.
$$

Let now $\{\Sigma_n'\}$, $\{\Sigma_m''\}$ constitute two sequences of nested regular domains for Σ. We may construct a third sequence of such domains, $\{\Sigma_j^0\}$, which contains an infinite number of domains of each of the sequences $\{\Sigma_n'\}$, $\{\Sigma_m''\}$; for, given any Σ_n', there is a Σ_m'' which contains Σ_n' and its boundary. In fact, for every m the set $(\Sigma_n'+s_n')\cdot(C\Sigma_m'')$ is closed and contains its successor when m is replaced by $m+1$; but since there is no point belonging to it for all m it must become empty for m sufficiently large.

It follows that the limiting functions $V_0'(M)$, $V_0''(M)$ are identical. In fact,

$$V_0'(M) = \lim_{n=\infty} V_n'(M) = \lim_{j=\infty} V_j^0(M) = \lim_{m=\infty} V_m''(M) = V_0''(M).$$

Hence the function $V_0(M)$ is unique.

But also, if $V(M)$ is the potential resulting from the weak convergence on any subsequence $\{\Sigma_{n_i}\}$ of any sequence $\{\Sigma_n\}$, we have, by (9), §2.2, and (4), §11,

$$V(M) = \lim_{\rho=0} A_V(\rho, M) = \lim_{\rho=0} A_{V_0}(\rho, M), \tag{6'}$$

where the function $V_0(M)$ is unique. Hence $V(M)$ is unique. Finally, if $V(M)$ is unique, the mass distribution $\mu(e)$, of which it is the potential, is uniquely determined on all sets measurable Borel.

We note finally that the inequality (5) is still valid. That is, if the potentials $U'(M)$, $U''(M)$, of positive mass distributions, are given, with $U'(M) \geqq U''(M)$, for all M, then $V_0'(M) \geqq V_0''(M)$ for all M. Moreover, for the resulting potentials $V'(M)$, $V''(M)$ we have, by means of (6'),

$$V'(M) \geqq V''(M), \qquad M \text{ in } W. \tag{5'}$$

12.2. **Alternative procedure for sweeping-out of discontinuous potentials.** The following method also extends the Poincaré sweeping-out process to apply to an arbitrary potential, and is more in line with the procedure of de la Vallée Poussin for continuous potentials. We write the given potential $U(M)$ in the form

$$U(M) = U'(M) + U''(M),$$

$$U'(M) = \int_W \frac{1}{MP}\, df(e_P \cdot \Sigma), \qquad U''(M) = \int_W \frac{1}{MP}\, df(e_P \cdot (s + B)),$$

and carry out the process on $U'(M)$. For this purpose we form a monotone-increasing sequence of continuous potentials $U'^{(p)}(M)$ of positive distributions on a bounded set (or potentials each continuous in a portion of $\Sigma + s$ neighboring s), such that

$$\lim_{p=\infty} U'^{(p)}(M) = U'(M), \qquad M \text{ in } W.$$

Let $V_0'^{(p)}(M)$, $V'^{(p)}(M)$ be the function and potential, respectively, generated by the sweeping-out of the mass from Σ of the continuous potentials $U'^{(p)}(M)$.

The functions $V_0'^{(p)}(M)$ form a monotone-increasing sequence dominated by $U'(M)$. Hence there exists the function

$$\overline{V}_0(M) = U''(M) + \overline{V}_0'(M) \tag{7}$$

with

$$\overline{V}_0'(M) = \lim_{p=\infty} V_0'^{(p)}(M). \tag{7'}$$

Moreover the total mass for $U'^{(p)}(M)$ is bounded, $\leqq f(\Sigma)$, and lies on a set which is bounded independently of p, and any closed set contained in B, ultimately, for sufficiently large p, bears no mass. Consequently the mass functions for $V'^{(p)}(M)$ converge in the weak sense, as p tends to ∞ on a subsequence, to a positive mass distribution $\bar{\mu}(e)$ which lies entirely on s. We define $\overline{V}(M)$ as the potential

$$\overline{V}(M) = U''(M) + \overline{V}'(M) \tag{8}$$

with

$$\overline{V}'(M) = \int_W \frac{1}{MP}\, d\bar{\mu}(e_P). \tag{8'}$$

Lemma I. *The function $\overline{V}_0(M)$ is independent of the choice of the sequence $U'^{(p)}(M)$.*

This lemma is verified by means of the relation (5) when the monotone-increasing sequence $U'^{(p)}(M)$ has been replaced by the strictly increasing sequence of potentials $(1-1/p)\ U'^{(p)}(M)$. Two such sequences may then be compared in the customary manner. In fact, let $U'^{(p)}(M)$ be a strictly increasing sequence of such continuous potentials, $u'^{(p)}(M)$ a strictly increasing sequence of such potentials, each continuous in a closed region σ_p comprising s and the points of Σ not distant from s by more than some $\delta_p > 0$. For p_1 given, since $u'^{(p)}(M)$ is lower semicontinuous the set of points where $u'^{(p)}(M) \leqq U'^{(p_1)}(M)$ is closed, and hence, for p sufficiently great, will vanish. Similarly for p_2 given, the set of points in σ_{p_2} where $U'^{(p)}(M) \leqq u'^{(p_2)}(M)$ is closed, and hence, for p sufficiently great, will vanish. Accordingly, for the corresponding $V'^{(p)}(M)$, $v_0'^{(p)}(M)$, obtained by the sweeping out, we have the analogous relations, by (5), and both sequences $V_0'^{(p)}(M)$, $v_0'^{(p)}(M)$ have the same limiting function $\overline{V}_0(M)$.

Lemma II. *The equation* (4), §11, *remains valid for the procedure of the present* §12.2.

In fact, we have merely to repeat the proof already given of (4).

Theorem. *Given $U(M)$, the potential of an arbitrary distribution of positive mass on F, the limiting functions $V_0(M)$, $\overline{V}_0(M)$, determined by the processes of* §§12.1, 12.2 *respectively, are identical*:

$$V_0(M) = \overline{V}_0(M), \qquad \textit{for all } M \textit{ in } W. \tag{9}$$

The statement is true for M in $s+B$. For, for M in $s+B$,

$$V_0(M) = U(M),$$
$$\overline{V}_0'(M) = \lim_{p=\infty} V_0'^{(p)}(M) = \lim_{p=\infty} U'^{(p)}(M) = U'(M),$$
$$\overline{V}_0(M) = \overline{V}_0'(M) + U''(M) = U'(M) + U''(M) = U(M).$$

For M in Σ, we have

$$U''(M) + V_0'^{(p)}(M) \leqq U''(M) + V_n'^{(p)}(M) \leqq V_n(M), \qquad n,\ p \text{ arbitrary},$$
$$\overline{V}_0(M) = U''(M) + \lim_{p=\infty} V_0'^{(p)}(M) \leqq V_n(M), \qquad n \quad \text{arbitrary}.$$

Hence

$$\overline{V}_0(M) \leqq V_0(M).$$

In order to establish the complementary inequality, let σ_δ be the portion of Σ at a distance from s not greater than δ, and Σ_δ the remaining open set. Write

$$U'(M) = U_\delta(M) + U'''(M)$$

where

$$U_\delta(M) = \int_W \frac{1}{MP}\, df(e_P\cdot\Sigma_\delta), \qquad U'''(M) = \int_W \frac{1}{MP}\, df(e_P\cdot\sigma_\delta).$$

Given $\epsilon>0$, and Q a fixed point in Σ, distant an amount κ from s, we choose a positive $\delta<\kappa$ so that $f(\sigma_\delta)/\kappa$ shall be $<\epsilon$; this is possible, since, Σ being an open set, $\lim\,(\delta=0)\, f(\sigma_\delta)=0$. But, evidently, with notation corresponding to that just used,

$$V_0(Q) = U''(Q) + V_{\delta 0}(Q) + V_0'''(Q) < U''(Q) + V_{\delta 0}(Q) + \epsilon,$$
$$\overline{V}_0(Q) = U''(Q) + \overline{V}_{\delta 0}(Q) + \overline{V}_0'''(Q) \geqq U''(Q) + \overline{V}_{\delta 0}(Q).$$

Now if we denote by $U_\delta^{(p)}(M)$ the average of $U_\delta(M)$ over a sphere of radius $\rho=1/p$, and take $p>1/\delta$, we shall have $U_\delta^{(p)}(M)=U_\delta(M)$ for M near enough to s. Hence, for the corresponding harmonic functions determined by their continuous values on s_n,

$$V_{\delta n}^{(p)}(Q) = V_{\delta n}(Q), \qquad n \text{ great enough},$$

whence

$$V_{\delta 0}^{(p)}(Q) = V_{\delta 0}(Q), \qquad \overline{V}_{\delta 0}(Q) = \lim_{p=\infty} V_{\delta 0}^{(p)}(Q) = V_{\delta 0}(Q).$$

Consequently, substituting in the above inequalities,

$$\overline{V}_0(Q) \geqq U''(Q) + V_{s0}(Q) > V_0(Q) - \epsilon.$$

This however yields the desired complementary inequality $\overline{V}_0(Q) \geqq V_0(Q)$, whence $\overline{V}_0(Q) = V_0(Q)$ for Q in Σ.

COROLLARY. *The potential $\overline{V}(M)$ is uniquely determined, and is the same as the potential $V(M)$ of* §12.1,

$$\overline{V}(M) = V(M), \qquad M \textit{ in } W. \tag{10}$$

In fact, by (4), §11,

$$A_{\overline{V}}(\rho, M) = A_V(\rho, M) = A_{V_0}(\rho, M),$$

whence the conclusion follows by letting ρ approach zero.

In particular, the process of §12.2 is instanced in the sweeping out of a general positive distribution on Σ by sweeping out successively the portions within the domains Σ_k, where these constitute a sequence of nested domains for Σ.

13. **Consistency theorems.** We may compare the potentials resulting from the succession of a generalized and a Poincaré sweeping-out process, or of two Poincaré sweeping-out processes.

LEMMA. *Let $U_1, U_2, \cdots$ be a monotone-decreasing sequence of potentials of positive mass distributions on F, with limit $U_0(M)$; let $f(e)$ be the mass distribution to which a subsequence of the mass distributions $f_i(e)$ of $U_i(M)$ converges in the weak sense, and $U(M)$ its potential. Let $V_{1,0}(M)$, $V_{2,0}(M), \cdots, V_0(M)$ be the limiting functions obtained by the sweeping out of the mass distributions $f_1(e), f_2(e), \cdots, f(e)$ from Σ; then*

$$V_0(M) = \lim_{i=\infty} V_{i,0}(M), \qquad M \textit{ in } \Sigma. \tag{11}$$

By (5), the functions $V_{i,0}(M)$ form a monotone-decreasing sequence, with limit, say, $\overline{V}_0(M)$. We show that $\overline{V}_0(M) = V_0(M)$, M in Σ. With the aid of (2) and (3), §11, we have

$$U(M) \leqq U_0(M) \leqq U_i(M), \qquad i = 1, 2, \cdots, M \text{ in } \Sigma.$$

Hence $V_0(M) \leqq V_{i,0}(M)$ and

$$V_0(M) \leqq \overline{V}_0(M), \qquad M \text{ in } \Sigma. \tag{12}$$

In order to establish the complementary inequality, let Σ_n be a set of nested regular domains for Σ, with the boundaries s_n of the Σ_n taken smooth enough so that $\lambda_n(M, P)$, the normal derivative of the Green's function for the Σ_n, with pole at M, will be continuous in P, for P on s_n. We carry out the

Poincaré process of §12.1 in terms of this set of nested domains, taking the monotone-increasing sequences $U_i^{(p)}(M)$ of that section as continuous potentials of positive mass. But for M in Σ_n,

$$V_{i,n}^{(p)}(M) = \frac{1}{4\pi}\int_{s_n} \lambda_n(M, P)U_i^{(p)}(P)dP,$$

whence

(13) $$V_{i,n}(M) = \frac{1}{4\pi}\int_{s_n} \lambda_n(M, P)U_i(P)dP.$$

Similarly for the same process carried out on $U(M)$,

$$V_n(M) = \frac{1}{4\pi}\int_{s_n} \lambda_n(M, P)U(P)dP.$$

We note that we have also

(13′) $$V_n(M) = \frac{1}{4\pi}\int_{s_n} \lambda_n(M, P)U_0(P)dP.$$

In fact, if σ is any regular surface element,

$$\int_\sigma U_0(P)dP = \lim_{i=\infty}\int_\sigma U_i(P)dP,$$

$$\int_\sigma U(P)dP = \int_W df(e_R)\int_\sigma \frac{1}{PR}\,dP.$$

The inside integral of the right hand member is however the potential at R of unit density distribution on σ, and is therefore continuous in R, for R in W. Hence by the weak convergence property there is the subsequence $\{i'\}$ of $\{i\}$ such that

$$\int_W df(e_R)\int_\sigma \frac{1}{PR}\,dP = \lim_{i'=\infty}\int_W df_{i'}(e_R)\int_\sigma \frac{1}{PR}\,dP.$$

Accordingly,

$$\int_\sigma U(P)dP = \lim_{i'=\infty}\int_\sigma U_{i'}(P)dP$$

and $\int_\sigma U(P)dP = \int_\sigma U_0(P)dP$; from which (13′) follows.

We have now what we need. Given Q in Σ and $\epsilon > 0$, we can find a stage n_1, of the Poincaré process, such that for $n \geqq n_1$, we have

$$V_0(Q) > V_n(Q) - \epsilon.$$

Consequently, since by (13), (13′), $V_n(Q) = \lim\,(i=\infty)\,V_{i,n}(Q)$,

$$V_0(Q) > \lim_{i=\infty} V_{i,n}(Q) - \epsilon,$$

and thus

$$V_0(Q) > \lim_{i=\infty} V_{i,0}(Q) - \epsilon,$$

for $V_{i,0}(Q) \leqq V_{i,n}(Q)$, the $V_{i,n}(M)$ forming a monotone-decreasing sequence in n, according to the definition of the Poincaré process as applied to $U_i(M)$. But then, $V_0(Q) > \overline{V}_0(Q) - \epsilon$ and

$$V_0(Q) \geqq \overline{V}_0(Q), \qquad Q \text{ in } \Sigma.$$

This is the complementary inequality, and the lemma is therefore established.

THEOREM I. *Let $U_1, U_2, \cdots$ be a monotone-decreasing sequence of potentials of positive mass distributions on F, generating a potential $U(M)$ by the generalized sweeping-out process, and let $v_1, v_2, \cdots, V$ be the potentials arising from the sweeping out of the above masses from Σ. Then $v_1, v_2, \cdots$ also constitute a generalized sweeping-out process of monotone-decreasing potentials, generating the same potential $V(M)$, for all M.*

By (5′), §12.1, the potentials $v_i(M)$ constitute a monotone-decreasing sequence. Let then E be any bounded set of positive spatial Lebesgue measure. With reference to the notation of the lemma, and writing $G = C\Sigma$, as before, we have by (4′), §11,

$$\begin{aligned}\int_E v_i(M)dM &= \int_E V_{i0}(M)dM \\ &= \int_{E\cdot G} V_{i0}(M)dM + \int_{E\cdot\Sigma} V_{i0}(M)dM.\end{aligned}$$

We denote $\lim\,(i=\infty)\; v_i(M)$ by $v_0(M)$ and the corresponding potential by $v(M)$, and by means of (4′), we obtain

$$\begin{aligned}\int_E v(M)dM &= \int_E v_0(M)dM = \lim_{i=\infty}\int_E v_i(M)dM \\ &= \lim_{i=\infty}\int_{E\cdot G} V_{i0}(M)dM + \lim_{i=\infty}\int_{E\cdot\Sigma} V_{i0}(M)dM.\end{aligned}$$

But in $E\cdot G$, $V_{i0}(M) = U_i(M)$, and in $E\cdot\Sigma$, $\lim\,(i=\infty)\; V_{i0}(M) = V_0(M)$ by the lemma of this section. Hence, since we are dealing with monotone sequences,

$$\int_E v(M)dM = \int_{E\cdot G} U_0(M)dM + \int_{E\cdot\Sigma} V_0(M)dM.$$

But now again we may apply (4′), and write

$$\int_{E\cdot G} U_0(M)dM = \int_{E\cdot G} U(M)dM = \int_{E\cdot G} V_0(M)dM,$$

so that

$$\int_E v(M)dM = \int_E V_0(M)dM = \int_E V(M)dM.$$

In particular,

$$a_v(\rho, M) = a_V(\rho, M),$$

whence, letting ρ approach zero,

$$v(M) = V(M), \qquad M \text{ in } W.$$

This is what was to be proved.

In particular, we may take, for the potentials $U_i(M)$, the sequence of potentials obtained by sweeping out a given potential $U_1(M)$ from Σ by means of a sequence of nested regular domains Σ_i for Σ, and for the $v_i(M)$, the sequence of potentials arising from the sweeping out of the $U_i(M)$ from a domain Σ' of which the boundary s' is a closed subset of $G = C\Sigma$. By the theorem of §12.1 the functions $v_1(M)$, $v_2(M)$, $\cdots$ are all identical. We deduce then, from the theorem of the present section, that the potential arising by sweeping out $U_1(M)$ from Σ' is the same as that obtained by first sweeping out $U_1(M)$ from Σ and then sweeping out the resulting potential from Σ'.

THEOREM II. *Let g be a closed subset of $G = C\Sigma$, and s' the external frontier of g, so that s' is the boundary of an infinite domain Σ' which contains Σ. Let $U(M)$ be a potential of a positive mass distribution $f(e)$ on F. Then the potential arising from the sweeping out of $f(e)$ from Σ' is everywhere the same as that obtained first, by sweeping out $f(e)$ from Σ, and second, by sweeping out the resulting distribution from Σ'.*

14. **Sweeping out of unit mass.** Consider a distribution $\mu(e, Q)$ arising from a sweeping out of unit mass at Q from the domain Σ of §1, and denote by $v_0(M, Q)$, $v(M, Q)$ the corresponding limiting and potential functions. For definiteness we take Σ as bounded; the unbounded region may be treated in the same manner.

As a first case we assume that Σ is normal for the Dirichlet problem (that is, corresponding to arbitrary continuous values assigned on s we assume that

there exists in Σ a solution of Laplace's equation which takes on continuously the assigned boundary value at every point of s), and that the boundary s is sufficiently smooth for applications of Green's theorem; in fact, that the normal derivative of the Green's function for Σ, with pole at Q in Σ, is continuous on s. We denote this derivative by $\lambda(Q, P)$. It is harmonic in Q for Q in Σ.

The function*

$$I = \frac{1}{4\pi}\int_s \frac{\lambda(Q, P)}{MP}\, dP \tag{14}$$

is harmonic as a function of M for M not on s, and is continuous in M for all M, vanishing continuously at ∞. For M fixed in $C(\Sigma+s)=B$, I is the value at Q of the function, harmonic in Σ, which takes on the value $1/(MP)$ as Q tends to a point P on s. Hence for Q in Σ, M in B,

$$I = 1/(QM),$$

and since both members are continuous, the same equation holds for M on s. Consequently, in Σ, as a function of M, I is the harmonic function which takes on continuously the values $1/(QP)$ as M tends to P on s. We deduce then that

$$v(M, Q) = v_0(M, Q) = \frac{1}{4\pi}\int_s \frac{\lambda(Q, P)}{MP}\, dP, \qquad M \text{ in } W. \tag{15}$$

The distribution of mass is uniquely determined on every set measurable Borel, if its potential is everywhere given. Hence, for our surface s, we have†

$$\mu(e, Q) = \frac{1}{4\pi}\int_{e\cdot s} \lambda(Q, P)dP. \tag{15'}$$

This is an absolutely continuous distribution of mass on s whose surface density at a point P of s is the normal derivative of the Green's function with pole at Q, divided by 4π. From (15), the Green's function itself is given by the equation

$$g(Q, M) = \frac{1}{QM} - v(M, Q). \tag{15''}$$

Let now $f(e)$ denote an arbitrary distribution of positive mass, lying on a

* A generalization of the corresponding function for the circle; see Picard, *Traité d'Analyse*, vol. 2, Paris, 1905, p. 91.

† The corresponding relation in two dimensions interprets the fact that the method of conformal mapping applies to the sweeping out of unit mass in the same way as to the normal derivative of the Green's function. See C. de la Vallée Poussin, *Extension de la méthode du balayage de Poincaré, et problème de Dirichlet*, Annales de l'Institut Henri Poincaré, vol. 2 (1932), pp. 169–232, at p. 190.

closed set in Σ, with a potential $U(M)$ which is therefore continuous on $C(\Sigma)=s+B$. The function $V_0(M)$ which is harmonic in Σ, identical with $U(M)$ in $C(\Sigma+s)=B$, and takes on continuously the values $U(M)$ on s, for approach from Σ, is therefore, by the mean value property (3), §11, identical with the potential $V(M)$ of the swept-out mass $\mu(e)$. This mass lies entirely on s. We have

$$\begin{aligned} V(M) = V_0(M) &= \frac{1}{4\pi}\int_W df(e_Q)\int_s \lambda(Q, P)\frac{1}{MP}dP \\ &= \int_W v(M, Q)df(e_Q). \end{aligned} \tag{16}$$

In fact, this last integral is a continuous function of M, $v(M, Q)$ being a continuous function of M in W and of Q in the closed set on which $f(e)$ lies. Moreover, for M in $s+B$, $v(M, Q)$ is $1/(MQ)$, so that the given integral reduces to $U(M)$. It is also harmonic in M for M in Σ, since $v(M, Q)$ has that property.

The function $\lambda(Q, P)$ is not negative, and therefore we may change the order of integration in (16) and write

$$V(M) = \frac{1}{4\pi}\int_s \frac{dP}{MP}\int_W \lambda(Q, P)df(e_Q). \tag{17}$$

That is, $V(M)$ is the potential of the distribution of positive mass

$$\begin{aligned} \mu(E) &= \frac{1}{4\pi}\int_{E\cdot s} dP\int_W \lambda(Q, P)df(e_Q) \\ &= \frac{1}{4\pi}\int_W df(e_Q)\int_{E\cdot s}\lambda(Q, P)dP = \frac{1}{4\pi}\int_\Sigma df(e_Q)\int_{E\cdot s}\lambda(Q, P)dP. \end{aligned} \tag{18}$$

The mass distribution is absolutely continuous on s, of surface density

$$\frac{1}{4\pi}\int_\Sigma \lambda(Q, P)df(e_Q),$$

and from (15′), (18)

$$\mu(E) = \int_\Sigma \mu(E, Q)df(e_Q). \tag{19}$$

This last equation includes as a special case the following, where s_1 is a regular surface bounding a domain Σ_1 interior to Σ, $\mu_\Sigma(e, Q)$ and $\mu_{\Sigma_1}(e, Q)$ denoting the respective swept-out unit masses:*

* Equation (19′) is given in the case of smooth boundaries by de la Vallée Poussin, loc. cit., p. 182.

$$\mu_\Sigma(E, P) = \int_\Sigma \mu_\Sigma(E, Q)d\mu_{\Sigma_1}(e_Q, P). \tag{19'}$$

We are able to extend the equation (19), and therefore of course (19′), to a general domain Σ, whose boundary is a closed bounded set. For the sake of definiteness we retain the hypothesis that Σ is a bounded set.

THEOREM. *Let $f(e)$ be a distribution of positive mass on a general* (*bounded*) *domain Σ whose boundary is s. If $\mu(e)$, $\mu(e, Q)$ are the mass distributions obtained by the sweeping out of $f(e)$ and of unit mass at Q, respectively, then* (19) *is valid.*

Suppose first that $f(e)$ is a distribution lying entirely on a closed set F interior to Σ; without loss of generality we may suppose F to be perfect. Let Σ_n be a sequence of nested regular domains for Σ, and $\mu_n(e)$, $\mu_n(e, Q)$ the sweeping-out distributions satisfying (19); let

$$\bar{\mu}(e) = \int_W \mu(e, Q)df(e_Q).$$

LEMMA I. *If the mass distribution $f(e)$ is swept out of Σ, by means of the domains Σ_n, then, for $\phi(P)$ continuous,*

$$\lim_{n=\infty} \int_W \phi(P)d\mu_n(e_P) \text{ exists, and equals } \int_W \phi(P)d\mu(e_P). \tag{20}$$

Otherwise there would be a subsequence $\{n_i\}$ such that $\int_W \phi d\mu_{n_i}$ would approach some value different from the right hand member. But this is impossible, since there would be a subsequence of the $\{n_i\}$ for which the mass distributions would converge weakly to a swept-out distribution, and the swept-out distribution is unique.

LEMMA II. *For each set E, measurable Borel, the function $\mu(E, Q)$ is harmonic in Q, for Q in Σ, and is $\leqq 1$.*

In fact, for Σ_n as above and $\phi(P)$ continuous, from Lemma I,

$$\lim_{n=\infty} \int_W \phi(P)d\mu_n(e_P, Q) = \int_W \phi(P)d\mu(e_P, Q).$$

The right hand member is harmonic in Q, for Q in Σ, for each continuous $\phi(P)$; for the integral of the left hand member is harmonic in Q, Q in Σ_n, and converges to the right hand member, remaining bounded.

Consequently if $\psi(P)$ is any bounded function, measurable Borel, the I-

integral $\int_W \psi(P) d\mu(e_P, Q)$ is harmonic in Q, Q in Σ. In fact, such a function is a (transfinite) limit, starting from continuous functions $\phi(P)$. In particular, if we take $\psi(P)=1$ on E and 0 elsewhere, the I-integral reduces to $\mu(E, Q)$, and this quantity is therefore harmonic in Q, Q in Σ.

Finally, $\mu(E, Q) \leqq 1$, since $\mu_n(W, Q) \leqq 1$.

LEMMA III.* *A sufficient condition that $\mu_n(e)$ converges to $\bar{\mu}(e)$ weakly is that*

$$\lim_{n=\infty} \int_W \phi(P) d\mu_n(e_P) = \int_W \phi(P) d\bar{\mu}(e_P)$$

for every continuous $\phi(P)$.

Consider in fact a rectangular net the boundaries of whose meshes bear none of the mass distribution $\bar{\mu}(e)$. Let ω be one such open mesh, Ω its closed cover. Let $\phi_1(P)=1$ in Ω and 0 at a distance $\geqq 1/k$ from ω, being continuous, $\leqq 1$, in W. Given $\epsilon>0$, by taking k large enough, we have

$$\int_W \phi_1(P) d\bar{\mu}(e) < \bar{\mu}(\omega) + \epsilon,$$

$$\limsup_{n=\infty} \mu_n(\Omega) \leqq \lim_{n=\infty} \int_W \phi_1(P) d\mu_n(e_P) < \bar{\mu}(\omega) + \epsilon.$$

On the other hand, if we take $\phi_2(P)$ continuous and $\leqq 1$ in W, zero outside ω, and unity in ω at a distance $\geqq 1/k$ from $C\omega$, we have similarly, taking k large enough,

$$\int_W \phi_2(P) d\bar{\mu}(e_P) > \bar{\mu}(\omega) - \epsilon,$$

$$\liminf_{n=\infty} \mu_n(\omega) \geqq \lim_{n=\infty} \int_W \phi_2(P) d\mu_n(e_P) > \bar{\mu}(\omega) - \epsilon.$$

In other words, $\mu_n(e)$ converges on each mesh of the net to $\bar{\mu}(e)$.

To return to the theorem, we take $\phi(P)$ continuous in W, and obtain

$$\begin{aligned} \int_W \phi(P) d\mu_n(e_P) &= \int_W \phi(P) d_P \left[\int_W \mu_n(e_P, Q) df(e_Q) \right] \\ &= \int_W \phi(P) d_P \left[\int_F \mu_n(e_P, Q) df(e_Q) \right]. \end{aligned}$$

But $\mu_n(W, Q) \leqq 1$, $\mu_n(e, Q)$ is continuous in Q for Q in F, and $\phi(P)$ is continuous; hence we may change the order of integration and write†

* See §2.1, footnote to (7).

† G. C. Evans, *Functionals and their Applications*, New York, 1918, p. 103.

$$\int_W \phi(P)d\mu_n(e_P) = \int_F df(e_Q)\int_W \phi(P)d\mu_n(e_P, Q).$$

Similarly,

$$\int_W \phi(P)d\bar{\mu}(e_P) = \int_F df(e_Q)\int_W \phi(P)d\mu(e_P, Q).$$

But this again, from the weak convergence on $\{n\}$, is equal to

$$\int_F df(e_Q)\lim_{n=\infty}\int_W \phi(P)d\mu_n(e_P, Q).$$

The function $\int_W\phi(P)d\mu_n(e_P, Q)$ is bounded, irrespective of n, is harmonic in Q for Q in F, and in fact approaches its limit uniformly for Q in F. Hence

$$\int_W \phi(P)d\bar{\mu}(e_P) = \lim_{n=\infty}\int_F df(e_Q)\int_W \phi(P)d\mu_n(e_P, Q)$$

$$= \lim_{n=\infty}\int_W \phi(P)d\mu_n(e_P).$$

By Lemma III then, $\mu_n(e)$ converges weakly to $\bar{\mu}(e)$, and therefore $\bar{\mu}(e)$ and $\mu(e)$ are identical, and $\mu(e)$ is given by (19).

In order to complete the proof of the theorem, let now $f(e)$ be any positive mass distribution, finite in total amount, on Σ. We have

$$f(e) = f(e\cdot F_\delta) + f(e\cdot[\Sigma - F_\delta])$$

where F_δ is the portion of Σ distant from s by at least as much as δ. For sufficiently large n, the region Σ_n contains in its interior any given F_δ, and therefore if we denote by $\mu_\delta(e)$ the mass distribution obtained by the sweeping out of the distribution $f(e\cdot F_\delta)$, we shall have

$$\mu_\delta(e) = \int_W \mu(e, Q)df(e_Q\cdot F_\delta).$$

But, according to the process of §12.2, the swept-out distribution for $f(e)$ is given by the formula

$$\mu(e) = \lim_{\delta=0}\mu_\delta(e)$$

$$= \lim_{\delta=0}\int_W \mu(e, Q)df(e_Q\cdot F_\delta),$$

and since

$$\int_W \mu(e, Q)df(e_Q) - \int_W \mu(e, Q)df(e_Q \cdot F_\delta) = \int_W \mu(e, Q)d[f(e_Q) - f(e_Q \cdot F_\delta)]$$
$$\leqq f(\Sigma) - f(\Sigma \cdot F_\delta),$$

this limit is precisely

$$\int_W \mu(e, Q)df(e_Q),$$

which is the fact which was to be proved.

V. Capacity and Kellogg's Lemma

15. **Conductor potential and capacity.** Let s be a closed bounded set, the boundary of an *infinite* domain Σ. Let Σ_1 be an infinite domain contained in Σ, of which the boundary s_1 is bounded and regular, and let $\xi_1(M)$ be the function which is continuous in W, harmonic in Σ_1 (vanishing continuously at ∞) and equal to 1 on $C\Sigma_1$. Then $\xi_1(M)$ is evidently superharmonic, and, by §2, the potential of some distribution of positive mass. This mass lies entirely on s_1.

Let $\eta_0(M)$ be the limiting function obtained by the sweeping out of the mass of $\xi_1(M)$ from Σ; that is, in Σ, $\eta_0(M)$ is the sequence solution for the values 1 on s. Let $\eta(M)$ be the potential of a distribution of positive mass $\nu(e)$ arising from the sweeping out, and K the total mass of this distribution. Both $\eta_0(M)$ and $\eta(M)$ are independent of the choice of the sequence of nested regular domains Σ_n, and $\eta(M) = \eta_0(M)$ in Σ. Moreover, K depends merely on the values of $\eta(M)$ in Σ, and is therefore uniquely determined.

The distribution $\nu(e)$ may be called a *conductor distribution*, and its potential a *conductor potential*. The quantity K is called the *capacity* of the closed set s and of the closed set $G = s + B$, in fact, of any closed set g whose external frontier is s. This is the value of the capacity as defined by Wiener.* In order to complete the definition for sets E which are bounded and measurable Borel, but not necessarily closed, we may write

$$K(E) = K(\overline{E}),$$

where $\overline{E}$ is the closed cover of E. The capacity K may of course in special cases have the value zero.

But other definitions of capacity are possible. We define $K_a(E)$, $K_b(E)$, $K_c(E)$ as the upper bounds of total masses of positive mass distributions on E

* N. Wiener, *The Dirichlet problem*, Journal of Mathematics and Physics of the Massachusetts Institute of Technology, vol. 3 (1924), pp. 24–51; see §4. In this paper the author discusses weight, capacity and conductor potential and arrives at a determination of the conductor distribution, which he calls the "outer charge."

of which the potentials do not surpass unity on the following portions, respectively, of space:

(*a*) on the complement $C\overline{E}$ of the closed cover of E;

(*b*) on the complement of E;

(*c*) on the whole space.

We say that K_a, K_b, or K_c is zero if no distribution exists for which the corresponding upper bound of potential is finite. The quantity K_c is the capacity as defined by de la Vallée Poussin.*

Evidently $K_c \leqq K_b \leqq K_a$. But also, if s is the exterior frontier of E and Σ the infinite domain bounded by s, and if Σ_n form a sequence of regular nested domains for Σ, we shall have

$$K(E) = \lim_{n=\infty} K(C\Sigma_n).$$

But $K_a(E) \leqq K_a(\overline{E}) \leqq K(C\Sigma_n)$. Accordingly

$$K_a(E) \leqq \lim_{n=\infty} K(C\Sigma_n) = K(E).$$

Hence

$$K_c(E) \leqq K_b(E) \leqq K_a(E) \leqq K(E). \tag{1}$$

The following properties may be mentioned as familiar, or directly verifiable.

(2) If E is a single point, $K(E)=0$, and similarly for K_a, K_b, K_c.

(2′) If E_1 is contained in E_2, $K(E_1) \leqq K(E_2)$, and similarly for K_a, K_b, K_c.

(2″) If $E=E_1+E_2+\cdots$, and $K_c(E_i)=0$ for all i, then $K_c(E)=0$.

We have also the theorem of de la Vallée Poussin† that for closed bounded sets $K_c=K$.

THEOREM. *For closed bounded sets* $\overline{E}$,

$$K = K_a = K_b = K_c. \tag{3}$$

With regard to (1) it follows that we need merely prove that $K_c(\overline{E}) \geqq K(\overline{E})$. This fact is evident if $K(\overline{E})=0$. And if $K(\overline{E})>0$, any conductor distribution $\nu(e)$ is itself a distribution on $\overline{E}$ of which the potential $\eta(M)$ nowhere exceeds unity; that is to say, $K_c(\overline{E})$ is at least as great as $K(\overline{E})$.

For sets which are not closed, however, the various definitions of capacity are not all equivalent. For instance, if E_1 is a denumerable set of points dense everywhere within the sphere of radius 1/2, it follows from (2″) that $K_c(E_1)=0$. Similarly $K_b(E_1)=0$. But evidently $K(E_1)=1/2$; and also $K_a(E_1)=1/2$, since a point mass as near 1/2 in value as desired may be placed on a

* C. de la Vallée Poussin, loc. cit., p. 225.

† Ibid., p. 226.

point of E_1 so near the center of the sphere that the potential outside the sphere does not exceed unity. Also if E_2 is a denumerable set, everywhere dense on the surface of the sphere, we have $K_c(E_2)=0$, $K(E_2)=1$. But it is clear that $K_b(E_2)=0=K_a(E_2)$, since if there is a positive mass on E_2, there will be a positive mass on some point Q of E_2, and its potential will be greater than N, N given arbitrarily, in a neighborhood of Q. This neighborhood includes points not in the closed cover of E_2. Similar reasoning establishes the fact that if $E=E_1+E_2$, then

$$0 = K_c(E) = K_b(E) < K_a(E) = \tfrac{1}{2} < K(E) = 1.$$

15.1. **Capacity of sets measurable Borel.** We prove the following

THEOREM. *For any bounded set E measurable Borel, $K_b(E)=K_c(E)$.*

On account of (1) it is sufficient to show that $K_c(E)\geqq K_b(E)$, where $K_b(E)>0$. Suppose the contrary, that $K_b(E)>K_c(E)$. Then there exists a distribution of positive mass $\nu(e)$ on E such that $\nu(E)>K_c(E)$ and such that the potential $V_\nu(M)$ of this mass is $\leqq 1$ on CE. For $K_b(E)$ is the upper bound of such $\nu(E)$.

There exists a closed set F, contained in E, such that $\nu(F)$ differs as little as we please from $\nu(E)$; for E, being measurable Borel, belongs to a normal family for $\nu(e)$ in the sense of de la Vallée Poussin.* We may assume then that $\nu(F)>K_c(E)$. Let $\mu(e)=\nu(e\cdot F)$ and let $V_\mu(M)$ be the potential of $\mu(e)$. Then $V_\mu(M)\leqq 1$ on CE, but is not everywhere $\leqq 1$. For in that case we should have $\nu(F)=\mu(F)\leqq K_c(F)\leqq K_c(E)$.

The open set e_0 on which $V_\mu(M)>1+\eta$, where η is chosen >0 and so that $\nu(F)>K_c(E)(1+\eta)$, lies in E. It is composed of at most a denumerable infinity of domains D_i, and is not vacuous. In fact, there is at least one of these domains whose boundary contains a point of CE. For otherwise, by sweeping out from these domains successively, we should obtain a monotone-decreasing sequence of potentials, and a potential corresponding to the limiting function would everywhere, by §11, be $\leqq 1+\eta$. Let the corresponding distribution be $\nu'(e)$. Its total mass would remain $\mu(F)=\nu(F)$, since this quantity remains fixed during the weak convergence. Hence the distribution $\nu''(e)=\nu'(e)/(1+\eta)$ would lie on E and would have a potential everywhere $\leqq 1$; its total mass would therefore be $\leqq K_c(E)$. But the total mass is $\nu(F)/(1+\eta)>K_c(E)$.

Let D then be so chosen from the D_i that its boundary contains a point Q of CE. Then Q does not lie in F and $V_\mu(M)$ is continuous at Q. Consequently there is a neighborhood of Q in which everywhere $V_\mu(M)<1+\eta$, since $V_\mu(Q)\leqq 1$. But this neighborhood contains points of D. And this is a contradiction. Thus the proof is complete.

* C. de la Vallée Poussin, *Intégrales de Lebesgue*, Paris, 1916, p. 85.

We may digress at this point to indicate still another possible definition of capacity, and the value may be determined at once in terms of Maria's result,* that if a positive mass is distributed on a closed bounded set F the upper bound of its potential on F is at least as great as its upper bound on CF.

We define, in fact, $K_d(E)$ as *the upper bound of* $\mu(E)$, *where* $\mu(e)$ *is a distribution of positive mass on* E, *of which the potential* $V_\mu(M)$ *is* $\leqq 1$ *on* E.

Obviously $K_d(E) \geqq K_c(E)$. But also $K_d(E) \leqq K_c(E)$. In fact, given such a distribution $\mu(e)$, $V_\mu(M)$ is $\leqq 1$ on $\overline{E}$, the closed cover of E; for since $V_\mu(M)$ is lower semicontinuous the set where $V_\mu(M) \leqq 1$ is closed. Hence, by Maria's result, $V_\mu(M) \leqq 1$, everywhere.

Our results may be summarized in the equation

$$K_d(E) = K_c(E) = K_b(E) \leqq K_a(E) \leqq K(E), \tag{4}$$

where E is a bounded set measurable Borel, the equality signs being valid throughout if E is closed.

15.2. **Capable points.** A point Q is said to be a *capable* point of a bounded set E, measurable Borel, if no matter how small $\rho > 0$, the portion of E within a sphere of radius ρ and center Q is of positive capacity. The subset E' of incapable points is open with respect to E; that is, there is a neighborhood about an incapable point Q' of E which contains no points of E which are not points of E'. The set E_1 of capable points is therefore closed with respect to E. We shall have possibly different definitions of the subsets E', E_1 according as we use one definition or another of capacity.

Lemma. *If every point of a subset* E' *of a bounded set* E, *measurable Borel, is an incapable point* (*according to any of our definitions of capacity*), *then* $K_c(E') = 0$.

In fact, as de la Vallée Poussin remarks, in the memoir cited, each such point may be enclosed in a sphere of rational radius with center of rational coordinates, which contains no capable points; and there are only a denumerable infinity of such spheres.

If the set E is closed, the definition of capable point is independent of the choice among the definitions of capacity, and therefore the subsets E', E_1 are also. The set E_1 is likewise closed. It is called the *reduced* set. If the set E bears any distribution of mass for which the potential is bounded, the mass lies entirely on the reduced subset E_1. It does not follow that $K_a(E') = 0$ or $K(E') = 0$.

We return now for the rest of this section to the closed bounded set g,

* See §6, Remark III. But Maria's result depends on using Kellogg's Lemma, so that consideration of it in this paper would properly come after §18.

whose external frontier is s, and give a brief proof of Vasilesco's theorem:*

THEOREM. *If Q is a capable point of g and $\eta(M)$ is a conductor potential for g, then*

$$\limsup_{M=Q} \eta(M) = 1, \qquad \text{for } M \text{ in } W. \tag{5}$$

Let g_ρ be the closed cover of the portion of g within a sphere $\Gamma(\rho, Q)$ of center Q and radius ρ, and let Σ_ρ be the domain which is bounded by s_ρ, the external frontier of g_ρ. Part of the mass for the conductor potential $\eta(M)$ of g may lie on Σ_ρ; if so, we sweep it out, and obtain by Theorem II, §13, the conductor distribution on g_ρ, of total mass $K(g_\rho)$. We denote the conductor potential of g_ρ by $\eta_\rho(M)$.

The set g_ρ, by hypothesis, is of positive capacity. It follows that the upper bound of $\eta_\rho(M)$ is 1; for if the upper bound were $r<1$, the set g_ρ would sustain a mass of total amount $K(g_\rho)/r$, such that the upper bound of its potential would be $=1$. But throughout W, $\eta_\rho(M) \leqq 1$, and since it is not constant, we must have $\eta_\rho(M)<1$ for M in Σ_ρ; moreover $\eta_\rho(M) \leqq \eta(M)$. Hence

$$\text{u.b. } \eta(M) = 1, \qquad M \text{ in } \Gamma(\rho, Q).$$

And this proves the theorem.

We note that if g reduces to s, the boundary of Σ, and Q is a capable point of s, it follows from the lower semicontinuity of $\eta(M)$ that

$$\limsup_{P=Q} \eta(P) \leqq \limsup_{M=Q} \eta(M), \qquad P \text{ in } s,\ M \text{ in } \Sigma + B, \tag{6}$$

so that the second member of the inequality must have the value 1.

COROLLARY. *If g is of positive capacity there is at least one point of g where the conductor potential $\eta(M)$ for g has the value unity.*

In fact, the reduced set g_1 of g is not vacuous and has no isolated points, and in the neighborhood of any point P of g_1 there is a point Q of g_1, where, by Corollary II of §5.1, $\eta(M)$ is continuous. But then

$$\eta(Q) = \lim_{M=Q} \eta(M) = 1.$$

16. **Points where a conductor potential has the value unity.** We prove the following

THEOREM. *Let $\eta(M)$ be the conductor potential of s, as before, and let H be the subset of $G=s+B$ where $\eta(M)=1$. Then*

$$K_e(H) = K(G) = K(s).$$

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* F. Vasilesco, *Sur les singularités des fonctions harmoniques*, Journal de Mathématiques, vol. 9 (1930), pp. 81–111; see p. 101.

In fact, $\eta(M)=\eta_0(M)$, $\eta_0(M)$ being the limiting function of the sweeping-out process, except for M on s. Hence $\eta(M)=1$ in B, if B is not vacuous. Let then t be the subset of s where $\eta(M)\leqq 1-\epsilon$, $1>\epsilon>0$. If t is not vacuous it is closed and bounded. We shall prove that $K_e(t)=0$.

Suppose that $K_e(t)$ is not zero. Let $\eta_t(M)$ be the potential obtained by sweeping out from the domain exterior to t the mass distribution of which $\eta(M)$ is the potential. Then $\eta_t(M)$ is the conductor potential for t, and $\eta_t(M)\leqq\eta(M)$. By the Corollary of §15, there is a point Q of t such that $\eta_t(Q)=1$. Hence $\eta(Q)=1$, which is a contradiction. Accordingly $K(t)=K_e(t)=0$.

The portion of s where $\eta(M)<1$ is the sum of a denumerable infinity of (overlapping) sets t, corresponding to decreasing values of ϵ, and therefore must have zero capacity K_e. Hence all of the mass of the conductor distribution must lie on H, and $K_e(H)=K(s)$, which was to be proved.

COROLLARY. *If $\eta(M)$ is a conductor potential for s, the conductor distribution lies entirely on that portion of s where $\eta(M)=1$.*

17. **Uniqueness of capacity potential.** We shall speak of a *capacity* distribution $\mu(e)$, for the moment, as any distribution of positive mass on G (G supposed to be of positive capacity), in total value equal to the capacity of s, provided that the upper bound of its potential $v(M)$ is less than or equal to unity. It cannot, in fact, be less than unity, from the definition of K_e, since $K_e(G)=K(G)=K(s)$. In particular, a conductor distribution for G is a capacity distribution. The following theorem was surmised by de la Vallée Poussin.*

THEOREM. *The potentials of all capacity distributions for G are identical in W; the capacity distributions are all identical on every set measurable Borel.*

We note first the following fact, which we may state as a lemma.

LEMMA I. *If E is a bounded set, measurable Borel, of positive spatial measure, $K_e(E)>0$.*

Let $m(e)$ be the measure of a Borel measurable set e, and define the mass distribution $\mu(e)$ by the equation

$$\mu(e)=m(e\cdot E).$$

The set function $\mu(e)$ is evidently additive and bounded, therefore completely additive, and represents a mass distribution on the bounded set E. Moreover,

* de la Vallée Poussin, memoir cited, p. 232.

its potential is everywhere $\leqq 2\pi d^2$, where d is the diameter of E; hence $K_e(E) \geqq 1/(2\pi d^2) > 0$. Thus the lemma is proved.

Let $\eta(M)$ be the conductor potential for s, and $\nu(e)$ the corresponding distribution of positive mass, and let $\mu(e)$ be a capacity distribution and $v(M)$ its potential. We have immediately the following lemma.*

LEMMA II. *For M in Σ, $v(M) = \eta(M)$.*

In fact, if $\{\Sigma_n\}$ is the sequence of nested domains employed in forming $\eta(M)$, and $\{\eta_n(M)\}$ the corresponding sequence of potentials, we have $v(M) \leqq \eta_n(M)$, for all n, whence $v(M) \leqq \eta(M)$ in Σ. But then either $v(M) \equiv \eta(M)$, in Σ, or else $v(M) < \eta(M)$, M in Σ. The latter case is impossible, since it would follow, as in §2, that $\mu(G) < K(G)$.

LEMMA III. *The Dirichlet integral for a conductor potential is given by the equation*

$$D(\eta) = 4\pi K(s) = \int_\Sigma [\nabla \eta(M)]^2 dM. \tag{7}$$

Writing H for the subset of G where $\eta(M) = 1$, we have by §10 that $D(\eta)$ exists, whence

$$D(\eta) = 4\pi \int_W \eta(P) d\nu(e_P \cdot H) + 4\pi \int_W \eta(P) d\nu(e_P \cdot [W - H]).$$

But the second integral of the right-hand member is zero, since there is no mass on $W - H$, and the first integral, by (4) of §1, reduces to $4\pi \int_W 1 d\nu(e_P \cdot H) = 4\pi\nu(H)$. This establishes the first of equations (7). In order to establish the second result, it is sufficient to consider the case where G is of positive measure and perfect. The function $\eta(M)$ then has the value unity at almost all points of G, by Lemma I and the results of §16.

The partial derivative $\partial\eta/\partial x$ is measurable spatially in the Lebesgue sense, and the function $\eta(M)$ itself is absolutely continuous in x, by §3, on almost all lines parallel to the x-axis. On such lines the set $E(y, z)$ where $\eta(M) = 1$ is closed, and the total variation of $\eta(M)$ over $E(y, z)$ is 0. Hence $\partial\eta/\partial x = 0$ for almost all x on $E(y, z)$, and this, for almost all y, z. That is, $\partial\eta/\partial x = 0$ almost everywhere in G. Similar results hold for $\partial\eta/\partial y$ and $\partial\eta/\partial z$. Consequently $(\nabla\eta)^2 = 0$ almost everywhere on G, and

$$\int_W [\nabla \eta(M)]^2 dM = \int_\Sigma [\nabla \eta(M)]^2 dM,$$

which was to be proved.

* Ibid., p. 228.

LEMMA IV. *The quantities $D(v)$ and $D(\eta)$ are the same.*

We have

$$D(v) = 4\pi \int_W v(P)d\mu(e_P)$$

$$\leqq 4\pi \int_W 1 d\mu(e_P) = 4\pi\mu(G) = 4\pi K(s).$$

Hence $D(v) \leqq D(\eta)$. But also

$$D(v) = \int_W (\nabla v)^2 dM$$

$$\geqq \int_\Sigma (\nabla v)^2 dM = \int_\Sigma (\nabla \eta)^2 dM = D(\eta), \quad \text{by Lemmas II, III,}$$

so that $D(v) \geqq D(\eta)$. Hence $D(v) = D(\eta)$.

We can now complete the proof of the theorem by showing that $v(M)$ and $\eta(M)$ are everywhere the same; for it will then follow that the corresponding mass distributions are identical on all sets measurable Borel.

From Lemmas II, III, IV it is evident that $(\nabla v)^2 = 0$ almost everywhere on G, and thus, that the partial derivatives of $v - \eta$ are almost everywhere 0. But on almost all lines parallel to the x-axis the function $v - \eta$ is absolutely continuous in x and vanishes outside G, so that $v - \eta$ is zero almost everywhere.

Accordingly, for the spherical averages of §4,

$$v_\rho(M) = \eta_\rho(M), \qquad \text{for all } M,$$

and by (9′), §4,

$$v(M) = \lim_{\rho=0} v_\rho(M) = \lim_{\rho=0} \eta_\rho(M) = \eta(M), \qquad \text{for all } M.$$

This is what was to be proved.

18. **Short proof of Kellogg's Lemma.*** This lemma may be stated in the following form.

THEOREM. *If g is a bounded closed set of positive capacity, s its external frontier, and Σ the infinite region of boundary s, then s contains at least one point which is a regular boundary point of Σ.*

* O. D. Kellogg, loc. cit., p. 337. The author acknowledges indebtedness in connection with this proof to discussion with members of the seminar of 1934–35 at the Rice Institute, particularly with Dr. A. J. Maria. For abstract see Bulletin of the American Mathematical Society, vol. 40 (1934), p. 665. The same proof is given independently by F. Vasilesco, Comptes Rendus de l'Académie des Sciences, vol. 200 (1935), pp. 1173–1174.

Consider a conductor potential $\eta(M)$ for s.* Its mass lies entirely on s. Hence the reduced set for s may be taken as the perfect set F of §5.1. By Corollary II of §5.1 there is thus a capable point Q of s, such that $\eta(M)$ is continuous at Q. Consequently $\eta(Q)=1$, for lim sup $(M=Q)$ $\eta(M)=1$, by Vasilesco's theorem given in §15.2. But it is also a theorem of Vasilesco that if lim $(M=Q)$ $\eta(Q)=1$, for M in Σ, then Q is a regular point of s for Σ.†

19. **Second proof of Kellogg's Lemma, independent of Green's function.** The proof of Vasilesco's theorem, just cited, involves the result that a sufficient condition for a regular boundary point is the continuous vanishing of the Green's function at the point. A method of treatment, which perhaps is more direct, is based on Lebesgue's concept of *barrier*. A barrier for Σ at Q is a function $V(M, Q)$ which is continuous and superharmonic in Σ, which approaches zero at Q and has a positive lower bound in Σ outside any sphere with center Q. The construction of a barrier is immediate if the conductor potential at Q of the closed cover $s(\rho, Q)$ of the portion of s within a sphere $\Gamma(\rho, Q)$ has the value unity.‡

We find such a point Q by means of the following proposition.

LEMMA. *Let $\eta(\rho, M)$ be the conductor potential of $s(\rho, Q_1)$, $\eta(M)$ the conductor potential of s. If Q_1 is a capable point of s, there is a closed reduced set s_ρ, contained in $s(\rho, Q_1)$, of capacity as near that of $s(\rho, Q_1)$ as we please, such that*

$$\eta(\rho, P)=\eta(P)=1, \quad P \text{ in } s_\rho. \tag{8}$$

In fact, if we sweep out the mass of the conductor distribution from the domain which is exterior to $s(\rho, Q_1)$ we obtain the unique conductor distribution, for $s(\rho, Q_1)$. The set of capable points of $s(\rho, Q_1)$ where $\eta(\rho, P)=1$ bears all the mass of this conductor distribution, and therefore contains a closed subset s_ρ on which the total mass μ is as close to $K(s(\rho, Q_1))$ as we please. But $K(s_\rho)\geqq\mu$. Moreover $\eta(\rho, M)\leqq\eta(M)$, and so $\eta(M)=1$ on s_ρ also.

With the lemma thus proved, let Q_1 be a capable point of s, and construct the sets $s(\rho, Q_1)$, s_ρ with $\rho=\rho_1$. We note, in particular, that if the conductor potential of a set has the value unity at a point, that point must be a capable point. Next take a point Q_2 of s_{ρ_1}, which is a capable point of s_{ρ_1} and distant from Q_1 by less than ρ_1, and construct $s(\rho_2, Q_2)$ from s_{ρ_1} in the same way that

* Kellogg's Lemma depends on §5.1 and known results, and might have been inserted in that section. Hence "a" rather than "the" conductor potential. The theorem is put late in the present memoir in order to separate the theorems which involve it explicitly from those which do not.

† Vasilesco, loc. cit., p. 94. This theorem was wrongly cited as in Kellogg, loc. cit., at p. 331, by the author, in his paper *Application of Poincaré's sweeping-out process*, Proceedings of the National Academy of Sciences, vol. 19 (1933), pp. 457–461.

‡ O. D. Kellogg, loc. cit., pp. 227, 331.

$s(\rho_1, Q_1)$ is formed from s, taking $\rho_2 < \rho_1 - Q_1Q_2$. Similarly form $s(\rho_k, Q_k)$, s_{ρ_k}, from $s_{\rho_{k-1}}$, with the values ρ_k tending to zero. Of the closed sets s_{ρ_k}, each contains the next and none is empty; hence there is at least one point common to all of them, say Q. The conductor potentials $\eta(\rho_k, M)$ all have the value unity at Q.

Let now ρ be any value >0. The set $s(\rho, Q)$ contains the sets $s(\rho_k, Q_k)$ for k sufficiently large. Hence the conductor potential of $s(\rho, Q)$ dominates those of the sets $s(\rho_k, Q_k)$, since the latter may be obtained by sweeping out the former. Hence the conductor potential of $s(\rho, Q)$ has the value 1 at Q. This is what was to be proved.

VI. Applications

20. **Necessary and sufficient condition for regular point.** We prove the following

Theorem. *A necessary and sufficient condition that Q be a regular point of s for Σ is that for every distribution of positive mass on a bounded set, the potential at Q be unchanged by the sweeping out of the portion of the mass in Σ.*

That the condition is sufficient is seen by the instance of the conductor potential, if Σ is an unbounded domain. If Σ is a bounded domain it is sufficient to consider the sweeping out of unit mass at a point M of Σ. From(15″), §14, if $g_n(P, M)$, $v_n(P, M)$ are respectively the Green's function and swept-out potential of unit mass at M, for Σ_n, $\{\Sigma_n\}$ being a sequence of nested regular domains for Σ, and if $g(P, M)$, $v(P, M)$ are the corresponding functions for Σ,

$$g_n(P, M) = \frac{1}{PM} - v_n(P, M),$$

$$g(P, M) = \frac{1}{PM} - v(P, M).$$

In fact, by definition, for P in Σ, $g(P, M) = \lim g_n(P, M)$; and the definition may be suitably extended to P in $C\Sigma$ by the above equation. But for P in Σ, $v(P, M) \leqq 1/(PM)$, so that if $v(Q, M) = 1/(QM)$ it follows that

$$\lim_{P=Q} v(P, M) = \frac{1}{QM}, \qquad \lim_{P=Q} g(P, M) = 0,$$

which is a sufficient condition for a regular point.*

* As is seen by means of a Kelvin transformation of the region into an infinite domain with bounded boundary. Or one may, with G. Bouligand (loc. cit), proceed directly from an analysis of the Green's function.

In order to prove the necessity of the condition, consider first the case where $U(M)$ is continuous on s and in its neighborhood. Then $V_0(M)$ is continuous at Q, being equal to $U(M)$ for M in $G=C\Sigma$, and taking on continuously the value $U(Q)$ as M tends to Q from Σ, as a property of the sequence solution at a regular point. But

$$V(Q) = \lim_{\rho=0} V(\rho, Q) = \lim_{\rho=0} V_0(\rho, Q) = V_0(Q) = U(Q). \tag{1}$$

In the more general case, where $U(M)$ is not necessarily continuous or bounded on s, we may write, recalling the notation of §12.2,

$$V'^{(p)}(Q) = V_0'^{(p)}(Q) = U'^{(p)}(Q),$$

$$V'(Q) = \lim_{p=\infty} V'^{(p)}(Q) = \lim_{p=\infty} U'^{(p)}(Q) = U'(Q),$$

$$V(Q) = U''(Q) + V'(Q) = U''(Q) + U'(Q) = U(Q),$$

which was to be proved.

Our theorem may be summarized by the equation

$$V(Q) = V_0(Q) = U(Q), \qquad Q \text{ a regular point of } s \text{ for } \Sigma, \tag{2}$$

since for all M, $V(M) \leqq V_0(M) \leqq U(M)$.

21. **The Dirichlet integral and the sweeping-out process.** The following theorem is a generalization of the statement that the value of the Dirichlet integral for the conductor potential is 4π times the capacity of the boundary set.

THEOREM. *Let $\{\Sigma_n\}$ be a sequence of nested regular domains for Σ, and $U(M)$ be a bounded potential of positive mass on a bounded set. If the sweeping-out process is carried out by means of the domains Σ_n, then the relation*

$$D(V) = \lim_{n=\infty} D(V_n) \tag{3}$$

holds for the Dirichlet integrals.

LEMMA. *The theorem is true if $U(M)$ is continuous for M on s and in its neighborhood.*

In fact, the irregular boundary points of Σ are points where the conductor potential has a value <1, and therefore, by §16, form a subset of zero capacity K_e, and can sustain no portion of a mass distribution of which the potential is bounded. Hence if we denote by G_0 the set of points of $G=C\Sigma$, which are not irregular points of s, we shall have

$$D(V) = 4\pi \int_W V d\mu = 4\pi \int_W V(P) d\mu(G_0 \cdot e_P) + 4\pi \int_W V(P) d\mu(CG_0 \cdot e_P)$$
$$= 4\pi \int_W V(P) d\mu(G_0 \cdot e_P),$$

where $CG_0 = \Sigma + (CG_0) \cdot s$, so that $\mu(CG_0) = 0$. But by the theorem of §20, $V(P) = U(P)$ on $G_0 \cdot s$, and as a result of the sweeping-out process $V(P) = U(P)$ on $B = C(\Sigma + s)$, so that, by the relation (4) of §1,

$$D(V) = 4\pi \int_W U(P) d\mu(G_0 \cdot e_P) = 4\pi \int_W U(P) d\mu(e_P).$$

Now $\mu_n(e)$ converges weakly to $\mu(e)$ and $U(M)$ is continuous on s and in its neighborhood, whence

$$\int_W U(P) d\mu(e_P) = \lim_{n=\infty} \int_W U(P) d\mu_n(e_P).$$

But $U(P) = V_n(P)$ on $C\Sigma_n$, so that $\int U d\mu_n = \int V_n d\mu_n$ and finally

$$D(V) = \lim_{n=\infty} 4\pi \int_W V_n(P) d\mu_n(e_P),$$

which was to be proved.

Returning to the theorem, we may assume without loss of generality that the mass distribution lies entirely in Σ.

The quantities $D(V)$, $D(V_n)$ converge, since $V(M)$, $V_n(M)$ are bounded (see §10). Moreover, since $V(M) \leqq V_n(M)$ it follows by Corollary II of §10 that $D(V) \leqq D(V_n)$; consequently

$$D(V) \leqq \liminf_{n=\infty} D(V_n). \tag{4}$$

In order to obtain the complementary inequality, let Σ_δ be the portion of Σ distant from the boundary s by as much as δ, $\mu_{\delta n}(e)$ the distribution obtained by sweeping from Σ_n the portion of mass in Σ_δ, and $V_{\delta n}(M)$ the potential of the distribution $\mu_{\delta n}(e)$. Then by (9), §10,

$$D(V_n) - D(V_{\delta n}) = 4\pi \int_W V_n d\mu_n - 4\pi \int V_{\delta n} d\mu_{\delta n}$$
$$= 4\pi \int_W (V_n - V_{\delta n}) d\mu_{\delta n} + 4\pi \int_W V_n d(\mu_n - \mu_{\delta n}).$$

But the first integral, which is $D(V_n - V_{\delta n}, V_{\delta n}) = D(V_n, V_{\delta n}) - D(V_{\delta n})$, may also be written in the form $4\pi \int_W V_{\delta n} d(\mu_n - \mu_{\delta n})$, so that

$$D(V_n) - D(V_{\delta n}) = 4\pi \int_W (V_n + V_{\delta n}) d(\mu_n - \mu_{\delta n}).$$

Let n_δ be a value of n such that Σ_n contains Σ_δ. Then, for every $n > n_\delta$, $\mu_n(e) \geqq \mu_{\delta n}(e)$, by the process of §12.2. Hence N exists so that

$$0 \leqq D(V_n) - D(V_{\delta n}) < 8\pi N(\mu_n(\Sigma) - \mu_{\delta n}(\Sigma)),$$

and given $\epsilon > 0$ we can choose $\delta > 0$ so that

$$0 \leqq D(V_n) - D(V_{\delta n}) < \epsilon, \qquad n > n_\delta.$$

Let $V_\delta(M)$ be the potential obtained by sweeping out from Σ the portion of the original mass distribution in Σ_δ, according to the process of §12.2. By the lemma, we have

$$D(V_\delta) = \lim_{n=\infty} D(V_{\delta n}),$$

since $V_{\delta n}(M)$ is continuous on s and in its neighborhood. Hence

$$D(V) \geqq \lim_{n=\infty} D(V_{\delta n}) \geqq \limsup_{n=\infty} D(V_n) - \epsilon,$$

and

(5) $$D(V) \geqq \limsup_{n=\infty} D(V_n).$$

From (4) and (5) we have (3), which is the statement to be proved. Incidentally, the inequality (4) shows that $D(V_n)$ is a decreasing function of n.

The theorem of this section is no longer true if the qualification "bounded" is removed from the hypothesis. In fact, if Σ is the domain exterior to a sphere and we are given a collection of point masses in Σ with limit point on the boundary s, such that the potential remains bounded on s, we shall have $D(U) = D(V_n) = \infty$, while $D(V)$ is finite.

22. **Condition that a function be a potential of positive mass.** We prove the following

THEOREM.* *Let $u(M)$ be harmonic in a domain Σ (with bounded boundary s), not identically zero, and, if Σ is an exterior domain, vanishing continuously at infinity. Let Σ' be a regular domain contained with its boundary s' in Σ, and let $V'(M)$ be the function constituted by the solutions of the Dirichlet problems (interior or exterior, as the case may be) for each of the domains comprising $B' = C(\Sigma' + s')$, with boundary values $u(M)$ on s'.*

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* Incidentally, this theorem provides an answer for G. Bouligand's Problem 2 (loc. cit., p. 16).

A necessary and sufficient condition that $u(M)$ be given for all M in Σ as a potential of some distribution of positive mass is that, for each Σ',

$$V'(M) \leqq u(M), \qquad M \text{ in } \Sigma - \Sigma'. \tag{6}$$

The mass may be distributed entirely on s.

If $u(M)$ is a potential of positive mass, the distribution lying accordingly on $C\Sigma$, it is superharmonic in each of the domains comprising $C(\Sigma'+s')$. Since the equation $V'(M)=u(M)$ is satisfied on each portion of s' which is the boundary of one of these domains, it follows that (6) is satisfied in the interior of the domain. Hence (6) is necessary.

In order to show that (6) is sufficient, consider a sequence $\{\Sigma_n\}$ of nested regular domains for Σ, and let $v_n(M)$ denote the corresponding functions $V'(M)$. We extend the definition of $v_n(M)$ by writing it equal to $u(M)$ in Σ_n. It is thus continuous in W. It possesses evidently the supermean property (see §2) for M in Σ_n and for M in $C(\Sigma_n+s_n)$. For points Q on s_n, we have, making use of (6),

$$v_n(Q) = u(Q) = A_u(\rho, Q) \geqq A_{v_n}(\rho, Q)$$

so that the supermean property holds there also. Hence $v_n(M)$ is superharmonic, and since it is not identically zero, is harmonic outside a bounded set and vanishes continuously at infinity, it is the potential of a positive distribution of mass. This mass is located entirely on s_n.

The functions $v_n(M)$ form a monotone-increasing sequence, their masses lie on sets which are bounded independently of n, and the limit function $v(M)=\lim v_n(M)$ is not identically infinite. In fact,

$$v_{n'}(M) \geqq v_n(M), \qquad \text{if } n' > n,$$

for $v_n(M)$ is harmonic in $C(\Sigma_n+s_n)$ and $v_{n'}(M)$ is superharmonic there, the two functions being identical in Σ_n+s_n. Moreover, the sets s_n are bounded, independently of n. Finally, the functions $v_n(M)$, forming an increasing sequence, are dominated by $u(M)$ in Σ, by hypothesis, and hence $v(M)$ is finite at every M in Σ.

It follows, by the theorem of §2.1, that the function $v(M)$ is a potential of positive mass, and since it is harmonic except on s the mass distribution must lie entirely on s. But, by construction, $v(M)$ is identical with $u(M)$ in Σ. This completes the proof.

23. **Sets of positive capacity.** Among other conditions, Wiener* gives the

* Wiener, loc. cit.; also Wiener, *The Dirichlet problem*, ibid., pp. 127–146. For a survey of this kind of problem and its extension to other special equations of elliptic type, see M. Brelot, *Le problème de Dirichlet sous sa forme moderne*, Mathematica, vol. 7 (1933), pp. 147–166.

following sufficient condition for the regularity of a point Q of s with respect to Σ. With our notation, G for the complement of Σ, $C(\rho, Q)$ for the spherical surface of center Q and radius ρ, and $\Gamma(\rho, Q)$ for the domain interior to $C(\rho, Q)$, it is expressed by the following statement:

The point Q is a regular point of s for Σ if there exists a sequence of values of r tending to zero and a constant $k>0$ such that the capacity of the set $G\cdot C(r, Q)$ is $\geqq kr$.

Likewise, it follows easily from the well known necessary and sufficient condition for a regular point, given by Wiener in the second of the memoirs just cited, that *Q is a regular point of s for Σ if*

$$K(G\cdot C(r, Q) + G\cdot \Gamma(r, Q)) \geqq kr$$

for a sequence of values of r tending to zero.

A point which satisfies this last condition may be called a *point of positive capacity density* in G. In particular, it follows from this capacity-density criterion that a point of s of positive spatial density in G is a regular boundary point with respect to Σ; and we have also the fact that if G is of positive capacity and contains a subset, similar to G, of diameter less than that of G, then it contains a point of positive capacity density, and its exterior frontier contains a regular boundary point for Σ. If it were true that every G of positive capacity contained a point of positive capacity density, we should have an independent proof of Kellogg's lemma.

In this section we content ourselves with proving the following theorem.

THEOREM. *Let g be a closed bounded set, g_0 its projection on any plane. If g_0 is of positive capacity (that is, with reference to Newtonian potential) then g is of positive capacity.*

The theorem will be proved if we can find a distribution of positive mass on g for which the potential is bounded. There exists such a distribution on g_0, by hypothesis; we represent it by $\mu^0(e)$. We take the plane of g_0 as the x, y plane.

Form a rectangular space net L, composed of a system of superimposed rectangular space lattices L_n, made by planes $x=$const., $y=$const., $z=$const., the meshes of L_n being mutually distinct point sets of diameter $\leqq \delta_n$, where $\lim (n=\infty)\ \delta_n=0$. The projection of L_n on the x, y plane is a lattice L_n^0, and these lattices form a plane net L^0. To each mesh of L_n we let belong the faces of lowest algebraic values x, y, z respectively, and the single vertex of lowest algebraic values x, y, z. Thus L_n^0 is composed also of mutually distinct meshes.

Let $\omega_{i,n}^0$ be a mesh of L_n^0 which contains a point of g_0, and $\omega_{i,n}$ a mesh of L_n, of which $\omega_{i,n}^0$ is a projection, which contains a point of g; for definite-

ness, $\omega_{i,n}$ may be the one with least z-coordinate for its vertex. To the face $z=$const., of this mesh, of least z-coordinate, transfer the mass distribution $\mu^0(e\cdot\overset{0}{\omega}_{i,n})$, forming on this face a distribution $\mu_{i,n}(e)$. We write

$$\mu_n(e) = \sum\nolimits_i \mu_{i,n}(e)$$

and thus obtain in space a bounded additive function of point sets measurable Borel.

There is a subsequence of these distributions $\mu_n(e)$ which converges in the weak sense to a distribution $\mu(e)$, and $\mu(e)$ lies entirely on g. In fact, if M is not on g, there will be a sphere of center M which contains no mesh $\omega_{i,n}$ for n sufficiently great. Without loss of generality we may restrict n to the sequence of the weak convergence.

Let M, P be points of space, Q, R their projections on the x, y plane, and write, with the notation of §1,

$$V^0(Q) = \lim_{N=\infty} \int_W h^N(Q, R)d\mu^0(e_R),$$

$$V(M) = \lim_{N=\infty} \int_W h^N(M, P)d\mu(e_P),$$

admitting the value $+\infty$, for the present, as a possible value of $V(M)$. Since $QR \leqq MP$, $h^N(M, P) \leqq h^N(Q, R)$, we note that

$$\int h^N(M, P)d\mu(e_P) \leqq \int h^N(Q, R)d\mu(e_P) = \lim_{n=\infty} \int h^N(Q, R)d\mu_n(e_P)$$

$$= \lim_{n=\infty} \sum\nolimits_i \int h^N(Q, R)d\mu_{i,n}(e_P)$$

$$= \lim_{n=\infty} \sum\nolimits_i \int h^N(Q, R)d\overset{0}{\mu}(e_R\cdot\overset{0}{\omega}_{i,n})$$

$$= \lim_{n=\infty} \int h^N(Q, R)d\mu^0(e_R).$$

Hence, since $\mu^0(e)$ does not involve n,

$$\int_W h^N(M, P)d\mu(e_P) \leqq \int_W h^N(Q, R)d\mu^0(e_R)$$

and

$$V(M) \leqq V^0(Q),$$

so that $V(M)$ is bounded for M in W. This is what was to be proved.

24. **Approximation on a closed set.*** Let g be a closed bounded set, and let the complement of g be written as an infinite domain Σ, plus possibly other domains $B_1, B_2, \cdots$. We speak of an exceptional point Q of g, as in §7.3, as a point of g such that in the neighborhood of Q there is contained in g a set of rectangles with sides parallel to arbitrary orthogonal directions x, y whose vertices constitute a set of positive spatial measure.

THEOREM. *If g contains no points which are exceptional, and $U(M)$ is given as superharmonic and continuous in a region with regular boundaries which encloses g in its interior, then there exists a sequence of functions $U_n(M)$, harmonic at all points of g, such that*

$$\lim_{n=\infty} U_n(M) = U(M), \quad \textit{uniformly for all } M \textit{ in } g.$$

In any bounded subregion Ω contained strictly in the region mentioned in the theorem, $U(M)$ is the sum of a harmonic function and a potential of positive mass, bounded in total amount and distributed on Ω (Riesz's theorem, §4). This potential function may be taken as continuous in all space.

In fact, if we take a subregion Ω_0 contained strictly in Ω, the potentials due to the masses on Ω_0 and $\Omega-\Omega_0$ respectively are continuous in Ω; for, since each potential is lower semicontinuous, the sum cannot be continuous at a point unless both terms are also. Hence the potential due to the mass on Ω_0 is continuous throughout all space; and, since the potential of the mass on $\Omega-\Omega_0$ is harmonic in Ω_0, the desired resolution is obtained for the region Ω_0. There is no loss in generality in substituting Ω for Ω_0.

There is thus no loss in generality in assuming that $U(M)$ of the theorem is a potential of positive mass on a bounded set F, and is continuous throughout all space. For, having proved the theorem for the potential $U(M)$ we may add again the harmonic function to $U(M)$, $U_1(M)$, $U_2(M)$, $\cdots$ and thus obtain the original theorem.

The points of g may be enclosed in a finite number of spheres, and therefore in a finite number of regions with regular boundaries, constituting in

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* J. L. Walsh, *The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions*, Bulletin of the American Mathematical Society, vol. 35 (1929), pp. 499–544. Walsh's principal theorem for three dimensions is for bounded closed regions, such that every ray from some point of the interior contains a single boundary point (the boundary therefore is of spatial measure zero), assuming that the given function is continuous over the region and harmonic in the interior. See also C. T. Holmes, *The Approximation of Harmonic Functions in Three Dimensions by Harmonic Polynomials*, Dissertation, Harvard University, 1931, Theorems I and III.

Replacing a continuous function by a superharmonic one is a well known device. Likewise, a potential which is harmonic in a bounded open region Σ_0 can be approximated uniformly in any closed region contained in Σ_0 by a harmonic polynomial (see Walsh, loc. cit., p. 542.)

this way a finitely multiple open region, say g_1, with boundary s_1. Similarly we complete a sequence of finitely multiple open regions $g_1, g_2, \cdots$, with boundaries $s_1, s_2, \cdots, g_{n+1}$ to be contained strictly in g_n, and with lim $(n=\infty)$ $g_n=g$. We form the functions $U_n(M)$ as follows:

(i) $U_n(M)$ is to be a solution of the Dirichlet problem in the regions composing g_n, with boundary values $U(M)$,

(ii) $$U_n(M) = U(M) \text{ for } M \text{ in } Cg_n.$$

Then $U_n(M)$ is continuous in W and superharmonic; in fact, the supermean property is satisfied at every point. Since it is harmonic outside a bounded set, vanishing continuously with $U(M)$ at ∞, it is a potential of positive mass distributed on a bounded set. The functions $U_n(M)$ are dominated by $U(M)$, for all n, and form a monotone-increasing sequence with n; in fact, $U_{n+1}(M)$ is identical with $U_n(M)$ in Cg_n and $\geqq U_n(M)$ in g_n. Moreover none of the mass distribution for $U_n(M)$ lies outside a sufficiently large sphere, independent of n. Hence by the theorem of §2.1 the limit function

$$u(M) = \lim_{n=\infty} U_n(M)$$

is itself a potential of a distribution of positive mass on a bounded set.

We note that $u(M)$ is identical with $U(M)$. In fact, both functions are identical in Cg since every point of Cg is ultimately a point where $U_n(M)$ remains equal to $U(M)$ for all values of n sufficiently great. Moreover, by the theorem of §7.3, of which the proof applies when the set s is replaced by g, if Q is a point of g and M tends to Q from Cg, then

$$\begin{aligned} u(Q) &= \liminf_{M=Q} u(M) \\ &= \liminf_{M=Q} U(M) = U(Q), \end{aligned}$$

so that $u(Q)=U(Q)$. We have therefore

$$U(M) = \lim_{n=\infty} U(M), \qquad M \text{ in } W.$$

Since $U_n(M)$, $U(M)$ are continuous and since the sequence is monotone-increasing, the limit must be uniform on any bounded region. Moreover $U_n(M)$ is harmonic at all points of g. This completes the proof.

It is to be noted that any bounded closed set of spatial measure zero

satisfies the conditions of the theorem: for example, a spherical surface with an isolated point in the interior, or a spherical surface supplemented with a Lebesgue spine, or a set consisting of a single point. The conditions that are given, however, are merely sufficient conditions. It is not presumed that the treatment of this problem is exhaustive, but merely that it shows an interesting application of the general methods.

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SOME GENERALIZATIONS OF PALEY'S THEOREMS ON FOURIER SERIES WITH POSITIVE COEFFICIENTS*

BY

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1. **Introduction.** Let $f(x)$ be a real-valued function of a real variable x, periodic with the period 2π and Lebesgue integrable. These properties will be assumed throughout, without being explicitly stated. Let, in addition, for all x,

$$|f(x)| \leqq L < \infty. \tag{1.1}$$

Let

$$f(x) \sim a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x) \tag{1.2}$$

be the Fourier series of $f(x)$,

$$s_0(x) = a_0,\ s_n(x) = a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x) \qquad (n = 1, 2, 3, \cdots) \tag{1.3}$$

the partial sums of (1.2), and

$$\sigma_n(x) = a_0 + \sum_{\nu=1}^{n} \left(1 - \frac{\nu}{n}\right)(a_\nu \cos \nu x + b_\nu \sin \nu x) \qquad (n = 1, 2, 3, \cdots) \tag{1.4}$$

their arithmetic means.

It is a classical result of the theory of Fejér and Lebesgue that the sequence $\{\sigma_n(x)\}$ is uniformly bounded and satisfies

$$|\sigma_n(x)| \leqq L, \tag{1.5}$$

and that, as $n \to \infty$, $\sigma_n(x) \to f(x)$ uniformly over $(-\pi, \pi)$ provided $f(x)$ is continuous in $(-\pi, \pi)$. As to the partial sums (1.3) themselves, as may be shown by suitable examples, they need not be uniformly bounded even when (1.1) is satisfied, and the sequence $\{s_n(x)\}$ need not converge uniformly to $f(x)$ even when $f(x)$ is continuous in $(-\pi, \pi)$.

Under these circumstances special attention should be given to a recent result of Paley† according to which the non-negativeness of the Fourier coeffi-

* Presented to the Society, February 23, 1935; received by the editors April 17, 1934.

† R. E. A. C. Paley, *On Fourier series with positive coefficients*, Journal of the London Mathematical Society, vol. 7 (1932), pp. 205–208. On the basis of (1.1) Paley derives the estimate $|s_n(x)| \leqq 10L$.

cients a_n, b_n of $f(x)$ combined with (1.1) implies the uniform boundedness of $\{s_n(x)\}$, while combined with the continuity of $f(x)$, it implies the uniform convergence of $s_n(x)$ to $f(x)$.

In a letter to Professor Fejér, written in the autumn of 1932,* Paley stated and gave a sketch of a proof of the fact that the same results hold if the condition of non-negativeness of a_n, b_n is replaced by a less restrictive one, viz.,

$$(1.6) \qquad a_n \geqq -K/n, \qquad b_n \geqq -K/n, \qquad 0 \leqq K < \infty.\dagger$$

After learning of these latter results of Paley's, the author of the present paper completed his proof with various improvements in the estimates‡ and also succeeded in extending these results to the generalized Fourier series of almost periodic functions of H. Bohr, $\sum(a_n \cos \lambda_n x + b_n \sin \lambda_n x)$. These investigations§ of the generalized series suggested, in the case of the ordinary series, the replacement of (1.6) by conditions

$$(1.7) \qquad a_n + \alpha_n \geqq 0, \qquad b_n + \beta_n \geqq 0 \qquad (n = 1, 2, 3, \cdots),$$

where

$$(1.8) \qquad \alpha_n \geqq 0, \qquad \beta_n \geqq 0,$$

while the series $\sum_1^\infty \alpha_n$, $\sum_1^\infty \beta_n$ are "slowly divergent" in the sense of one of the definitions which follow.

Definition 1′. *A series $\sum_1^\infty c_n$ with non-negative terms is said to be (at most) slowly divergent if there exist two positive numbers P and p, and a positive integer N, such that*

$$\sum_{\nu=n}^{n+q} c_\nu \leqq P, \quad for\ n \geqq N, \quad q \leqq pn.$$

Definition 1″. *A series $\sum_1^\infty c_n$ with non-negative terms is said to be (at most) slowly divergent if for an arbitrarily given positive P there exist a positive number p and a positive integer N, both depending on P, such that*

* This letter is reproduced in a note by Fejér, *On a theorem of Paley*, Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 469–475, especially pp. 474–475. On the basis of (1.1) and (1.6) Paley derives the estimate $|s_n(x)| \leqq K\epsilon + M_\epsilon L$ where ϵ is an arbitrary positive number while M_ϵ is a positive number which depends on ϵ but not on K and L.

† Analogous results have been found independently by Szász, *Zur Konvergenztheorie der Fourierschen Reihen*, Acta Mathematica, vol. 61 (1933), pp. 185–201.

‡ M. Fekete, *Proof of three propositions of Paley*, Bulletin of the American Mathematical Society, vol. 41 (1935), pp. 138–144.

§ The previous results of Paley on Fourier series (of purely periodic functions) with positive coefficients have been already extended by the author in his paper *On generalized Fourier series with non-negative coefficients*, presented to the London Mathematical Society on November 16, 1933, forthcoming in their Proceedings.

$$\sum_{\nu=n}^{n+q} c_\nu \leqq P \ for \ n \geqq N, \ q \leqq pn.^*$$

Since the harmonic series $\sum 1/n$ is slowly divergent in the sense of both Definitions 1′ and 1″ we thus obtain in (1.7) a generalization of conditions (1.6). It will be shown (Theorems 1 and 2 below) that if conditions (1.7), (1.8) are satisfied, the boundedness of $f(x)$ combined with the slow divergence of the series $\sum_1^\infty \alpha_n, \sum_1^\infty \beta_n$ in the sense of Definition 1′ implies the uniform boundedness of the sequence $\{s_n(x)\}$, while the continuity of $f(x)$ together with the slow divergence of $\sum_1^\infty \alpha_n, \sum_1^\infty \beta_n$ in the sense of Definition 1″ implies the uniform convergence to $f(x)$ of the sequence $\{s_n(x)\}$.

These results can be established in the same fashion as in the special case $\alpha_n = K/n$, $\beta_n = K/n$, but the method applied here has proved, after a slight modification, to be adequate to cope with a more general situation as far as the coefficients a_n, b_n are concerned. The modified, more general, conditions on a_n, b_n are suggested by the fact that conditions (1.8) together with the slow divergence of $\sum_1^\infty \alpha_n, \sum_1^\infty \beta_n$ in the sense of Definition 1′ implies the "slow oscillation" of the series $\sum_1^\infty \alpha_n \cos nx, \sum_1^\infty \beta_n \sin nx$, uniformly in x, in the sense of Definition 2′ below and the Remark appended, while (1.8) together with the slow divergence in the sense of Definition 1″ implies the slow oscillation of $\sum_1^\infty \alpha_n \cos nx, \sum_1^\infty \beta_n \sin nx$, uniformly in x, in the sense of Definition 2″ below and the Remark.

DEFINITION 2′. *A series $\sum_1^\infty c_n$ with real terms is said to be slowly oscillating if there exist two positive numbers P and p and a positive integer N such that*

$$\left|\sum_{\nu=n}^{n+q} c_\nu\right| \leqq P, \ for \ n \geqq N, \ q \leqq pn.$$

DEFINITION 2″. *A series $\sum_1^\infty c_n$ with real terms is said to be slowly oscillating if for an arbitrarily given positive P there exist a positive number p and a positive integer N, both depending on P, such that*

$$\left|\sum_{\nu=n}^{n+q} c_\nu\right| \leqq P, \ for \ n \geqq N, \ q \leqq pn.$$

Remark. If the terms c_n of the series $\sum_1^\infty c_n$ depend on a parameter t which ranges over an interval $c \leqq t \leqq d$, we shall say that the slow oscillation of $\sum_1^\infty c_n(t)$ is uniform in t over (c, d) if the series oscillates slowly (in the sense

* It is clear that slow divergence in the sense of Definition 1″ implies that of Definition 1′, but the converse is not true even if the general term of the series should tend to zero, as may be shown by examples. Incidentally the property required in Definition 1′ is equivalent to

$$\sum_{\nu=n}^{2n} c_\nu \leqq K < \infty \text{ for all } n.$$

of either Definition 2′ or 2″), the characteristic data of the slow oscillation being independent of t.

This raises the question whether the results above concerning the sequence $\{s_n(x)\}$ still hold if, without changing the hypotheses on $f(x)$, and retaining (1.7), we replace conditions (1.8) together with the slow divergence of $\sum_1^\infty \alpha_n, \sum_1^\infty \beta_n$ by the condition of the uniform slow oscillation over $(-\pi, \pi)$ of the series $\sum_1^\infty \alpha_n \cos nx, \sum_1^\infty \beta_n \sin nx$ in the sense of Definitions 2′ or 2″. That this question can be answered in the affirmative is shown by Theorems 5 and 6 below. The modified conditions deserve special attention, for they lead to conditions which are not only sufficient but also necessary for the behavior under consideration of the sequence of partial sums $\{s_n(x)\}$ (Theorems 7 and 8). On the other hand, while conditions (1.8) together with the slow divergence of $\sum_1^\infty \alpha_n, \sum_1^\infty \beta_n$ bear only upon the negative ones among the Fourier coefficients a_n, b_n, our modified conditions are of a more complicated nature and involve all the coefficients a_n, b_n, negative as well as positive.

In concluding this introduction the author wishes to state that he owes the notion of slow divergence to Professor Fejér, who defined and used the notion of slow oscillation in the sense of Definition 2″ in his investigations on summability.* The method used by Fejér, after suitable modifications, proved effective in deriving the sharpest result of the present paper, embodied in Theorems 9 and 10 below, where necessary and sufficient conditions for the uniform boundedness or uniform convergence of the sequence of partial sums $\{s_n(x)\}$ are obtained in terms of the "one-sided" uniform oscillation (from below) of the cosine and sine components of the Fourier series of $f(x)$.

2. The present section is devoted to a proof of the two following propositions.

THEOREM 1. *Let conditions* (1.1), (1.7), *and* (1.8) *be satisfied and let the series* $\sum_1^\infty \alpha_n, \sum_1^\infty \beta_n$ *be slowly divergent in the sense of Definition* 1′. *Then the partial sums* (1.3) *of the Fourier series of* $f(x)$ *are uniformly bounded and the upper bound of* $|s_n(x)|$ *can be expressed in terms of* L *and of the characteristic data of the slow divergence of* $\sum_1^\infty \alpha_n, \sum_1^\infty \beta_n$.

THEOREM 2. *If* $f(x)$ *is continuous in* $(-\pi, \pi)$ *and if conditions* (1.7) *and* (1.8) *are satisfied with* $\sum_1^\infty \alpha_n, \sum_1^\infty \beta_n$ *slowly divergent in the sense of Definition* 1″, *then the Fourier series* (1.2) *of* $f(x)$ *converges to* $f(x)$ *uniformly for all* x.

* As yet unpublished; cf. a reference in the paper by M. Fekete and C. E. Winn, *On the connection between the limits of oscillation of a sequence and its Cesàro and Riesz means*, Proceedings of the London Mathematical Society, (2), vol. 35 (1933), pp. 488–513, especially p. 490. Since receiving the proof sheets, I have noticed that conditions equivalent to those required in Definitions 1′, 1″, 2′, 2″ occurred also in investigations of Landau and Schnee. Cf. Schnee's paper in the Proceedings of the London Mathematical Society, vol. 23 (1924), pp. 172–184.

The proof of these propositions is based on the following

LEMMA. *If, under conditions* (1.1), (1.7), *and* (1.8), *the series* $\sum_1^\infty \alpha_n$, $\sum_1^\infty \beta_n$ *satisfy the conditions*

$$\sum_{\nu=k}^{k+q} \alpha_\nu \leqq A, \tag{2.1}$$

$$\text{for } k \geqq N \geqq 1,\ 0 \leqq q \leqq pk,\ p>0,$$

$$\sum_{\nu=k}^{k+q} \beta_\nu \leqq B, \tag{2.2}$$

then the partial sums (1.3) *admit of the estimate*

$$|s_n(x)| \leqq (5 + 2/p)L + 3(A + B),\ \textit{provided that } n \geqq N(1 + p). \tag{2.3}$$

To prove this lemma we start with the identity used by Paley*

$$\begin{aligned} s_n(x) &= \{n\sigma_n(x) - m\sigma_m(x)\}/(n - m) \\ &\quad + \sum_{\nu=m+1}^{n} (\nu - m)(a_\nu \cos \nu x + b_\nu \sin \nu x)/(n - m),\quad 1 \leqq m < n. \end{aligned} \tag{2.4}$$

In view of (1.5) we have for the first term of the right-hand member of (2.4),

$$|n\sigma_n(x) - m\sigma_m(x)|/(n - m) \leqq (n + m)L/(n - m). \tag{2.5}$$

To evaluate the second term we consider, following Fejér and Paley,* the $(n-m)$th arithmetic means of the Fourier series of the functions $[f(x) + f(-x)] \cos nx$ and $[f(x) - f(-x)] \sin nx$, for $x=0$. Thus we obtain†

$$\left| \sum_{\nu=m+1}^{n} (\nu - m)a_\nu + \sum_{\nu=n+1}^{2n-m} (2n - m - \nu)a_\nu \right| \leqq 2L(n - m), \tag{2.6}$$

$$\left| \sum_{\nu=m+1}^{n} (\nu - m)b_\nu + \sum_{\nu=n+1}^{2n-m} (2n - m - \nu)b_\nu \right| \leqq 2L(n - m). \tag{2.7}$$

Consequently, in view of (1.7) and (1.8),

$$\begin{aligned} 0 &\leqq \sum_{\nu=m+1}^{n} (\nu - m)(a_\nu + \alpha_\nu) + \sum_{\nu=n+1}^{2n-m} (2n - m - \nu)(a_\nu + \alpha_\nu) \\ &\leqq (n - m)\left(2L + \sum_{\nu=m+1}^{2n-m} \alpha_\nu\right), \end{aligned} \tag{2.8}$$

$$0 \leqq \sum_{\nu=m+1}^{n} (\nu - m)(a_\nu + \alpha_\nu) \leqq (n - m)\left(2L + \sum_{\nu=m+1}^{2n-m} \alpha_\nu\right). \tag{2.9}$$

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* Loc. cit., footnote ‡ on p. 238.

† Cf. our note referred to in footnote ‡ on p. 238 where more details are given.

On combining (2.9) with (1.7), (1.8), we get

$$\begin{aligned}
&\left|\sum_{\nu=m+1}^{n}(\nu-m)a_\nu\cos\nu x\right| \\
(2.10)\qquad &\leqq \left|\sum_{\nu=m+1}^{n}(\nu-m)(a_\nu+\alpha_\nu)\cos\nu x\right| + \left|\sum_{\nu=m+1}^{n}(\nu-m)\alpha_\nu\cos\nu x\right| \\
&\leqq \sum_{\nu=m+1}^{n}(\nu-m)(a_\nu+\alpha_\nu) + \sum_{\nu=m+1}^{n}(\nu-m)\alpha_\nu \\
&\leqq (n-m)\left(2L+\sum_{\nu=m+1}^{2n-m}\alpha_\nu+\sum_{\nu=m+1}^{n}\alpha_\nu\right),
\end{aligned}$$

and similarly

$$(2.11)\qquad \left|\sum_{\nu=m+1}^{n}(\nu-m)b_\nu\sin\nu x\right| \leqq (n-m)\left(2L+\sum_{\nu=m+1}^{2n-m}\beta_\nu+\sum_{\nu=m+1}^{n}\beta_\nu\right).$$

Now assume that the integers n and m satisfy the conditions

$$(2.12)\quad n \geqq N(1+p),\ m = [n/(1+p)],\ \text{i.e.},\ n/(1+p)-1 < m \leqq n/(1+p).$$

Then

$$(2.13)\qquad \begin{aligned} &m \geqq N \geqq 1,\ m \leqq n-1, \\ &n-m-1 < p(m+1) < p(n+1). \end{aligned}$$

This enables us to apply (2.1) and (2.2) to estimate the sums of the right-hand members of (2.10), (2.11), with the result

$$(2.14)\qquad \left|\sum_{\nu=m+1}^{n}(\nu-m)a_\nu\cos\nu x\right| \leqq (n-m)(2L+3A),$$

$$(2.15)\qquad \left|\sum_{\nu=m+1}^{n}(\nu-m)b_\nu\sin\nu x\right| \leqq (n-m)(2L+3B).$$

Furthermore, for the values of m and n in question, (2.5) gives

$$(2.16)\quad |n\sigma_n(x)-m\sigma_m(x)|/(n-m) \leqq (1+2/(n/m-1))L \leqq (1+2/p)L.$$

Inequality (2.3) follows by an easy combination of (2.4), (2.16), (2.14), and (2.15).

The proof of Theorem 1 is now easily derived from the lemma above, whose conditions obviously are satisfied on the hypotheses of Theorem 1. If $n \geqq N(1+p)$ we use the estimate (2.3) directly. If $n < N(1+p)$ then by the Cauchy-Schwarz inequality and Parseval's theorem,

$$|s_n(x)| \leqq (1+2n^{1/2})L \leqq \{1+(2N(1+p))^{1/2}\}L.$$

Thus we obtain our final estimate

$$|s_n(x)| \leqq \{5 + 2/p + 2(N(1+p))^{1/2}\}L + 3(A+B), \tag{2.17}$$

valid for all values of $n \geqq 0$.

We pass on to the proof of Theorem 2 and first observe that estimates (2.3) and (2.17), in view of the hypotheses of Theorem 2, will hold for an arbitrary choice of $A>0$ and $B>0$, provided N and p are suitably fixed as functions of A and B. Next we remark that the Fourier coefficients of

$$\begin{aligned} f(x) - \sigma_n(x) \sim \sum_{\nu=1}^{n} (\nu/n)(a_\nu \cos \nu x + b_\nu \sin \nu x) \\ + \sum_{\nu=n+1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x) \end{aligned} \tag{2.18}$$

satisfy inequalities like (1.7) with the same α_ν's and β_ν's that accompanied the Fourier coefficients a_ν, b_ν of $f(x)$. Hence, on putting

$$M_n = \max_x |f(x) - \sigma_n(x)|, \quad m_n = \max_x |f(x) - s_n(x)|,$$

we derive the estimate corresponding to (2.17) for the nth partial sum of (2.18), namely

$$\begin{aligned} |s_n(x) - \sigma_n(x)| &= \left| \sum_{\nu=1}^{n} (\nu/n)(a_\nu \cos \nu x + b_\nu \sin \nu x) \right| \\ &\leqq 3(A+B) + \{5 + 2/p + 2(N(1+p))^{1/2}\} M_n. \end{aligned} \tag{2.19}$$

Since

$$|f(x) - s_n(x)| \leqq |f(x) - \sigma_n(x)| + |s_n(x) - \sigma_n(x)|,$$

and, by Fejér's theorem, $M_n \to 0$ as $n \to \infty$, we see that

$$0 \leqq \limsup_{n\to\infty} m_n \leqq 3(A+B).$$

As $A>0$ and $B>0$ are arbitrary, we get finally $m_n \to 0$ as $n \to \infty$, which proves Theorem 2.

3. From Definitions 1′ and 1″ of slow divergence it is immediately seen that, if there exist at all sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying the requirements of Theorems 1 or 2, then the particular sequences

$$\{\alpha_n^*\} = \{\tfrac{1}{2}(|a_n| - a_n)\}, \quad \{\beta_n^*\} = \{\tfrac{1}{2}(|b_n| - b_n)\}$$

will evidently satisfy these requirements. Consequently our Theorems 1 and 2 may be restated in the following form:

THEOREM 3. *If $f(x)$ satisfies condition* (1.1) *and if the series*

$$\sum_{n=1}^{\infty}(|a_n|-a_n), \qquad \sum_{n=1}^{\infty}(|b_n|-b_n) \tag{3.1}$$

are slowly divergent in the sense of Definition 1′, *then the partial sums $s_n(x)$ of the Fourier series of $f(x)$ are uniformly bounded.*

THEOREM 4. *If $f(x)$ is continuous and the series* (3.1) *are slowly divergent in the sense of Definition* 1″, *then the Fourier series of $f(x)$ converges to $f(x)$ uniformly.*

4. We now pass on to the generalizations of Theorems 1 and 2 mentioned in the Introduction. Using the notion of the slow oscillation of Definitions 2′, 2″ and of the uniform slow oscillation of the Remark following these definitions, we can enunciate the generalizations in questions as follows.

THEOREM 5. *Let $f(x)$ satisfy condition* (1.1) *and let its Fourier coefficients satisfy conditions* (1.7) *where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are such that the series $\sum_1^\infty \alpha_n \cos nx$ and $\sum_1^\infty \beta_n \sin nx$ are slowly oscillating* (*uniformly in x*) *in the sense of Definition* 2′. *Then the partial sums $s_n(x)$ of the Fourier series of $f(x)$ are uniformly bounded and the upper bound of $|s_n(x)|$ is expressible in terms of L and of the characteristic data of the slow oscillation of the series $\sum_1^\infty \alpha_n \cos nx$, $\sum_1^\infty \beta_n \sin nx$.*

THEOREM 6. *If $f(x)$ is continuous and if its Fourier coefficients satisfy conditions* (1.7) *where the sequence $\{\alpha_n\}$ and $\{\beta_n\}$ are such that the series $\sum_1^\infty \alpha_n \cos nx$ and $\sum_1^\infty \beta_n \sin nx$ are slowly oscillating* (*uniformly in x*) *in the sense of Definition* 2″, *then the Fourier series of $f(x)$ converges to $f(x)$ uniformly.*

The proof of these propositions is based on a lemma analogous to that of §2.

LEMMA. *If, under the assumptions of Theorem* 5, *the trigonometric series $\sum_1^\infty \alpha_n \cos nx$, $\sum_1^\infty \beta_n \sin nx$ satisfy the conditions*

$$\left|\sum_{\nu=k}^{k+q} \alpha_\nu \cos \nu x\right| \leqq A, \tag{4.1}$$

$$\left|\sum_{\nu=k}^{k+q} \beta_\nu \sin \nu x\right| \leqq B, \tag{4.2}$$

for $k \geqq N \geqq 1$, $0 \leqq q \leqq pk$,

then the partial sums $s_n(x)$ admit of the estimate

$$|s_n(x)| \leqq (5 + 2/p)L + 6(A + B), \quad \textit{provided that } n \geqq N(1 + p). \tag{4.3}$$

Using the notation and proceeding in the same fashion as in the proof of the lemma of §2, we now have, instead of (2.9),

$$
\begin{aligned}
0 &\leqq \sum_{\nu=m+1}^{n} (\nu - m)(a_\nu + \alpha_\nu) \\
&\leqq 2(n-m)L + \left| \sum_{\nu=m+1}^{n} (\nu - m)\alpha_\nu + \sum_{\nu=n+1}^{2n-m} (2n - m - \nu)\alpha_\nu \right|.
\end{aligned}
\tag{4.4}
$$

In view of (2.13) conditions (4.1) yield

$$\left| \sum_{\nu=m+1}^{n} \alpha_\nu \cos \nu x \right| \leqq A, \qquad \left| \sum_{\nu=n+1}^{2n-m} \alpha_\nu \cos \nu x \right| \leqq A,$$

whence it follows that

$$\left| \sum_{\nu=m+1}^{2n-m} \alpha_\nu \cos \nu x \right| \leqq 2A. \tag{4.5}$$

On applying estimate (2.6) to the trigonometric polynomial of the left-hand member of (4.5) we have

$$\left| \sum_{\nu=m+1}^{n} (\nu - m)\alpha_\nu + \sum_{\nu=n+1}^{2n-m} (2n - m - \nu)\alpha_\nu \right| \leqq 4A(n-m). \tag{4.6}$$

Being combined with (4.4) this gives

$$0 \leqq \sum_{\nu=m+1}^{n} (\nu - m)(a_\nu + \alpha_\nu) \leqq 2(n-m)(L + 2A). \tag{4.7}$$

Hence

$$
\begin{aligned}
&\left| \sum_{\nu=m+1}^{n} (\nu - m) a_\nu \cos \nu x \right| \\
&\qquad \leqq \sum_{\nu=m+1}^{n} (\nu - m)(a_\nu + \alpha_\nu) + \left| \sum_{\nu=m+1}^{n} (\nu - m)\alpha_\nu \cos \nu x \right| \\
&\qquad \leqq 2(n-m)(L + 2A) + \left| \sum_{\nu=m+1}^{n} (\nu - m)\alpha_\nu \cos \nu x \right|.
\end{aligned}
\tag{4.8}
$$

We now consider Paley's identity (2.4) with $f(x)$ replaced by the trigonometric polynomial

$$t(x) = \sum_{\nu=m+1}^{n} \alpha_\nu \cos \nu x.$$

If we denote by $\tau_\nu(x)$ the νth arithmetic mean associated with $t(x)$, we have

$$\frac{1}{n-m}\sum_{\nu=m+1}^{n}(\nu-m)\alpha_\nu\cos\nu x = t(x) - (n\tau_n(x) - m\tau_m(x))/(n-m).$$

In view of (2.13) and (4.1),

$$|n\tau_n(x) - m\tau_m(x)| = \left|\sum_{q=0}^{n-m-1}\sum_{\nu=m+1}^{m+1+q}\alpha_\nu\cos\nu x\right| \leqq (n-m)A.$$

Since also $|t(x)| \leqq A$, we have

$$\left|\sum_{\nu=m+1}^{n}(\nu-m)a_\nu\cos\nu x\right| \leqq 2(n-m)(L+3A), \tag{4.9}$$

and similarly

$$\left|\sum_{\nu=m+1}^{n}(\nu-m)b_\nu\sin\nu x\right| \leqq 2(n-m)(L+3B). \tag{4.10}$$

From this point on the argument proceeds in precisely the same fashion as in §2, and the proof of our lemma is complete. Theorem 5 now follows from the above lemma in precisely the same way as Theorem 1 was derived from the lemma of §2. In the present case we obtain the estimate

$$|s_n(x)| \leqq \{5 + 2/p + 2(N(1+p))^{1/2}\}L + 6(A+B), \quad n \geqq 0. \tag{4.11}$$

As to Theorem 6, it will be also proved on the basis of our lemma, but the method used in §2 to prove Theorem 2 can not be applied here and we have to resort to the original argument of Paley, which he used in discussing the uniform convergence of a Fourier series with positive coefficients.* Let $t_r(x; n)$ denote the partial sums of the Fourier series of

$$\begin{aligned} f(x) - \sigma_n(x) \sim &\sum_{\nu=1}^{n}(\nu/n)(a_\nu\cos\nu x + b_\nu\sin\nu x) \\ &+ \sum_{\nu=n+1}^{\infty}(a_\nu\cos\nu x + b_\nu\sin\nu x). \end{aligned} \tag{4.12}$$

Under the hypotheses of Theorem 6 the series $\sum_1^\infty \alpha_\nu\cos\nu x$, $\sum_1^\infty \beta_\nu\sin\nu x$ are slowly oscillating in the sense of Definition 2″. It follows that the series

$$\sum_{\nu=1}^{\infty}\gamma_\nu^{(n)}\cos\nu x \equiv \sum_{\nu=1}^{n}(\nu/n)\alpha_\nu\cos\nu x + \sum_{\nu=n+1}^{\infty}\alpha_\nu\cos\nu x,$$

$$\sum_{\nu=1}^{\infty}\delta_\nu^{(n)}\sin\nu x \equiv \sum_{\nu=1}^{n}(\nu/n)\beta_\nu\sin\nu x + \sum_{\nu=n+1}^{\infty}\beta_\nu\sin\nu x$$

* Loc. cit. in footnote † on p. 237.

are also slowly oscillating in the sense of Definition 2″ and that the constants $\gamma_\nu^{(n)}$, $\delta_\nu^{(n)}$ play the same part relative to the series (4.12) as α_ν, β_ν do relative to the Fourier series of $f(x)$. Indeed, for an arbitrary choice of $A>0, B>0$, there exist an integer $N=N(A, B)$ and a positive $p=p(A, B)$ such that

$$(4.13)\qquad \left|\sum_{\nu=k}^{k+q}\gamma_\nu^{(n)}\cos \nu x\right| \leqq A,$$

$$\text{for } k \geqq R = \max(n, N),\ 0 \leqq q \leqq pk.$$

$$(4.14)\qquad \left|\sum_{\nu=k}^{k+q}\delta_\nu^{(n)}\sin \nu x\right| \leqq B,$$

On applying the lemma above to the series (4.12) we conclude

$$(4.15)\qquad |\,t_r(x; n)\,| \leqq (5 + 2/p)M_n + 6(A + B), \quad r \geqq (1 + p)R,$$

where, as before,

$$M_n = \max_x |\,f(x) - \sigma_n(x)\,|.$$

Since

$$t_r(x; n) = \sum_{\nu=1}^{n}(\nu/n)(\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x) + \sum_{\nu=n+1}^{r}(\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x),$$
$$r > n \geqq 1,$$

we have

$$|\,s_{r_2}(x) - s_{r_1}(x)\,| \leqq (10 + 4/p)M_n + 12(A + B), \quad r_2 > r_1 \geqq (1 + p)R.$$

Now, given any $\epsilon>0$, set $48A=48B=\epsilon$, which fixes also p and N. Choose n_0 so that $(10+4/p)M_n \leqq \epsilon/2$ when $n \geqq n_0$. Then $|s_{r_2}(x)-s_{r_1}(x)| \leqq \epsilon$, provided $r_2>r_1 \geqq (1+p)R_0$, $R_0=\max(n_0, N)$. Thus the Fourier series of $f(x)$ converges uniformly. The fact that its sum is $f(x)$ is implied by the classical theory of Fourier series. The proof of Theorem 6 is now complete.

5. It is clear that the uniform boundedness of the partial sums (1.3) implies the uniformly slow oscillation of the series

$$(5.1)\qquad a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

$$(5.2)\qquad \sum_{n=1}^{\infty} b_n \sin nx,$$

in the sense of Definition 2′, while the uniform convergence of the series (1.2) implies the uniformly slow oscillation of (5.1), (5.2) in the sense of Definition 2″. This leads to the following two propositions, the proof of which is obvious in view of Theorems 5 and 6.

THEOREM 7. *The uniformly slow oscillation of the series* (5.1) *and* (5.2), *in the sense of Definition* $2'$, *is necessary and sufficient for the uniform boundedness of the partial sums of the Fourier series of a bounded (measurable and real-valued) function* $f(x)$.

THEOREM 8. *The uniformly slow oscillation of the series* (5.1) *and* (5.2), *in the sense of Definition* $2''$, *is necessary and sufficient for the uniform convergence of the Fourier series of a continuous function* $f(x)$.

6. Our final generalization of the results obtained heretofore is based on the notion of the one-sided slow oscillation (from above or from below) of a series.

DEFINITION $3'$. *A series* $\sum_1^\infty c_n$ *with real terms is said to be slowly oscillating from below if there exist two positive numbers* P *and* p *and a positive integer* N *such that*

$$\sum_{\nu=k}^{k+q} c_\nu \geqq -P, \quad \text{for } k \geqq N, \quad 0 \leqq q \leqq pk. \tag{6.1}$$

DEFINITION $3''$. *A series* $\sum_1^\infty c_n$ *with real terms is said to be slowly oscillating from below if for an arbitrarily given positive* P *there exist a positive number* p *and a positive integer* N, *both depending on* P, *such that* (6.1) *holds.*

It is clear how these definitions should be modified in order to characterize slow oscillation from above, and also uniform slow oscillation, from below or from above.

We are in a position to state and prove

THEOREM 9. *A necessary and sufficient condition that the partial sums of the Fourier series of a bounded (measurable and real-valued) function* $f(x)$ *be uniformly bounded, is that the series* (5.1) *and* (5.2) *be slowly oscillating from below, in the sense of Definition* $3'$, *uniformly in* x.

THEOREM 10. *A necessary and sufficient condition that the Fourier series of a continuous function* $f(x)$ *converge uniformly to* $f(x)$ *is that the series* (5.1) *and* (5.2) *be slowly oscillating from below, in the sense of Definition* $3''$, *uniformly in* x.

The necessity of the conditions of Theorems 9 and 10 is obvious. The proof of sufficiency is based on the following

LEMMA. *Assume that the series* $\sum_1^\infty c_n$ *satisfies* (6.1) *and, in addition, that the arithmetic means of its partial sums* $s_n = c_1 + c_2 + \cdots + c_n$,

$$S_n = (s_1 + s_2 + \cdots + s_n)/n,$$

satisfy

$$|S_n| \leqq L < \infty, \qquad n \geqq 1. \tag{6.2}$$

Then the partial sums themselves satisfy

$$|s_n| \leqq (3 + 2/p)L + P, \textit{ provided that } n \geqq N(1 + p). \tag{6.3}$$

To prove this lemma we use the identities

$$\begin{aligned}(n + m)S_{n+m} - nS_n - ms_n &= s_{n+1} + \cdots + s_{n+m} - ms_n \\ &= c_{n+1} + (c_{n+1} + c_{n+2}) + \cdots + (c_{n+1} + \cdots + c_{n+m}), \qquad m \geqq 1,\end{aligned} \tag{6.4}$$

$$\begin{aligned}ms_n - nS_n + (n - m)S_{n-m} &= ms_n - (s_{n-m+1} + \cdots + s_n) \\ &= c_n + (c_n + c_{n-1}) + \cdots + (c_n + \cdots + c_{n-m+2}), \qquad 1 \leqq m \leqq n.\end{aligned} \tag{6.5}$$

Now, assuming

$$n \geqq (1 + p)N, \quad pn/(1 + p) < m \leqq 1 + pn/(1 + p), \tag{6.6}$$

we have

$$\begin{gathered} n > N, \quad n/m < 1 + 1/p, \\ 0 \leqq m - 1 \leqq pn/(1 + p) < p(n + 1), \\ n - m + 2 \geqq n/(1 + p) + 1 \geqq N + 1 \geqq 2, \quad m - 2 < p(n - m + 2). \end{gathered}$$

We then may apply (6.1), which, being combined with (6.4), (6.5), gives

$$(n + m)S_{n+m} - nS_n - ms_n \geqq - mP, \tag{6.7}$$

$$ms_n - nS_n + (n - m)S_{n-m} \geqq - (m - 1)P > - mP, \tag{6.8}$$

respectively. From (6.2), (6.7), (6.8) we derive

$$\begin{aligned} - P - (1 + 2n/m)L &< - P + (n/m)S_n - (n/m - 1)S_{n-m} \\ &\leqq s_n \leqq P + (1 + n/m)S_{n+m} - (n/m)S_n \leqq P + (1 + 2n/m)L. \end{aligned} \tag{6.9}$$

Since $n/m<1+1/p$ the desired inequality (6.3) follows at once.

The proof of Theorem 9 is now readily obtained from this lemma, since series (5.1) and (5.2), in view of (1.1) and Fejér's theorem, clearly satisfy all the requirements of the lemma. For $s_n(x)$ we get an estimate of the type (4.3). As to the proof of Theorem 10, it is easily obtained by an argument analogous to that used in proving Theorem 6, the details of which may be left to the reader.

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ON FOURIER TRANSFORMS. III†

BY

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1. Introduction. In this paper we are concerned with functions connected by the relations

$$(1.11)\qquad F(x) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} f(u)e^{-ixu}du,$$

$$(1.12)\qquad f(x) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} F(u)e^{ixu}du.$$

The integrals will usually be interpreted in the sense of Cesàro, i.e., the integral $\int_{-\infty}^{\infty}\phi(u)du$ is said to be summable (C, k), $k>0$, if

$$\int_{-\omega}^{\omega}\left(1-\frac{|u|}{\omega}\right)^k\phi(u)du$$

tends to a limit as ω tends to infinity.

DEFINITION 1. *Write*

$$(1.2)\qquad F(x,\omega) = (2\pi)^{-1/2}\int_{-\omega}^{\omega}\left(1-\frac{|u|}{\omega}\right)f(u)e^{-ixu}du.$$

Suppose that $F(x,\omega)$ *is in* $L^p(-\infty,\infty)$,§ $1\leqq p\leqq\infty$, *for all* ω *and that there is a function* $F(x)$ *such that, if* $1\leqq p<\infty$,

$$\lim_{\omega\to\infty}\int_{-\infty}^{\infty}|F(x)-F(x,\omega)|^p dx = 0,$$

or in case $p=\infty$,

$$\lim_{\omega\to\infty} \textit{essential upper bound } |F(x)-F(x,\omega)| = 0.$$

Then we say that $F(x)$ *is the Fourier transform in* L^p *of* $f(x)$.

This definition is shown to be consistent with the usual one‖ except in the cases $p=1$, ∞.

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§ A measurable function $\phi(x)$ is in $L^p(a, b)$, $1\leqq p<\infty$, if $\int_a^b|\phi(x)|^p dx$ is finite. It is in $L^\infty(a, b)$ if it is equivalent to a function which is bounded in (a, b).

‖ Cf. Wiener, 12, p. 67; Berry, 2, p. 227. (See Bibliography, on p. 266, for references.)

DEFINITION 2. *A measurable function $f(x)$ is said to belong to the class H^p, $1<p<\infty$, when it is integrable L in every finite range and such that*

$$\int_{-\infty}^{\infty} |F(x, \omega)|^p dx \leqq M^p, \tag{1.3}$$

where $F(x, \omega)$ is given by (1.2) and M† is a number independent of ω. The function $f(x)$ belongs to the class H^1 if it satisfies (1.3) for $p=1$ and in addition

$$\int_e |F(x, \omega)| dx \leqq \epsilon \tag{1.31}$$

whenever the measure of the set e is less than some positive number $\delta(\epsilon)$. When $p=\infty$ the condition (1.3) is to be interpreted as

$$|F(x, \omega)| \leqq M, \tag{1.32}$$

for all x and ω.

DEFINITION 3. *A measurable function $F(x)$ belongs to the class L_p^* if it is in $L^p(-\infty, \infty)$, $1 \leqq p < \infty$, and such that*

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{e^{ixu}-1}{iu} F(u)du$$

converges to an indefinite Lebesgue integral. The class L_∞^* is the class of all the bounded functions $F(x)$ for which the expression

$$(2\pi)^{-1/2}\left\{ \int_{-1}^{1} \frac{e^{ixu}-1}{iu} F(u)du + \int_{1}^{\infty} \frac{e^{ixu}}{iu} F(u)du + \int_{-\infty}^{-1} \frac{e^{ixu}}{iu} F(u)du \right\}$$

is summable (C, 1) to an indefinite Lebesgue integral.

DEFINITION 4. *A measurable function belongs to the class H^pL^q, $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$, if it belongs to both H^p and L^q.*

The results of this paper can now be summarized as follows:

(i) If $f(x)$ belongs to H^p, $1 \leqq p \leqq \infty$, then it has a Fourier transform $F(x)$ in L^p and the inverse formula (1.12) holds (Theorems 1 and 2).

(ii) If $F(x)$ belongs to L_p^*, $1 \leqq p \leqq \infty$, then it is the Fourier transform in L^p of a function $f(x)$ which belongs to H^p (Theorem 5).

† Throughout this paper we shall use M to denote a number not necessarily the same at each occurrence but always independent of the variables under consideration.

(iii) If $f(x)$ belongs to H^pL^q, then it has a Fourier transform $F(x)$ in L^p. $F(x)$ belongs to H^q and $f(-x)$ is the Fourier transform in L^q of $F(x)$ (Theorem 10).

(iv) There is complete reciprocity in the class $H^pL^p, 1 \leqq p \leqq \infty$ (Theorem 12).

It is not assumed in (i) that $f(x)$ belongs to a Lebesgue class and this need not be the case. Hence although the inverse formula (1.12) holds it does not follow that $f(-x)$ is the Fourier transform of $F(x)$ in the sense of Definition 1. The result (ii) is the converse of (i). It is also shown in Theorem 6 that if $f(x)$ belongs to H^p then its Fourier transform $F(x)$ belongs to L_p^*.

The results of (iii) and (iv) complete those of (i). A result of particular interest in the Plancherel-Titchmarsh theory of Fourier transforms is the reciprocity in the class L^2. In (iv) we assert the existence of other classes of functions which also possess this reciprocity. When $1 \leqq p \leqq 2$ the class H^pL^p is contained in L^2 but for $p > 2$ this need not be the case.

There is an interesting connection between the case $p = \infty$ and functions which are bounded and harmonic in a half plane. We have already discussed this case from this point of view.† However, the arguments there employed are hardly suited for the case $1 \leqq p < \infty$. Accordingly, in this paper we employ an entirely different method using the notion of weak convergence where before we used Fatou's theorem. We shall, however, refer to our previous paper for the proofs of some of the theorems concerning the case $p = \infty$ when they differ from the case $1 \leqq p < \infty$.

One further point requires explanation. It is the substitution of Definition 1 for the definition of the Fourier transform ordinarily given. There are two reasons for this. First, with the ordinary definition, some of the results of this paper, noticeably Theorem 5, would not be true in the case $p = 1$ as has been shown by Hille and Tamarkin.‡ Secondly, with Definition 1, not only does $F(x, \omega)$ converge in mean to $F(x)$ but it also converges in the ordinary sense almost everywhere. This enables us to avoid using convergence in mean which is in some respects an advantage. In fact some of the functions we employ do not belong to Lebesgue classes and so the notion of convergence in mean is not always applicable. However, it can be shown by the argument used by Hille and Tamarkin§ that the two definitions are equivalent except when $p = 1, \infty$. This we show in Theorem 7.

2. **The class H^p.** In this section we shall prove some of the fundamental results of the paper. In each case we shall have recourse to a series of lemmas.

† Offord, 7. The classes H^∞ and L_∞^* were described in this paper as the classes H and B^* respectively. Some of these results have been extended to Hankel transforms. See 8, and 9.

‡ Hille and Tamarkin, 4, p. 773.

§ Loc. cit.

THEOREM 1. *If $f(x)$ belongs to H^p, $1 \leqq p < \infty$, then the integral*

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} f(u) e^{-ixu} du \tag{2.1}$$

is summable $(C, 1)$ almost everywhere, to a function $F(x)$ which belongs to the class $L^p(-\infty, \infty)$. The function $F(x)$ is the Fourier transform in L^p of $f(x)$ and

$$f(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} F(u) e^{ixu} du \qquad (C, 1)$$

almost everywhere.

THEOREM 2. *If $f(x)$ belongs to H^∞, then the integral (2.1) is boundedly summable $(C, 1)$ almost everywhere to a bounded function $F(x)$ which is the Fourier transform in L^∞ of $f(x)$ and*

$$f(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} F(u) e^{ixu} du \qquad (C, 1) \tag{2.2}$$

almost everywhere.

Theorem 1 follows from Lemmas 3 and 4. The proof of Theorem 2 is given elsewhere.† Possibly the most important result in this connection is Parseval's theorem which takes the following forms according to the value of p.

THEOREM 3. *Let $f(x)$ belong to H^p, $2 \leqq p \leqq \infty$. Let $G(x)$ belong to $L^{p'}$ $(-\infty, \infty)$, where $p' = p/(p-1)$. Let $g(-x)$ be the Fourier transform‡ in L^p of $G(x)$ and let $g(x)$ be bounded in every finite range. Then*

$$\int_{-\infty}^{\infty} F(x) G(x) dx = \int_{-\infty}^{\infty} f(x) g(-x) dx,$$

where the first integral is convergent and the second summable $(C, 1)$.

THEOREM 4. *Let $f(x)$ belong to H^p, $1 \leqq p \leqq 2$. Let $g(x)$ belong to L^p $(-\infty, \infty)$ and let $g(x)$ be bounded in every finite range. Let $G(x)$ be the Fourier transform‡ in $L^{p'}$ of $g(x)$. Then*

$$\int_{-\infty}^{\infty} F(x) G(x) dx = \int_{-\infty}^{\infty} f(x) g(-x) dx$$

where the first integral is convergent and the second summable $(C, 1)$.

---

† Offord, 7, Theorem 1. This theorem only asserts that (2.2) is summable $(C, 2)$ almost everywhere. The complete result is given in Offord, 10.

‡ In the ordinary sense. It is, of course, a classical result that, if $G(x)$ belongs to $L^{p'}$, $1 \leqq p' \leqq 2$, then $G(x)$ has a Fourier transform in L^p. A similar remark applies also to Theorem 4.

The proofs of Theorems 3 and 4 are given after Lemma 3. We employ these theorems in establishing the inverse relations in Theorems 1 and 2.

LEMMA 1. *Let $f(u)$ belong to H^p, $1 \leqq p \leqq \infty$, and let*

$$F(x, \nu) = (2\pi)^{-1/2} \int_{-\nu}^{\nu} \left(1 - \frac{|u|}{\nu}\right) f(u) e^{-ixu} du.$$

Then there exists a sequence $\{\nu_j\}$ such that $F(x, \nu_j)$ converges weakly, with exponent p, to some function $F(x)$ of the class L^p.

By the definition of the class H^p the functions $F(x, \nu)$ are uniformly bounded in L^p. The lemma is thus a classical result† of the theory of weak convergence.

In the following lemma we consider a function $G(x)$ which is bounded and integrable L in $(-\infty, \infty)$. This function belongs to every Lebesgue class and so has a Fourier transform $g(-u)$ in the classical sense.

LEMMA 2. *Let $f(x)$ belong to H^p, $1 \leqq p \leqq \infty$, and let a sequence $\{\nu_j\}$ and a function $F(x)$ be defined as in Lemma 1. Let $G(x)$ be integrable L and bounded in $(-\infty, \infty)$ and let $g(-x)$ be the Fourier transform of $G(x)$. Then*

$$\int_{-\infty}^{\infty} F(u)G(u)du = \lim_{j\to\infty} \int_{-\nu_j}^{\nu_j} \left(1 - \frac{|u|}{\nu_j}\right) f(u)g(-u)du.$$

Since $G(x)$ belongs to $L(-\infty, \infty)$,

$$\int_{-\nu_j}^{\nu_j} \left(1 - \frac{|u|}{\nu_j}\right) f(u)g(-u)du$$

$$= (2\pi)^{-1/2} \int_{-\nu_j}^{\nu_j} \left(1 - \frac{|u|}{\nu_j}\right) f(u)du \int_{-\infty}^{\infty} G(x)e^{-ixu}dx$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty} G(x)dx \int_{-\nu_j}^{\nu_j} \left(1 - \frac{|u|}{\nu_j}\right) f(u)e^{-ixu}du = \int_{-\infty}^{\infty} F(x, \nu_j)G(x)dx.$$

Now $G(x)$ is bounded and belongs to $L(-\infty, \infty)$. Hence it belongs to $L^{p'}(-\infty, \infty)$, $p' = p/(p-1)$. Therefore, by Lemma 1 and the general convergence theorem of Hobson and Lebesgue,‡

$$\int_{-\infty}^{\infty} F(x)G(x)dx = \lim_{\nu_j\to\infty} \int_{-\nu_j}^{\nu_j} \left(1 - \frac{|u|}{\nu_j}\right) f(u)g(-u)du.$$

The reader will observe that, in the proof of the case $p = 1$, it is at this stage that we require (1.31).

† Cf. Hobson, 5, p. 253, and Banach, 1.

‡ Hobson, 5, p. 422, and Lebesgue, 6, p. 52.

LEMMA 3. *If $f(x)$ belongs to H^p, $1 \leqq p \leqq \infty$, then the integral*

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} f(u)e^{-ixu}du$$

is summable $(C, 1)$ almost everywhere to a function $F(x)$ which is the Fourier transform in L^p of $f(x)$.

In Lemma 2 take

$$g(u) = \begin{cases} (2\pi)^{-1/2}\left(1 - \dfrac{|u|}{\omega}\right) e^{ixu}, & -\omega \leqq u \leqq \omega, \\ 0, & u \leqq -\omega,\ \omega \leqq u. \end{cases}$$

Then

$$G(y) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} g(u)e^{-iyu}du = \frac{2}{\pi\omega} \frac{\sin^2 \frac{1}{2}\omega(x-y)}{(x-y)^2}.$$

The function $G(y)$ obviously satisfies the hypotheses of Lemma 2. Hence, if $F(y)$ is defined as in Lemma 1, we have

$$(2\pi)^{-1/2} \int_{-\omega}^{\omega} \left(1 - \frac{|u|}{\omega}\right) f(u)e^{-ixu}du = \frac{2}{\pi\omega} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{1}{2}\omega(x-y)}{(x-y)^2} F(y)dy.$$

Now this is Fejér's integral and it is well known that for $1 \leqq p \leqq \infty$ the integral tends to $F(x)$ almost everywhere as ω tends to infinity. Hence

$$F(x) = (2\pi)^{-1/2} \lim_{\omega\to\infty} \int_{-\omega}^{\omega} \left(1 - \frac{|u|}{\omega}\right) f(u)e^{-ixu}du$$

almost everywhere, the convergence being in L^p. This is the desired result.

We can now complete the proofs of Theorems 3 and 4.

Proof of Theorem 3. By hypothesis, $f(x)$ belongs to H^p and $G(x)$ to $L^{p'}$. Hence, by Lemma 3,

$$\begin{aligned} \int_{-\infty}^{\infty} G(x)F(x)dx &= (2\pi)^{-1/2} \lim_{\omega\to\infty} \int_{-\infty}^{\infty} G(x)dx \int_{-\omega}^{\omega} \left(1 - \frac{|u|}{\omega}\right) f(u)e^{-ixu}du \\ &= (2\pi)^{-1/2} \lim_{\omega\to\infty} \lim_{X\to\infty} \int_{-X}^{X} \left(1 - \frac{|x|}{X}\right) G(x)dx \int_{-\omega}^{\omega} \left(1 - \frac{|u|}{\omega}\right) f(u)e^{-ixu}du \\ &= \lim_{\omega\to\infty} I, \end{aligned}$$

in virtue of the consistence theorem for Cesàro summation. We use I as an abbreviation.

Now $G(x)$ belongs to $L^{p'}$, $1 \leqq p' \leqq 2$. Hence $G(x)$ has a Fourier transform $g(-u)$ in L^p. Also†

† This follows from a classical argument due to Plancherel. See Hobson, 5, pp. 748–49. The proof for bounded convergence is of course the same as the proof for uniform convergence.

$$(2.21) \qquad g(-u) = \lim_{X\to\infty} (2\pi)^{-1/2} \int_{-X}^{X} \left(1 - \frac{|x|}{X}\right) G(x) e^{-ixu} dx.$$

Further the convergence is bounded in every interval in which $g(-u)$ is bounded and so, by hypothesis, in every finite range $|u| \leq \omega$. Therefore

$$\begin{aligned} I &= \lim_{X\to\infty} (2\pi)^{-1/2} \int_{-\omega}^{\omega} \left(1 - \frac{|u|}{\omega}\right) f(u) du \int_{-X}^{X} \left(1 - \frac{|x|}{X}\right) G(x) e^{-ixu} dx \\ &= \int_{-\omega}^{\omega} \left(1 - \frac{|u|}{\omega}\right) f(u) g(-u) du. \end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} G(x) F(x) dx = \lim_{\omega\to\infty} \int_{-\omega}^{\omega} \left(1 - \frac{|u|}{\omega}\right) f(u) g(-u) du$$

and this is the desired result.

Proof of Theorem 4. The proof of this theorem is very similar to that of Theorem 3. Here $g(u)$ belongs to L^p, $1 \leq p \leq 2$. Hence, with the usual terminology, $g(u)$ has a Fourier transform $G(x)$ in $L^{p'}$. Further, by Plancherel's argument,† the relation (2.21) holds, the convergence being bounded in every interval in which $g(-u)$ is bounded. The desired conclusion therefore follows by the preceding argument.

The following lemma together with Lemma 3 completes the proof of Theorem 1.

LEMMA 4. *If $f(x)$ belongs to H^p, $1 \leq p \leq \infty$, and if $F(x)$ is defined as in Lemma 3, then*

$$f(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} F(u) e^{ixu} du \qquad (C, 1)$$

almost everywhere.

In Theorems 3 and 4 take

$$g(u) = \begin{cases} 1, & -x \leq u \leq 0, \\ 0, & u < -x,\ 0 < u. \end{cases}$$

Then

$$G(y) = (2\pi)^{-1/2} \int_{-x}^{0} e^{-iyu} du = \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin \frac{1}{2}xy}{y} e^{ixy/2},$$

and the hypotheses of Theorems 3 and 4 are obviously satisfied.

Hence

† Hobson, 5, p. 750.

$$\int_0^x f(u)du = \left(\frac{2}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} \frac{\sin \frac{1}{2}xy}{y} e^{ixy/2} F(y) dy. \tag{2.31}$$

Consider first the case $p=1$. In this case $F(y)$ belongs to $L(-\infty, \infty)$ and the second member of (2.31) may be written in the form

$$(2\pi)^{-1/2} \int_0^x du \int_{-\infty}^{\infty} e^{iuy} F(y) dy.$$

The desired conclusion now follows at once.

Now suppose $p>1$. From (2.31) and Hölder's inequality,

$$\begin{aligned} \left| \int_0^x f(u)du \right| &\leqq \left(\frac{2}{\pi}\right)^{1/2} \left[\int_{-\infty}^{\infty} \left| \frac{\sin \frac{1}{2}xy}{y} \right|^p dy \right]^{1/p'} \left[\int_{-\infty}^{\infty} | F(y) |^p dp \right]^{1/p} \\ &\leqq M x^{1/p}. \end{aligned} \tag{2.32}$$

Take

$$G(u) = \begin{cases} (2\pi)^{-1/2} \left(1 - \dfrac{|u|}{\omega}\right) e^{ixu}, & -\omega \leqq u \leqq \omega, \\ 0, & u \leqq -\omega, \ \omega \leqq u. \end{cases}$$

Then

$$g(-y) = \frac{2}{\pi\omega} \frac{\sin^2 \frac{1}{2}\omega(x-y)}{(x-y)^2}$$

and the hypotheses of Theorems 3 and 4 are again satisfied. Hence

$$\begin{aligned} &(2\pi)^{-1/2} \int_{-\omega}^{\omega} \left(1 - \frac{|u|}{\omega}\right) F(u) e^{ixu} du \\ &= \frac{2}{\pi\omega} \int_{-\infty}^{\infty} f(x+y) \frac{\sin^2 \frac{1}{2}xy}{y^2} dy \qquad (C, 1). \end{aligned} \tag{2.33}$$

But it is easy to see that, in virtue of (2.32), the integral in the second member of (2.33) exists as an improper Lebesgue integral. For, writing

$$f_1(y) = \int_0^y f(u)du$$

we have

$$\begin{aligned} \int_0^Y f(x+y) \frac{\sin^2 \frac{1}{2}\omega y}{y^2} dy &= f_1(x+Y) \frac{\sin^2 \frac{1}{2}\omega Y}{Y^2} - \frac{1}{4}\omega^2 f_1(x) \\ &\quad - \int_0^Y f_1(x+y) \frac{d}{dy}\left(\frac{\sin^2 \frac{1}{2}\omega y}{y^2}\right) dy, \end{aligned}$$

and by (2.32) this obviously tends to a limit as Y tends to infinity.

Divide the integral in the second member of (2.33) into three terms, viz.,

$$\frac{2}{\pi\omega}\left\{\int_{-\infty}^{-K}+\int_{-K}^{K}+\int_{K}^{\infty}\right\} = I_1 + I_2 + I_3,$$

where K is a positive constant such that $-K \leqq x \leqq K$. Then

$$I_3 = -\frac{2}{\pi\omega}\frac{\sin^2 \frac{1}{2}\omega K}{K^2} f_1(x+K) + \frac{4}{\pi\omega}\int_K^\infty \frac{\sin^2\frac{1}{2}\omega y}{y^3} f_1(x+y)dy$$
$$-\frac{2}{\pi}\int_K^\infty \frac{\sin \omega v}{y^2} f_1(x+y)dy.$$

Now, by (2.32),

$$|f_1(x+y)| \leqq M|x+y|^{1/p}.$$

Therefore, if K and x are fixed,

$$\lim_{\omega\to\infty} I_3 = -\frac{2}{\pi}\lim_{\omega\to\infty}\int_K^\infty \frac{\sin\omega y}{y^2} f_1(x+y)dy = 0$$

by the Riemann-Lebesgue theorem. Similarly $\lim_{\omega\to\infty} I_1 = 0$ while I_2 is Fejér's integral and so tends to $f(x)$ as ω tends to infinity. This yields the desired result.

3. **The class L_p^*.** The fundamental property of functions of this class is given in the following theorem. The corresponding result for the case $p=\infty$ is given elsewhere.†

THEOREM 5. *If $F(x)$ belongs to L_p^*, $1 \leqq p < \infty$, and if*

$$f(x) = (2\pi)^{-1/2}\frac{d}{dx}\int_{-\infty}^{\infty}\frac{e^{ixu}-1}{iu}F(u)du, \tag{3.1}$$

then $f(x)$ belongs to H^p and $F(x)$ is the Fourier transform in L^p of $f(x)$.

We begin by showing that $f(x)$ belongs to H^p. Since $F(x)$ belongs to L_p^*, $1 \leqq p < \infty$, the integral

$$\int_{-\infty}^{\infty}\frac{e^{ixu}-1}{iu}F(u)du$$

converges uniformly in every finite range. We have

$$F(x,\omega) = (2\pi)^{-1/2}\int_{-\omega}^{\omega}\left(1-\frac{|u|}{\omega}\right)f(u)e^{-ixu}du$$

† Offord, 7, Theorem 5.

$$
(3.11)\qquad
\begin{aligned}
&= (2\pi)^{-1/2}\int_{-\omega}^{\omega}\frac{d}{du}\left\{\left(1-\frac{|u|}{\omega}\right)e^{-ixu}\right\}du\int_{-\infty}^{\infty}\frac{e^{iut}-1}{it}F(t)dt\\
&= (2\pi)^{-1/2}\int_{-\infty}^{\infty}F(t)dt\int_{-\omega}^{\omega}\frac{d}{du}\left\{\left(1-\frac{|u|}{\omega}\right)e^{-ixu}\right\}\frac{e^{itu}-1}{it}\,du\\
&= \frac{2}{\pi\omega}\int_{-\infty}^{\infty}\frac{\sin^2\frac{1}{2}\omega t}{t^2}F(t+x)dt.
\end{aligned}
$$

Consider first the case $1<p<\infty$. By Hölder's inequality

$$
\begin{aligned}
|F(x,\omega)|^p &\leqq \left(\frac{2}{\pi\omega}\right)^p\left[\int_{-\infty}^{\infty}|F(t+x)|^p\left|\frac{\sin\frac{1}{2}\omega t}{t}\right|^p dt\right]\left[\int_{-\infty}^{\infty}\left|\frac{\sin\frac{1}{2}\omega t}{t}\right|^{p'}dt\right]^{p/p'}\\
&\leqq M\omega^{1-p}\int_{-\infty}^{\infty}|F(t+x)|^p\left|\frac{\sin\frac{1}{2}\omega t}{t}\right|^p dt.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_{-\infty}^{\infty}|F(x,\omega)|^p dx &\leqq M\omega^{1-p}\int_{-\infty}^{\infty}\left|\frac{\sin\frac{1}{2}\omega t}{t}\right|^p dt\int_{-\infty}^{\infty}|F(t+x)|^p dx\\
&\leqq M\int_{-\infty}^{\infty}\left|\frac{\sin\theta}{\theta}\right|^p d\theta\int_{-\infty}^{\infty}|F(\xi)|^p d\xi = M,
\end{aligned}
$$

the desired result. Now suppose $p=1$. Then

$$
\int_{-\infty}^{\infty}|F(x,\omega)|\,dx \leqq \frac{2}{\pi\omega}\int_{-\infty}^{\infty}\frac{\sin^2\frac{1}{2}\omega t}{t^2}dt\int_{-\infty}^{\infty}|F(t+x)|\,dx \leqq M.
$$

We have further to show that $F(x,\omega)$ satisfies (1.31). Let e be any measurable set in $(-\infty,\infty)$. Then

$$
\int_e|F(x,\omega)|\,dx \leqq \frac{2}{\pi\omega}\int_{-\infty}^{\infty}\frac{\sin^2\frac{1}{2}\omega t}{t^2}dt\int_e|F(x+t)|\,dx.
$$

But since $F(x)$ belongs to $L(-\infty,\infty)$ it is possible to find a δ independently of t such that, if $m(e)\leqq\delta$, then

$$
\int_e|F(x+t)|\,dt \leqq \epsilon.
$$

Hence $f(x)$ belongs to H^1 as desired. It only remains to show that $F(x,\omega)$ tends to $F(x)$ almost everywhere. This, however, follows at once from the last equality in (3.11). This completes the proof of the theorem.

THEOREM 6. *If $f(x)$ belongs to H^p, $1\leqq p\leqq\infty$, then its Fourier transform $F(x)$ belongs to L_p^*.*

The proof for the case $p=\infty$ is given elsewhere.† When $1 \leqq p < \infty$, we have

$$
\begin{aligned}
(2\pi)^{-1/2} & \int_{-\infty}^{\infty} \frac{e^{ixu}-1}{iu} F(u)du \\
&= (2\pi)^{-1/2} \lim_{\omega\to\infty} \int_{-\omega}^{\omega} \left(1 - \frac{|y|}{\omega}\right) f(y)dy \int_{-\infty}^{\infty} \frac{(e^{ixu}-1)e^{-iyu}}{iu} du \\
&= \int_0^x f(y)dy.
\end{aligned}
$$

This is the desired result.

The next theorem connects the theory developed here with the ordinary theory of Fourier transforms.

THEOREM 7. *Let $f(x)$ belong to H^p, $1<p<\infty$, and let $F(x)$ be its Fourier transform in L^p as defined in Theorem* 1. *Write*

$$I(x, \omega) = (2\pi)^{-1/2} \int_{-\omega}^{\omega} f(u) e^{-ixu} du;$$

then

$$\lim_{\omega\to\infty} \int_{-\infty}^{\infty} |F(x) - I(x,\omega)|^p dx = 0, \tag{3.12}$$

i.e., $F(x)$ is the Fourier transform in L^p of $f(x)$ in the ordinary sense. Conversely if $f(x)$ is such that (3.12) *is satisfied, then $f(x)$ belongs to H^p and $F(x)$ is its Fourier transform in L^p in the sense of Definition* 1.

To prove the first part of the theorem, in Theorems 3 and 4, take

$$g(u) = \begin{cases} (2\pi)^{-1/2} e^{ixu}, & -\omega \leqq u \leqq \omega, \\ 0, & u < -\omega,\ \omega < u. \end{cases}$$

Then

$$G(y) = \frac{1}{\pi} \frac{\sin \omega(x-y)}{x-y}.$$

Hence

$$I(x,\omega) = (2\pi)^{-1/2} \int_{-\omega}^{\omega} f(u) e^{-ixu} du = \frac{1}{\pi} \int_{-\infty}^{\infty} F(y) \frac{\sin \omega(x-y)}{x-y} dy.$$

Now Hille and Tamarkin‡ have shown that the integral in the second member converges in mean of order p, $1<p<\infty$, to $F(x)$ as ω tends to infinity. This proves the first part of the theorem. As regards the second part, we have

† Offord, 7, Theorem 2.

‡ Hille and Tamarkin, 4, pp. 770–771.

$$F(t) = \underset{\omega\to\infty}{\mathrm{l.i.m}} \int_{-\omega}^{\omega} f(u)e^{-itu}du.$$

Hence, if $1<p<\infty$,

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty}\frac{e^{ixt}-1}{it}F(t)dt = (2\pi)^{-1/2}\int_{-\infty}^{\infty}f(u)du\int_{-\infty}^{\infty}\frac{e^{ixt}-1}{it}e^{-itu}dt$$
$$= \int_0^x f(u)du.$$

Therefore $F(t)$ belongs to L_p^* and so, by Theorem 5, $f(x)$ must belong to H^p. Again, by Theorem 5, it follows that $F(t)$ is the Fourier transform in L^p of $f(x)$. This is the desired result.

We now give some sufficient conditions for functions to belong to the classes H^p and L_p^*.

THEOREM 8. *If $F(x)$ belongs to $L^p(-\infty,\infty)$, $1\leqq p\leqq\infty$, and if*

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty}F(u)e^{ixu}du \tag{3.2}$$

converges everywhere to a function $f(x)$ which is everywhere finite and integrable L in every finite range, then $F(x)$ belongs to L_p^* and is the Fourier transform in L^p of $f(x)$.

Since $F(x)$ belongs to $L^p(-\infty,\infty)$, $1\leqq p\leqq\infty$,

$$\int_{-\infty}^{\infty}\frac{|F(x)|}{1+x^2}dx \tag{3.3}$$

is finite. Now Pollard† has shown that if (3.2) converges everywhere to a function $f(x)$ which is everywhere finite and integrable L in every finite range and if (3.3) is finite, then

$$\int_0^x dt\int_0^t f(u)du = (2\pi)^{-1/2}\int_{-1}^{1}\frac{e^{ixu}-1-ixu}{-u^2}F(u)du$$
$$-(2\pi)^{-1/2}\left\{\int_1^{\infty}+\int_{-\infty}^{-1}\right\}\frac{e^{ixu}}{u^2}F(u)du + \text{constant}.$$

Hence, since $F(x)$ belongs to $L^p(-\infty,\infty)$, $1\leqq p\leqq\infty$,

$$f(x) = (2\pi)^{-1/2}\frac{d}{dx}\int_{-\infty}^{\infty}\frac{e^{ixu}-1}{iu}F(u)du,$$

which is the desired result.

† Pollard, 11, p. 455.

The following theorem also follows from Pollard's methods.

THEOREM 9. *If $f(x)$ is such that*

$$\int_{-\infty}^{\infty} \frac{|f(x)|}{1+x^2}\,dx \tag{3.41}$$

is convergent and if

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty} f(u)e^{-ixu}du$$

converges everywhere to a function $F(x)$ which is everywhere finite and belongs to $L^p(-\infty,\infty)$, $1\leqq p<\infty$, then $f(x)$ belongs to H^p and $F(x)$ is its Fourier transform.

Without loss of generality we may suppose $f(u)=0$, $0\leqq u\leqq 1$. Then it follows by Pollard's theorem that

$$\frac{f(u)}{u^2} = -(2\pi)^{-1/2}\lim_{Y\to\infty}\int_{-Y}^{Y}\left(1-\frac{|y|}{Y}\right)F_2(y)e^{iuy}dy \tag{3.42}$$

where

$$F_2(y) = \int_0^y F_1(t)dt = \int_0^y dt\int_0^t F(u)du.$$

Now Pollard has shown that

$$F_2(y)\to 0 \tag{3.43}$$

as y tends to infinity. Again, since $F(y)$ belongs to L^p, $1\leqq p<\infty$,

$$|F_1(y)| \leqq y^{1/p'}\left[\int_{-\infty}^{\infty}|F(t)|^p dt\right]^{1/p} = o(y). \tag{3.44}$$

Hence, from (3.41), (3.42), (3.43) and (3.44),

$$\begin{aligned}\int_0^x f(u)du = \int_1^x f(u)du &= -(2\pi)^{-1/2}\lim_{Y\to\infty}\int_{-Y}^{Y}\left(1-\frac{|y|}{Y}\right)F_2(y)dy\int_1^x u^2e^{iyu}du\\
&= (2\pi)^{-1/2}\lim_{Y\to\infty}\int_{-Y}^{Y}\left(1-\frac{|y|}{Y}\right)F_2(y)\frac{d^2}{dy^2}\left(\frac{e^{ixy}-e^{iy}}{iy}\right)dy\\
&= (2\pi)^{-1/2}\lim_{Y\to\infty}\int_{-Y}^{Y}\frac{e^{ixy}-e^{iy}}{iy}\,\frac{d^2}{dy^2}\left\{\left(1-\frac{|y|}{Y}\right)F_2(y)\right\}dy\\
&= (2\pi)^{-1/2}\int_{-\infty}^{\infty}\frac{e^{ixy}-1}{iy}F(y)dy - (2\pi)^{-1/2}\int_{-\infty}^{\infty}\frac{e^{iy}-1}{iy}F(y)dy,\end{aligned}$$

the integrals being convergent. It follows that $F(x)$ belongs to L_p^* and conquently, by Theorem 5, $f(x)$ belongs to H^p. This is the desired result.

4. **The class H^pL^q.** Consider the class H^pL^q of all the functions which belong to both H^p, $1 \leqq p \leqq \infty$, and L^q, $1 \leqq q \leqq \infty$. The following theorem holds for functions of this class.

THEOREM 10. *If $f(x)$ belongs to H^pL^q, $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$, then it has a Fourier transform $F(x)$ in L^p. $F(x)$ belongs to H^qL^p and $f(-x)$ is the Fourier transform in L^q of $F(x)$.*

Consider first the case $1 \leqq p < \infty$, $1 \leqq q \leqq \infty$. We can apply Theorems 3 and 4 with

$$G(u) = \begin{cases} 1, & 0 \leqq u \leqq x, \\ 0, & \text{elsewhere}. \end{cases}$$

Then

$$\int_0^x F(u)du = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{1 - e^{-ixu}}{iu} f(u)du. \tag{4.1}$$

But $f(x)$ belongs to L^q and so by (4.1) to L_q^*. Hence, if $1 \leqq q < \infty$, it follows from Theorem 5 that $F(x)$ belongs to H^q. Again, if $q = \infty$ then $F(x)$ belongs to H^∞ by the analogue† of Theorem 5 for this case. Now $f(x)$ belongs to H^p and so must have a Fourier transform in L^p, i.e., $F(x)$ belongs to H^qL^p. Also, since $F(x)$ belongs to H^q, it has a Fourier transform $f(-x)$ in L^q. This proves the theorem for the case $1 \leqq p < \infty$. Now suppose $p = \infty$. Then

$$F(x) = (2\pi)^{-1/2} \lim_{\omega \to \infty} \int_{-\omega}^{\omega} \left(1 - \frac{|u|}{\omega}\right) f(u) e^{-ixu} du.$$

The convergence is bounded so that

$$\int_0^x F(t)dt = (2\pi)^{-1/2} \lim_{\omega \to \infty} \int_{-\omega}^{\omega} \left(1 - \frac{|u|}{\omega}\right) \frac{1 - e^{-ixu}}{iu} f(u)du.$$

The desired result now follows as before.

From this result we can deduce at once the following theorem.

THEOREM 11. *If $f(x)$ has a Fourier transform $F(x)$ in the class L^p, $1 \leqq p \leqq \infty$, then the necessary and sufficient condition that $f(-x)$ should be the Fourier transform of $F(x)$ in L^q, $1 \leqq q \leqq \infty$, is that $f(x)$ should belong to L^q.*

It must be remarked that this theorem is true only when the Fourier transform of a function is defined as in Definition 1.

† Offord, 7, p. 208.

5. **The reciprocal class H^pL^p.** A very interesting special case of the class H^pL^q is the class H^pL^p of all the functions which belong to both H^p and L^p. We obtain the following theorem by putting $q=p$ in Theorem 10.

THEOREM 12. *If $f(x)$ belongs to H^pL^p, $1 \leqq p \leqq \infty$, then its Fourier transform also belongs to H^pL^p.*

We have already discussed† the case $p=\infty$ under name of the class HB. We now add an example of a function which is bounded and belongs to H^∞ but does not belong to any Lebesgue class. Take

$$f(u) = \cos\left\{\left[\frac{u}{2\pi}\right]u\right\} = \begin{cases} 1, & 0 \leqq u < 2\pi, \\ \cos nu, & 2n\pi \leqq u < 2(n+1)\pi. \end{cases}$$

We shall show that

$$\left|\int_0^\omega f(u)\cos xu\,du\right| \leqq M$$

for all x and ω and from this it follows that

$$\left|\int_{-\omega}^{\omega}\left(1 - \frac{|u|}{\omega}\right)e^{-ixu}f(u)du\right| \leqq M,$$

i.e., $f(u)$ belongs to $H^\infty L^\infty$ as desired. Suppose that $2(m+1)\pi \leqq \omega < 2(m+2)\pi$ and that $r-\frac{1}{2} \leqq x < r+\frac{1}{2}$. Then, using $O(1)$ to denote an expression which is uniformly bounded in x and ω, we get

$$\begin{aligned}\int_0^\omega f(u)\cos xu\,du &= \sum_{\substack{n=0 \\ n\neq r-1,r}}^{m} \int_{2n}^{2(n+1)\pi} \cos xu \cos nu\,du + O(1) \\ &= \frac{1}{2}\sum_{\substack{n=0 \\ n\neq r-1,r}}^{m}\left[\frac{\sin 2(n+1)\pi x - \sin 2n\pi x}{n+x}\right. \\ &\qquad\qquad \left. - \frac{\sin 2(n+1)\pi x - \sin 2n\pi x}{n-x}\right] + O(1) \\ &= \frac{1}{2}\sum_{\substack{n=1 \\ n\neq r-1,r,r+1}}^{m} \sin 2n\pi x\left\{\frac{1}{(n+x)(n+x-1)}\right. \\ &\qquad\qquad \left. - \frac{1}{(n-x)(n-x-1)}\right\} + O(1).\end{aligned}$$

Therefore

† Offord, 7, p. 211.

$$\left|\int_0^\infty f(u)\cos xu\,du\right|$$

$$\leqq \frac{1}{2}\sum_{\substack{n=1\\ n\neq r-1,r,r+1}}^{m}\left\{\frac{1}{(n+x)(n+x-1)}+\frac{1}{|n+x|\,|n-x-1|}\right\}+O(1)$$

$$\leqq \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{2}\sum_{n=r+2}^{\infty}\frac{1}{(n-x)(n-x-1)}$$

$$+\frac{1}{2}\sum_{n=1}^{r-2}\frac{1}{(x-n)(x-n+1)}+O(1)$$

$$\leqq \frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}+O(1)\leqq M.$$

It is easily verified that

$$F(x)=(2\pi)^{-1/2}\int_0^\infty \cos xu\cos\left\{\left[\frac{u}{2\pi}\right]u\right\}du$$

$$=(2\pi)^{-1/2}2x\sin\pi x\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi x}{x^2-n^2}.$$

Corresponding to each class H^pL^p there is a class of self-reciprocal functions and we conclude this paper by giving two theorems for these functions.

Suppose that $\chi(t)$ is integrable L in every finite range and such that

$$\int_{-\infty}^{\infty}\left|\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)\chi(t)x^{-1/2-it}dt\right|^p dx\leqq M, \tag{5.1}$$

for all $T\geqq 0$. Further let

$$\frac{\chi(t)}{2^{1/4+it/2}\Gamma(\frac{1}{4}+\frac{1}{2}it)}=\text{even function of } t. \tag{5.2}$$

THEOREM 13. *A necessary and sufficient condition that an even function $f(x)$ of H^pL^p, $1\leqq p\leqq\infty$, should be its own Fourier transform is that it should be of the form*

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\chi(t)x^{-1/2-it}dt, \tag{5.3}$$

where the integral is summable $(C, 1)$ *almost everywhere and $\chi(t)$ satisfies* (5.1) *and* (5.2).

THEOREM 14. *A necessary and sufficient condition that an even function $f(x)$ of L^p, $1 \leqq p < \infty$, should be a solution of the equation*

$$f(x) = (2\pi)^{-1/2} \frac{d}{dx} \int_{-\infty}^{\infty} \frac{e^{ixu} - 1}{iu} f(u) du$$

is that it should be of the form (5.3), *where the integral is summable* $(C, 1)$ *almost everywhere and* $\chi(t)$ *satisfies* (5.1) *and* (5.2).

These theorems are the analogues for the class H^pL^p of the theorem of Hardy and Titchmarsh† for the class L^2. They can be proved by their argument and we shall not give the proof here. It is interesting to notice that the function $x^{1/2}J_{-1/4}(\frac{1}{2}x^2)$ is a self-reciprocal function of the class H^pL^p, for all $p > 2$, but it does not belong to L^2.

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† Hardy and Titchmarsh, 3, p. 201. See also Offord, 7, Theorem 9.

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DIE DIFFERENTIALGEOMETRIE DER UNTER-MANNIGFALTIGKEITEN DES R_n KONSTANTER KRÜMMUNG*

BY

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Einleitung: Die Stellung des Problems

Wie eine Strecke durch ihre Länge, ein Dreieck durch seine Seiten und eine Kurve durch ihre Bogenlänge und die Krümmungen bis auf eine Kongruenz-Transformation im R_n konstanter Krümmung bestimmt sind, so ist auch die e-dimensionale Fläche (F_e, $e=1, 2, \cdots, n-1$) des R_n konstanter Krümmung durch ein System von invarianten Formen bis auf ihre Lage im R_n festgelegt.

Wir meinen damit den Kongruenz-Satz:

Kongruente F_e haben gleiche Formen-Systeme und umgekehrt.

Zu der Aufgabe der Herstellung eines die F_e vollständig bestimmenden Formen-Systems tritt ganz natürlich die, *die Bedingungen dafür anzugeben, dass es zu einem gegebenen System von Formen eine F_e dieses Formen-Systems gibt.* (Für die F_2 des euklidischen R_3: die Gauss'schen resp. Codazzi'schen Relationen.)

Die beiden so skizzierten Probleme wurden 1924 von C. Burstin und dem Verfasser gelöst (das zweite Problem für das vollständige System der "Masstensoren"); die Darstellung (siehe Duschek-Mayer, *Lehrbuch der Differentialgeometrie*, in der Folge als *Lehrbuch* zitiert) benutzt aber Hilfs-Beine, die die Normal-Vektorräume der F_e aufspannen, und die zwar ohne Schaden eingeführt werden können, aber als "fremdes Element" einen Schönheitsfehler für die Darstellung bedeuten.

In der nun vorliegenden Arbeit wird dieser "Schönheitsfehler" beseitigt.† Die neue Darstellung hat damit den Vorzug, mit den Koordinaten des Raumes R_n und den Parametern der Fläche F_e allein auszukommen: Bein-Indizes gibt es keine.

Es ist natürlich klar, dass sich in dieser Darstellung dann alle geometrischen Verhältnisse im Formalismus klarer zu erkennen geben, unverwischter, da zwischen Objekt und Symbol sich nichts fremdes mehr einschiebt.

* Presented to the Society, April 20, 1935; received by the editors December 12, 1934.

† Es soll damit nicht behauptet sein, dass die ursprüngliche Darstellung (*Lehrbuch*) nun überflüssig geworden ist, da gerade für die Behandlung spezieller Probleme sich die Einführung von Hilfs-Beinen als zweckmässig erweist.

Die vorliegende Arbeit hat bereits einen Vorgänger: Eine mit C. Burstin gemeinsam verfasste, aber vom Schreiber allein ausgearbeitete Schrift (Monatshefte für Mathematik und Physik, vol. 35 (1928), pp. 87–110: *Über das vollständige Formensystem* . . .) enthält eine solche Hilfs-Bein-freie Darstellung der Theorie.

Methodische Unzulänglichkeiten aber, aus dem natürlichen Bestreben entstanden, für die Darstellung der Vektoren von Vektorräumen ausschliesslich *unabhängige* Basis-Vektoren zu verwenden, hatten eine Unsymmetrie in der Behandlung gleichartiger Objekte zur Folge, die nicht in der Natur des Problemes liegt.

Es galt also das Widerstreben zu beseitigen, eine nicht linear unabhängige Vektor-Basis für die Beschreibung zu verwenden, und damit jenes Basis-Bein voll zu benutzen, das sich völlig natürlich dem Geometer darbietet.

Eine Neubearbeitung der erwähnten Schrift erschien uns auch umso erstrebenswerter, als die hier gebotene Theorie keine triviale Verallgemeinerung der Verhältnisse des Dreidimensionalen darstellt, sondern im Gegenteil ganz neue Erkenntnisse vermittelt.

Dies klar hervortreten zu lassen war auch unser Hauptbestreben. Wir gingen daher auch viel genauer ein auf die geometrische Natur aller auftretenden Grössen, als dies in der erwähnten Schrift und in der Darstellung des *Lehrbuchs* geschah.

Es war dabei nicht immer leicht, sich der Lockung eines allzuliebevollen Eingehens in die Details zu entziehen, doch geschah dies im Interesse der Einheitlichkeit der Darstellung.

Was das verwendete Formen-System betrifft, so kann eine solches auch für die F_e eines beliebigen Riemannschen Raumes definiert werden (Schlussparagraph).*

Da es aber in einem solchen Raume den Begriff einer Kongruenz nicht gibt, so hat das so definierte Formen-System keine besonders tiefe Bedeutung.

Aus didaktischen Gründen werden in der vorliegenden Arbeit (wie im *Lehrbuch*) zuerst die Verhältnisse im Euklidischen besprochen. Der Leser braucht dann, sobald rechtwinklige kartesische Koordinaten eingeführt werden, von einem "verallgemeinerten Ricci-Kalkül" nichts zu wissen.

Es genügen gerade jene Kenntnisse des Tensor-Kalküls, die dem feldtheoretisch orientierten mathematischen Physiker heutzutage geläufig sind.

Den R_n konstanter Krümmung erledigen wir im Schlussparagraphen.

* I. A. Schouten und E. R. van Kampen, *Über die Krümmung einer V_m in V_n*, Mathematische Annalen, vol. 105 (1931).

1. Die Schmieg- und Normal-Vektorräume der F_e im euklidischen R_n

Wie in der Einleitung erwähnt, benutzen wir in unserer Darstellung die sogenannten rechtwinklig-kartesischen Koordinaten zur Beschreibung des euklidischen R_n. Die zulässigen Koordinatentransformationen, nämlich die, welche den Masstensor

$$g_{ik} = \delta_{ik} = \begin{cases} 0, & i \neq k, \\ 1, & i = k, \end{cases} \tag{1}$$

numerisch invariant lassen, nennt man ortogonale Transformationen. Als Punkt-Transformationen aufgefasst, stellen sie die Kongruenz-Transformationen des euklidischen R_n dar.

Die F_e liege in der Parameterdarstellung vor

$$x_i = x_i(y_1, \cdots, y_e) \qquad (i = 1, \cdots, n). \tag{2}$$

Was Stetigkeit und Differenzierbarkeit der in der Folge auftretenden Funktionen betrifft, so setzen wir sie voraus, soweit unser Problem es erheischt. Es wäre ja von gar keinem Nutzen, den Gedankengang jedesmal zu unterbrechen, um diese differentialgeometrisch unwesentlichen Voraussetzungen an jeder Stelle genau zu fixieren.

Wir betrachten nun den beliebigen Punkt P der F_e; in ihm sind durch die e Raum-Vektoren*

$$\frac{\partial x_i}{\partial y_p} \qquad (p = 1, \cdots, l) \tag{3}$$

ein l-Bein aufgespannt: *der erste Schmieg-Vektorraum oder Tangential-Vektorraum I_1 der F_e.*

Der Tangential-Vektorraum I_1 hängt von der Wahl der Flächenparameter $(y_1, \cdots, y_e)$ nicht ab. Ist nämlich

$$\begin{aligned} \bar{y}_p &= \bar{y}_n(y_1, \cdots, y_e), \\ y_p &= y_p(\bar{y}_1, \cdots, \bar{y}_e) \end{aligned} \tag{4}$$

---

* Wenn wir von einer geometrischen Grösse als Raum-Tensor sprechen, so meinen wir den Transformationscharakter bei Veränderung der Raumkoordinaten allein (also bei Fixierung der Flächenparameter $y_1, \cdots, y_e$).

Ebenso wollen wir von einer Grösse als Flächen-Tensor sprechen, sobald wir ihren Transformationscharakter bei Veränderung der Flächenparameter allein beschreiben.

So werden wir ein und dieselbe Grösse je nach Notwendigkeit einmal als Raum-Tensor und einmal als Flächen-Tensor ansprechen.

Wenn der gemeinte Tensorcharakter aber ohne weiteres einzusehen ist, werden wir die nähere Angabe (Raum- resp. Flächentensor) späterhin unterlassen. (Also besonders bei jenen Grössen, die nur Raum- oder nur Flächenindizes enthalten.)

eine Parametertransformation, so folgt aus

$$\frac{\partial x_i}{\partial \bar{y}_p} = \frac{\partial x_i}{\partial y_r}\frac{\partial y_r}{\partial \bar{y}_p}, \qquad \frac{\partial x_i}{\partial y_p} = \frac{\partial x_i}{\partial \bar{y}_r}\frac{\partial \bar{y}_r}{\partial y_p} \tag{5}$$

sofort die Identität der Vektorräume

$$\left\{\frac{\partial x_i}{\partial y_p}\right\} \quad \text{und} \quad \left\{\frac{\partial x_i}{\partial \bar{y}_p}\right\}. \tag{6}$$

Als Flächengrössen aufgefasst, d.h. bei Transformation der y, sind die $\partial x_i/\partial y_p$ $(i=1, \cdots, n)$ ein System von n kovarianten (Flächen) Vektoren. Der zweite Schmieg-Vektorraum I_{12} im Punkte P der F_s ist definiert durch Gesamtheit der Raum-Vektoren

$$\left\{\frac{\partial x_i}{\partial y_p}, \frac{\partial^2 x_i}{\partial y_p \partial y_q}\right\}. \tag{7}$$

Wie der Tangential-Vektorraum I_1 ist auch dieser Vektorraum von der Wahl der Flächenparameter unabhängig.

In der Tat gilt ja neben (5) das durch Differentiation von (5) abgeleitete System

$$\begin{aligned} \frac{\partial^2 x_i}{\partial \bar{y}_p \partial \bar{y}_q} &= \frac{\partial x_i}{\partial y_r}\frac{\partial^2 y_r}{\partial \bar{y}_p \partial \bar{y}_q} + \frac{\partial^2 x_i}{\partial y_r \partial y_s}\frac{\partial y_r}{\partial \bar{y}_p}\frac{\partial y_s}{\partial \bar{y}_q}, \\ \frac{\partial^2 x_i}{\partial y_p \partial y_q} &= \frac{\partial x_i}{\partial \bar{y}_r}\frac{\partial^2 \bar{y}_r}{\partial y_p \partial y_q} + \frac{\partial^2 x_i}{\partial \bar{y}_r \partial \bar{y}_s}\frac{\partial \bar{y}_r}{\partial y_p}\frac{\partial \bar{y}_s}{\partial y_q}. \end{aligned} \tag{8}$$

Aber aus (5) und (8) folgt die Invarianz des I_{12} gegen Parametertransformationen.

Die Relationen (5) und (8) sind nur Spezialfälle einer durch Rekursion herzuleitenden allgemeinen Formel

$$\begin{aligned} \frac{\partial^h x_i}{\partial \bar{y}_{p_1} \cdots \partial \bar{y}_{p_h}} &= \frac{\partial^h x_i}{\partial y_{r_1} \cdots \partial y_{r_h}}\frac{\partial y_{r_1}}{\partial \bar{y}_{p_1}} \cdots \frac{\partial y_{r_h}}{\partial \bar{y}_{p_h}} + \sum_{k=1}^{h-1} \frac{\partial^k x_i}{\partial y_{r_1} \cdots \partial y_{r_k}}(\cdots), \\ \frac{\partial^h x_i}{\partial y_{p_1} \cdots \partial y_{p_h}} &= \frac{\partial^h x_i}{\partial \bar{y}_{r_1} \cdots \partial \bar{y}_{r_h}}\frac{\partial \bar{y}_{r_1}}{\partial y_{p_1}} \cdots \frac{\partial \bar{y}_{r_h}}{\partial y_{p_h}} + \sum_{k=1}^{h-1} \frac{\partial^k x_i}{\partial \bar{y}_{r_1} \cdots \partial \bar{y}_{r_k}}(\cdots). \end{aligned} \tag{9}$$

Definieren wir als *kten Schmieg-Vektorraum* $I_{12\cdots k}$ den Vektorraum

$$\left\{\frac{\partial x_i}{\partial y_{p_1}}, \frac{\partial^2 x_i}{\partial y_{p_1} \partial y_{p_2}}, \cdots, \frac{\partial^k x_i}{\partial y_{p_1} \cdots \partial y_{p_k}}\right\} \tag{10}$$

so folgt aus (9) für $h=1, 2, \cdots k$ die Invarianz des $I_{12\cdots k}$ gegenüber Parametertransformation.

Nachdem wir so die (invarianten) Schmieg-Vektorräume im Punkt P der F_e definiert haben, kommen wir zu weiteren (invarianten) Vektorräumen, den *Normal-Vektorräumen der* F_e.

Die Schmieg-Vektorräume sind so definiert, dass der $I_{12\cdots k}$ den $I_{12\cdots k-1}$ enthält.

Die Gesamtheit der Vektoren des $I_{12\cdots k}$ die auf den $I_{12\cdots k-1}$ normalstehen, bildet ebenfalls einen linearen Vektorraum, den wir den Normal-Vektorraum I_k der F_e nennen.

So ist der I_2 definiert als der grösste Unter-Vektorraum des I_{12}, der auf den Tangential-Vektorraum I_1 normal steht u. s. w.

Wir führen jetzt die folgende Bezeichnung ein:

Die Projektion irgend eines im Punkte P der F_e definierten Raum-Vektors λ_i $(i=1, \cdots, n)$ in den Vektorraum $I_{12\cdots k}$ (resp. I_k) bezeichnen wir $\underline{\lambda_{i_{12\cdots k}}}$ (resp. $\underline{\lambda_{i_k}}$).

Den Raum-Vektor

$$\underline{\frac{\partial^k x_i}{\partial y_{p_1}\cdots\partial y_{p_k}}}_{k}$$

aber werden wir in der Folge stets ohne das am Querstrich angehängte k, also

$$\underline{\frac{\partial^k x_i}{\partial y_{p_1}\cdots\partial y_{p_k}}}$$

schreiben.

Betrachten wir jetzt die Relation (9) als eine zwischen den Raum-Vektoren, die in ihr auftreten, so ergibt die Projektion in den I_h-Vektorraum, da dieser zum $I_{12\cdots h-1}$ normal steht:

$$\text{(11)}\qquad \begin{aligned} \underline{\frac{\partial^h x_i}{\partial \bar{y}_{p_1}\cdots\partial \bar{y}_{p_h}}} &= \underline{\frac{\partial^h x_i}{\partial y_{r_1}\cdots\partial y_{r_h}}}\,\frac{\partial y_{r_1}}{\partial \bar{y}_{p_1}}\cdots\frac{\partial y_{r_h}}{\partial \bar{y}_{p_h}},\\ \underline{\frac{\partial^h x_i}{\partial y_{p_1}\cdots\partial y_{p_h}}} &= \underline{\frac{\partial^h x_i}{\partial \bar{y}_{r_1}\cdots\partial \bar{y}_{r_h}}}\,\frac{\partial \bar{y}_{r_1}}{\partial y_{p_1}}\cdots\frac{\partial \bar{y}_{r_h}}{\partial y_{p_h}}. \end{aligned}$$

Die Raum-Vektoren

$$\text{(12)}\qquad \underline{\frac{\partial^h x_i}{\partial y_{p_1}\cdots\partial y_{p_h}}}$$

die ersichtlich den I_k-Vektorraum aufspannen, verhalten sich also in bezug auf die Transformation der y wie ein System von n $(i=1, \cdots, n)$ *symmetrischen kovarianten Flächentensoren hter Stufe.*

(Die oben gegebene Konstruktion der Normal-Vektorräume der F_e,

durchgeführt für die F_1 (Kurve), führt natürlich auf ihre Normal-Vektoren.) Bei der Konstruktion der $I_{12\cdots k}$-Schmieg-Vektorräume werden wir einmal zu einem $I_{12\cdots m}$ gelangen, der Eigenschaft

$$
(13) \qquad \begin{aligned} I_{12\cdots m} &\neq I_{12\cdots m-1}, \qquad \text{aber} \\ I_{12\cdots m+1} &\equiv I_{12\cdots m}. \end{aligned}
$$

Wir nennen dann $I_{12\cdots m}$ *den "letzten" oder "grössten" Schmieg-Vektorraum der* F_e. In der Tat folgt aus (13)

$$
(14) \qquad I_{12\cdots m+2} \equiv I_{12\cdots m+1} \equiv I_{12\cdots m} \qquad \text{u. s. w.}
$$

Ist $I_{12\cdots m}$ der letzte Schmieg-Vektorraum, so folgt aus der zweiten Gleichung (13)

$$
(15) \qquad I_{m+1} \equiv 0,
$$

d. h. der I_{m+1} ist leer, er existiert nicht. Ebenso folgt aus (14) $I_{m+2} \equiv 0$ u. s. w.

Wir können statt (15) auch schreiben

$$
(16) \qquad \underline{\frac{\partial^{m+1} x_i}{\partial y_{p_1} \cdots \partial y_{p_{m+1}}}} \equiv 0
$$

und in der Folge

$$
(17) \qquad \underline{\frac{\partial^{N} x_i}{\partial y_{p_1} \cdots \partial y_{p_N}}} \equiv 0 \qquad \text{für } N > m.
$$

Das Raum-Vektorbein

$$
(18) \qquad \frac{\partial x_i}{\partial y_{p_1}}, \quad \underline{\frac{\partial^2 x_i}{\partial y_{p_1} \partial y_{p_2}}}, \cdots, \underline{\frac{\partial^m x_i}{\partial y_{p_1} \cdots \partial y_{p_m}}}
$$

nennen wir kurz eine *Basis der Schmieg-Vektorräume der* F_e. In Bezug auf die Transformationen der Parameter stellt die Basis ein System von kovarianten und symmetrischen Flächentensoren erster bis $m$ter Stufe dar.

2. Das System der Grundformen der F_e; die Masstensoren der I_h-Räume

Ein Raum-Vektor, der ganz im I_h liegt, hat als Darstellung

$$
(1) \qquad \lambda_i = \underline{\frac{\partial^h x_i}{\partial y_{p_1} \cdots \partial y_{p_h}}}\, l^{p_1 \cdots p_h}.
$$

Die Darstellungsgrössen $l^{p_1\cdots p_h}$, die wir symmetrisch in allen Indizes wählen, bilden ihrem Transformationscharakter nach einen *symmetrischen kontravarianten Flächentensor hter Stufe.*

Ein solcher Tensor hat

$$L_h = \binom{e+h-1}{h}$$

verschiedene Komponenten.

Da im allgemeinen die den I_h aufspannenden Raumvektoren

$$\frac{\partial^h x_i}{\partial y_{p_1} \cdots \partial y_{p_h}}$$

nicht linear unabhängig sein werden, hat auch der Nullvektor ($\lambda_i = 0$) Darstellungen

$$(2) \qquad 0 = \frac{\partial^h x_i}{\partial y_{p_1} \cdots \partial y_{p_h}} \theta^{p_1 \cdots p_h}$$

mit $\theta^{p_1 \cdots p_h}$, die nicht alle verschwinden. Die Anzahl linear unabhängiger Lösungen von (2), d_h, ist gegeben durch

$$(3) \qquad d_h = L_h - l_h,$$

wenn l_h die Dimension des I_h ist, d. h. der Rang der Matrix

$$\left\| \frac{\partial^h x_i}{\partial y_{p_1} \cdots \partial y_{p_h}} \right\|.$$

In der Darstellung (1) des Vektors λ_i des I_h sind die $l^{p_1 \cdots p_h}$ bis auf eine additive *Null Lösung* $\theta^{p_1 \cdots p_h}$ von (2) bestimmt.

Für die Länge $(\lambda_i \lambda_i)^{1/2}$ des Vektors λ_i gibt (1)

$$(4) \qquad \begin{aligned} \lambda_i \lambda_i &= \frac{\partial^h x_i}{\partial y_{p_1} \cdots \partial y_{p_h}} \frac{\partial^h x_i}{\partial y_{q_1} \cdots \partial y_{q_h}} l^{p_1 \cdots p_h} l^{q_1 \cdots q_h} \\ &= E_{p_1 \cdots p_h | q_1 \cdots q_h} l^{p_1 \cdots p_h} l^{q_1 \cdots q_h}, \end{aligned}$$

wo der kovariante Flächentensor

$$(5) \qquad E_{p_1 \cdots p_h | q_1 \cdots q_h} = \frac{\partial^h x_i}{\partial y_{p_1} \cdots \partial y_{p_h}} \frac{\partial^h x_i}{\partial y_{q_1} \cdots \partial y_{q_h}}$$

nach (4) der Masstensor für die durch die $l^{p_1 \cdots p_h}$ dargestellten Vektoren des I_h ist. Der Masstensor (5) enthält nur Flächen-Indizes mehr, er ist in den

(durch den Querstrich getrennten) beiden Indizes-h-Tupel symmetrisch und ausserdem symmetrisch in bezug auf die Indizes eines jeden der beiden h-Tupel. Da er zudem als inneres Produkt zweier Raum-Vektoren bei ortogonalen (Kongruenz) Transformationen invariant bleibt, so folgt, *dass kongruente Flächen F_e äquivalente Masstensoren der I_h Räume haben* (d. h. bei geeigneter Wahl der Parameter *gleiche* Masstensoren).

Multiplizieren wir (2) mit

$$\underline{\frac{\partial^h x_i}{\partial y_{q_1} \cdots \partial y_{q_h}}}$$

so gewinnen wir

$$0 = E_{q_1 \cdots q_h | p_1 \cdots p_h} \theta^{p_1 \cdots p_h}. \tag{6}$$

Somit ist jede Null Lösung $\theta^{p_1 \cdots p_h}$ von (2) eine Lösung von (6). Ist dann umgekehrt der Tensor $\theta^{p_1 \cdots p_h}$ eine Lösung von (6), so folgt nach Multiplikation mit $\theta^{q_1 \cdots q_h}$ nach (5)

$$\begin{aligned} 0 &= \left(\underline{\frac{\partial^h x_i}{\partial y_{q_1} \cdots \partial y_{q_h}}} \theta^{q_1 \cdots q_h} \right) \left(\underline{\frac{\partial^h x_i}{\partial y_{p_1} \cdots \partial y_{p_h}}} \theta^{p_1 \cdots p_h} \right) \\ &= \sum_{i=1}^{n} \left(\underline{\frac{\partial^h x_i}{\partial y_{p_1} \cdots \partial y_{p_h}}} \theta^{p_1 \cdots p_h} \right)^2, \end{aligned} \tag{7}$$

d. h. es gilt (2).

Wir haben damit das wichtige Resultat gewonnen:

Die Gleichungen (2) *und* (6) *für die $\theta^{p_1 \cdots p_h}$ haben dieselben Lösungen.*

Das bedeutet aber, dass die Matrizen*

$$\left\| \underline{\frac{\partial^h x_i}{\partial y_{p_1} \cdots \partial y_{p_h}}} \right\| \quad \text{und} \quad \left\| E_{p_1 \cdots p_h | q_1 \cdots q_h} \right\| \tag{8}$$

den gleichen Rang haben: *Die Dimension des I_h-Vektorraums l_h ist also zugleich der "Rang" des Masstensors $E_{p_1 \cdots p_h | q_1 \cdots q_h}$ dieses Vektorraums.* Ist $I_{12 \cdots m}$ der letzte Schmieg-Vektorraum, so ist das vollständige System der Masstensoren der F_e das System der m Tensoren

$$E_{p_1 | q_1}, \; E_{p_1 p_2 | q_1 q_2}, \; \cdots, \; E_{p_1 \cdots p_m | q_1 \cdots q_m}. \tag{9}$$

Der symmetrische Masstensor des I_1

* In der Matrix $\|E_{p_1 \cdots p_h | q_1 \cdots q_h}\|$ entspricht einem bestimmten p-Tupel eine Zeile und einem bestimmten q-Tupel eine Spalte.

$$E_{p|q} = \frac{\partial x_i}{\partial y_p}\frac{\partial x_i}{\partial y_q} \tag{10}$$

ist der metrische Tensor der F_e (als l-dimensionaler Riemannscher Raum betrachtet).

Wir werden im Folgenden zeigen können, dass die Masstensoren (9) die F_e bis auf ihre Lage im R_n, also bis auf Kongruenz, bestimmen. Aber wir zeigen mehr!

Definieren wir nämlich durch

$$E_{p_1\cdots p_k|p_{k+1}\cdots p_{2k}} l^{p_1}\cdots l^{p_{2k}} = B_{p_1\cdots p_{2k}} l^{p_1}\cdots l^{p_{2k}} \tag{10'}$$

die in allen Indizes $p_1, p_2, \cdots, p_{2k}$ *symmetrischen kovarianten Flächentensoren, die Grundtensoren* $B_{p_1\cdots p_{2k}}$, $k=1, 2, \cdots, m$, *so können wir zeigen, dass bereits durch die Grundtensoren*

$$B_{pq} = E_{p|q},\ B_{p_1\cdots p_4},\ \cdots,\ B_{p_1\cdots p_{2m}} \tag{11}$$

die F_e *bis auf Kongruenz festgelegt ist.*

Der Vergleich der Koeffizienten von $l^{p_1}\cdots l^{p_{2k}}$ in (10) führt auf

$$(2k)!B_{p_1\cdots p_{2k}} = \sum E_{c_1\cdots c_k|c_{k+1}\cdots c_{2k}}, \tag{12}$$

wo in der Summe rechts die $c_1, c_2, \cdots, c_{2k}$ alle $(2k)!$ Permutationen von $p_1p_2\cdots p_{2k}$, also auch gleiche, durchlaufen.*

3. Die Frenet-Gleichungen für die F_e des R_n

Um das in der Einleitung gestellte Problem zu lösen, müssen wir ein System totaler Differential-Gleichungen für die Grössen

$$x_i,\ \frac{\partial x_i}{\partial y_p},\ \frac{\partial^2 x_i}{\partial y_{p_1}\partial y_{p_2}},\ \cdots,\ \frac{\partial^m x_i}{\partial y_{p_1}\cdots\partial y_m} \tag{1}$$

als Funktionen der $y_1, \cdots, y_e$ aufstellen: *die Frenet-Gleichungen der* F_e.

* In der Tat: Enthält $p_1, p_2, \cdots, p_{2k}$ der Reihe nach $a, b, \cdots, f$ gleiche Indizes, so ist

$$\frac{(2k)!}{a!b!\cdots f!}B_{p_1\cdots p_{2k}}$$

der Koeffizient von $l^{p_1}\cdots l^{p_{2k}}$ in (10′) rechts.

Der entsprechende Koeffizient links hat die Form der rechten Seite von (12), wobei aber in der Summe nur die verschiedenen Permutationen von $p_1, p_2, \cdots, p_{2k}$ auftreten. Da aber die rechte Seite diese $(a!\,b!\cdots f!)$-fach enthält, ist

$$\frac{1}{a!b!\cdots f!}\sum E_{c_1\cdots c_k|c_{k+1}\cdots c_{2k}}$$

der Faktor links.

Für x_i gilt

$$dx_i = \frac{\partial x_i}{\partial y_p} dy_p. \tag{2}$$

Wir gehen nun daran, die Differentiale der übrigen Grössen der Reihe (1) zu bestimmen.

Aus der Formel

$$\frac{\partial^k x_i}{\underline{\partial y_{p_1} \cdots \partial y_{p_k}}} = \frac{\partial^k x_i}{\partial y_{p_1} \cdots \partial y_{p_k}} + \text{Vektor des } I_{12\cdots k-1} \tag{3}$$

gewinnen wir

$$\frac{\partial}{\partial y_t}\left(\frac{\partial^k x_i}{\underline{\partial y_{p_1} \cdots \partial y_{p_k}}}\right) = \frac{\partial^{k+1} x_i}{\partial y_{p_1} \cdots \partial y_{p_k} \partial y_t} + \text{Vektor des } I_{12\cdots k}. \tag{4}$$

Somit gibt die Projektion in den I_{k+1}

$$\underline{\frac{\partial}{\partial y_t}\left(\frac{\partial^k x_i}{\underline{\partial y_{p_1} \cdots \partial y_{p_k}}}\right)}_{k+1} = \frac{\partial^{k+1} x_i}{\underline{\partial y_{p_1} \cdots \partial y_{p_k} \partial y_t}}, \tag{5}$$

und in den I_{k+r}, $r=2, 3, \cdots$,

$$\underline{\frac{\partial}{\partial y_t}\left(\frac{\partial^k x_i}{\underline{\partial y_{p_1} \cdots \partial y_{p_k}}}\right)}_{k+r} = 0 \qquad (r = 2, 3, \cdots). \tag{6}$$

Also gilt die Darstellung für den Raum-Vektor

$$\frac{\partial}{\partial y_{p_{k+1}}}\left(\frac{\partial^k x_i}{\underline{\partial y_{p_1} \cdots \partial y_{p_k}}}\right):$$

$$\begin{aligned} \frac{\partial}{\partial y_{p_{k+1}}}\left(\frac{\partial^k x_i}{\underline{\partial y_{p_1} \cdots \partial y_{p_k}}}\right) &= \Gamma^{r_1}_{p_1 \cdots p_k p_{k+1}} \frac{\partial x_i}{\partial y_{r_1}} + \Gamma^{r_1 r_2}_{p_1 \cdots p_k p_{k+1}} \frac{\partial^2 x_i}{\underline{\partial y_{r_1} \partial y_{r_2}}} \\ &+ \cdots + \Gamma^{r_1 \cdots r_k}_{p_1 \cdots p_k p_{k+1}} \frac{\partial^k x_i}{\underline{\partial y_{r_1} \cdots \partial y_{r_k}}} + \frac{\partial^{k+1} x_i}{\underline{\partial y_{p_1} \cdots \partial y_{p_k} \partial y_{p_{k+1}}}}. \end{aligned} \tag{7}$$

Wir zeigen weiter, dass in (7)

$$\Gamma^{r_1 \cdots r_t}_{p_1 \cdots p_k p_{k+1}} \frac{\partial^t x_i}{\underline{\partial y_{r_1} \cdots \partial y_{r_t}}} = 0 \quad \text{ist für } t = 1, 2, \cdots, k-2. \tag{7'}$$

Aus

$$\frac{\partial^k x_i}{\partial y_{p_1}\cdots\partial y_{p_k}}\,\frac{\partial^h x_i}{\partial y_{r_1}\cdots\partial y_{r_h}}=0, \tag{8}$$

für $h=1, 2, \cdots, k-2$ (allgemein für $h\neq k$) folgt durch Differentiation

$$\begin{aligned}&\frac{\partial}{\partial y_{p_{k+1}}}\left(\frac{\partial^k x_i}{\partial y_{p_1}\cdots\partial y_{p_k}}\right)\frac{\partial^h x_i}{\partial y_{r_1}\cdots\partial y_{r_h}}\\ &\quad=-\frac{\partial^k x_i}{\partial y_{p_1}\cdots\partial y_{p_k}}\,\frac{\partial}{\partial y_{p_{k+1}}}\left(\frac{\partial^h x_i}{\partial y_{r_1}\cdots\partial y_{r_h}}\right),\ \text{für } h=1,2,\cdots,k-2.\end{aligned} \tag{8'}$$

Nach (7) ist

$$\frac{\partial}{\partial y_{p_{k+1}}}\left(\frac{\partial^h x_i}{\partial y_{r_1}\cdots\partial y_{r_h}}\right)$$

ein Raum-Vektor des $I_{12\cdots h+1}$, also wegen $h=1, 2, \cdots, k-2$, ein Raum-Vektor des $I_{12\cdots k-1}$.

Dagegen liegt

$$\frac{\partial^k x_i}{\partial y_{p_1}\cdots\partial y_{p_k}}$$

im I_k. Da aber die Vektorräume $I_{12\cdots k-1}$ und I_k normal stehen, ist die rechte Seite (8′) und somit die linke Seite (8′) Null. Multiplizieren wir also (7) mit

$$\frac{\partial^h x_i}{\partial y_{s_1}\cdots\partial y_{s_h}}\qquad (h=1,2,\cdots,k-2),$$

so erhalten wir

$$\Gamma^{r_1\cdots r_h}_{p_1\cdots p_k p_{k+1}}E_{r_1\cdots r_h|s_1\cdots s_h}=0. \tag{9}$$

Aus (9) aber folgt nach §2 die Behauptung (7′).

Wir schreiben somit statt (7)

$$\begin{aligned}\frac{\partial}{\partial y_{p_{k+1}}}\left(\frac{\partial^k x_i}{\partial y_{p_1}\cdots\partial y_{p_k}}\right)&=\Gamma^{r_1\cdots r_{k-1}}_{p_1\cdots p_k p_{k+1}}\frac{\partial^{k-1} x_i}{\partial y_{\ 1}\cdots\partial y_{r_{k-1}}}\\ &\quad+\Gamma^{r_1\cdots r_k}_{p_1\cdots p_k p_{k+1}}\frac{\partial^k x_i}{\partial y_{r_1}\cdots\partial y_{r_k}}+\frac{\partial^{k+1}x_i}{\partial y_{p_1}\cdots\partial y_{p_k}\partial y_{p_{k+1}}}.\end{aligned} \tag{10}$$

Für $k=m$ tritt noch hinzu §1 (16)

$$\frac{\partial^{m+1} x_i}{\partial y_{p_1} \cdots \partial y_{p_{m+1}}} = 0. \tag{10'}$$

Wir können nun das System der Frenet-Gleichungen der F_e anschreiben:

$$\begin{aligned}
dx_i &= \frac{\partial x_i}{\partial y_p} dy_p, \\
d\left(\frac{\partial x_i}{\partial y_p}\right) &= \left(\Gamma^r_{pq} \frac{\partial x_i}{\partial y_r} + \frac{\partial^2 x_i}{\partial y_p \partial y_q}\right) dy_q, \\
&\vdots \\
d\left(\frac{\partial^k x_i}{\partial y_{p_1} \cdots \partial y_{p_k}}\right) &= \left(\Gamma^{r_1 \cdots r_{k-1}}_{p_1 \cdots p_k p_{k+1}} \frac{\partial^{k-1} x_i}{\partial y_{r_1} \cdots \partial y_{r_{k-1}}}\right. \\
&\quad \left. + \Gamma^{r_1 \cdots r_k}_{p_1 \cdots p_k p_{k+1}} \frac{\partial^k x_i}{\partial y_{r_1} \cdots \partial y_{r_k}} + \frac{\partial^{k+1} x_i}{\partial y_{p_1} \cdots \partial y_{p_k} \partial y_{p_{k+1}}}\right) dy_{p_{k+1}}, \\
&\vdots \\
d\left(\frac{\partial^m x_i}{\partial y_{p_1} \cdots \partial y_{p_m}}\right) &= \left(\Gamma^{r_1 \cdots r_{m-1}}_{p_1 \cdots p_m p_{m+1}} \frac{\partial^{m-1} x_i}{\partial y_{r_1} \cdots \partial y_{r_{m-1}}}\right. \\
&\quad \left. + \Gamma^{r_1 \cdots r_m}_{p_1 \cdots p_m p_{m+1}} \frac{\partial^m x_i}{\partial y_{r_1} \cdots \partial y_{r_m}}\right) dy_{p_{m+1}}.
\end{aligned} \tag{11}$$

Die in (11) auftretenden Koeffizienten

$$\Gamma^{r_1 \cdots r_{k-1}}_{p_1 \cdots p_k p_{k+1}}, \qquad \Gamma^{r_1 \cdots r_k}_{p_1 \cdots p_k p_{k+}}, \tag{12}$$

seien in den oberen Indizes symmetrisch angenommen; sie sind bis auf Null Lösungen $\theta^{r_1 \cdots r_{k-1}}$ resp. $\theta^{r_1 \cdots r_k}$ von

$$\frac{\partial^{k-1} x_i}{\partial y_{r_k} \cdots \partial y_{r_{k-1}}} \theta^{r_1 \cdots r_{k-1}} = 0 \quad \text{resp.} \quad \frac{\partial^k x_i}{\partial y_{r_1} \cdots \partial y_{r_k}} \theta^{r_1 \cdots r_k} = 0$$

fixiert.

Ausser der (*angenommenen*) *Symmetrie in den oberen Indizes folgt für die Grössen* (12) *die Symmetrie in den ersten k unteren Indizes aus* (11). Was nun den Charakter der Grössen (12) in bezug auf Parametertransformation betrifft (Raumtransformationen lassen sie invariant), untersuchen wir einen längs eines Kurvenstücks der F_e definierten Raum-Vektor λ_i des I_h

$$\lambda_i = \frac{\partial^k x_i}{\partial y_{p_1} \cdots \partial y_{p_k}} l^{p_1 \cdots p_k}. \tag{13}$$

Bilden wir $d\lambda_i$, so gewinnen wir nach (11)

$$\begin{aligned} d\lambda_i = {} & \frac{\partial^{k-1} x_i}{\partial y_{r_1} \cdots \partial y_{r_{k-1}}} \Gamma^{r_1 \cdots r_{k-1}}_{p_1 \cdots p_k p_{k+1}} l^{p_1 \cdots p_k} dy_{p_{k+1}} \\ & + \frac{\partial^k x_i}{\partial y_{r_1} \cdots \partial y_{r_k}} (dl^{r_1 \cdots r_k} + \Gamma^{r_1 \cdots r_k}_{p_1 \cdots p_k t} l^{p_1 \cdots p_k} dy_t) \\ & + \frac{\partial^{k+1} x_i}{\partial y_{p_1} \cdots \partial y_{p_k} \partial y_{p_{k+1}}} l^{p_1 \cdots p_k} dy_{p_{k+1}}. \end{aligned} \tag{14}$$

Da $d\lambda_i$ invariant in bezug auf eine Parameteränderung ist, und die

$$\frac{\partial^k x_i}{\partial y_{r_1} \cdots \partial y_{r_k}}, \quad l^{p_1 \cdots p_k}$$

symmetrische Tensoren in bezug auf diese Transformationen sind, folgt, dass (bis auf ihre Unbestimmtheit) die

$$\Gamma^{r_1 \cdots r_{k-1}}_{p_1 \cdots p_k p_{k+1}}$$

Tensoren sind, wogegen die

$$\Gamma^{r_1 \cdots r_k}_{p_1 \cdots p_k p_{k+1}}$$

den Charakter von Christoffel-Symbolen haben.

In der Tat ist mit dem symmetrischen Tensor $l^{r_1 \cdots r_k}$ nach (14)

$$Dl^{r_1 \cdots r_k} = dl^{r_1 \cdots r_k} + \Gamma^{r_1 \cdots r_k}_{p_1 \cdots p_k t} l^{p_1 \cdots p_k} dy_t \tag{15}$$

ebenfalls ein symmetrischer Tensor, *den wir das I_k-Differential von $l^{r_1 \cdots r_k}$ nennen wollen.*

Will man die Relation (10) in eine Form bringen, die auch den tensoriellen Charakter der in ihr eintretenden Grössen in bezug auf die Parametertransformationen zur Geltung kommen lässt, hat man zu schreiben

$$\begin{aligned} & \frac{\partial}{\partial y_{p_{k+1}}} \left(\frac{\partial^k x_i}{\partial y_p \cdots \partial y_{p_k}} \right) - \Gamma^{r_1 \cdots r_k}_{p_1 \cdots p_k p_{k+1}} \frac{\partial^k x_i}{\partial y_{r_1} \cdots \partial y_{r_k}} \\ & = \Gamma^{r_1 \cdots r_{k-1}}_{p_1 \cdots p_k p_{k+1}} \frac{\partial^{k-1} x_i}{\partial y_{r_1} \cdots \partial y_{r_{k-1}}} + \frac{\partial^{k+1} x_i}{\partial y_{p_1} \cdots \partial y_{p_k} \partial y_{p_{k+1}}}. \end{aligned} \tag{16}$$

Mit der rechten Seite hat jetzt auch die linke Seite Tensorcharakter sowohl in bezug auf die x- als auch auf die y-Transformationen.

Das Differential des Raumvektors λ_i des I_h liegt nach (14) nicht im I_h mehr. Dagegen gilt für seine Projektion in den I_h (14), (15)

$$
(17) \qquad \underset{\longrightarrow h}{d\lambda_i} = \underline{\frac{\partial^h x_i}{\partial y_{r_1} \cdots \partial y_{r_h}}}\, D l^{r_1 \cdots r_h}.
$$

Das Verschwinden der I_h-Ableitung des Darstellungstensors $l^{r_1 \cdots r_h}$ eines Vektors λ_i des I_h bedeutet also, dass der Raumzuwachs $d\lambda_i$ auf den I_h normal steht. Einen Vektor λ_i des I_h dieser Eigenschaft nennen wir I_h-parallel.

Die I_1-Parallelverschiebung ist die von Levi-Civita.

In derselben Art, in der wir die I_h-Parallelverschiebung definierten, können wir eine Parallelverschiebung für Raum-Vektoren beliebiger Vektorräume definieren. Ist z. B. jetzt λ_i ein Vektor des $I_{12 \cdots h}$, so definiert

$$
(18) \qquad \underset{\longrightarrow 12 \ldots h}{d\lambda_i}
$$

das $I_{12 \cdots h}$-Differential von λ_i. Wir nennen λ_i nun $I_{12 \cdots h}$-parallel, längs einer Kurve der F_e, wenn längs dieser das $I_{12 \cdots h}$-Differential (18) verschwindet, also, wenn der räumliche Zuwachs von λ_i längs dieser Kurve stets normal steht zum $I_{12 \cdots h}$. Die näheren Ausführungen bringen wir im §6.

Zuvor aber seien einige für diese Zwecke notwendigen Formeln hergeleitet. Ist in (13) $\lambda_i = 0$, also $l^{p_1 \cdots p_h} = \theta^{p_1 \cdots p_h}$, so gibt (14) ($\tilde{\theta}$ bedeuten Nulltensoren):

$$
(19) \qquad \Gamma^{r_1 \cdots r_{h-1}}_{p_1 \cdots p_h p_{h+1}} \theta^{p_1 \cdots p_h} = \tilde{\theta}^{r_1 \cdots r_{h-1}}_{(p_{h+1})},
$$

$$
(20) \qquad d\theta^{r_1 \cdots r_h} + \Gamma^{r_1 \cdots r_h}_{p_1 \cdots p_h t} \theta^{p_1 \cdots p_h} dy_t = \tilde{\theta}^{r_1 \cdots r_h}
$$

und

$$
(21) \qquad \underline{\frac{\partial^{h+1} x_i}{\partial y_{p_1} \cdots \partial y_{p_h} \partial y_{p_{h+1}}}}\, \theta^{p_1 \cdots p_h} = 0, \quad \text{resp.} \quad E_{a_1 \cdots a_{h+1} | p_1 \cdots p_h p_{h+1}} \theta^{p_1 \cdots p_h} = 0,
$$

welche Relationen für jede Null Lösung $\theta^{p_1 \cdots p_h}$ gelten von

$$
(22) \qquad \underline{\frac{\partial^h x_i}{\partial y_{p_1} \cdots \partial y_{p_h}}}\, \theta^{p_1 \cdots p_h} = 0.
$$

4. Berechnung der in den Frenet-Gleichungen eintretenden Γ-Koeffizienten und der Masstensoren der I_h-Räume aus den Grundformen der F_e. Beweis der Theoreme

Die innere Orientierung des Basis-Beins des $I_{12\cdots m}$

$$\frac{\partial x_i}{\partial y_{p_1}}, \quad \frac{\partial^2 x_i}{\partial y_{p_1}\partial y_{p_2}}, \quad \cdots, \quad \frac{\partial^m x_i}{\partial y_{p_1}\cdots\partial y_{p_m}} \tag{1}$$

d. h. die Längen und Winkel der Raum-Vektoren (1) ist völlig bestimmt durch das System

$$\begin{aligned} \frac{\partial^h x_i}{\partial y_{p_1}\cdots\partial y_{p_h}} \frac{\partial^k x_i}{\partial y_{q_1}\cdots\partial y_{q_k}} &= 0, \quad \text{für } h \neq k, \\ \frac{\partial^h x_i}{\partial y_{p_1}\cdots\partial y_{p_h}} \frac{\partial^h x_i}{\partial y_{q_1}\cdots\partial y_{q_h}} &= E_{p_1\cdots p_h|q_1\cdots q_h}. \end{aligned} \tag{2}$$

Wenn wir die Koeffizienten Γ der Frenet-Gleichungen und die Masstensoren E aus den Grundformen berechnet haben, so können wir das System der Frenet-Gleichungen als System totaler Differentialgleichungen für die Grössen

$$x_i, \quad \frac{\partial x_i}{\partial y_p}, \quad \frac{\partial^2 x_i}{\partial y_{p_1}\partial y_{p_2}}, \quad \cdots, \quad \frac{\partial^m x_i}{\partial y_{p_1}\cdots\partial y_{p_m}} \tag{1'}$$

betrachten mit dem System (2) als zusätzliche Bedingungen (Nebenbedingungen). Wir haben zwei Probleme zu lösen.

Das erste ist der Beweis des *Kongruenz Satzes*:

Kongruente F_e haben gleiche Grundformen und umgekehrt sind F_e mit gleichen Grundformen kongruent.

Um das zu zeigen, genüget es zu wissen, dass die Koeffizienten der Frenet-Gleichungen wie die in (2) eintretenden E-Grössen durch die Grundformen eindeutig bestimmt sind.*

In der Tat folgt aus der Definition der Grundformen als innere Produkte von Raumvektoren ohne weiteres, dass kongruente F_e dieselben Grundformen haben. Haben aber zwei F_e gleiche Grundformen, und bestimmen diese eindeutig die Γ und E, so haben sie dasselbe System der Frenet-Gleichungen mit Nebenbedingungen (2).

Wir können also durch eine Kongruenztransformation erreichen, dass die

* Die Γ natürlich nur bis auf ihre Unbestimmtheit (Nulltensor), die in das Frenet-System wegen der Nebenbedingung (2) aber nicht mehr eintritt.

eine F_e in eine solche Lage kommt, dass in einem gemeinsamen Punkt der beiden F_e die Basis-Beine (1) zur Deckung kommen.

Wir haben damit zwei Lösungen (die erste F_e und die kongruent verpflanzte zweite) des Frenet-Systems mit gleichen Anfangsbedingungen.

Diese zwei Lösungen müssen daher ganz zusammenfallen (die Stetigkeit der Γ vorausgesetzt).

Damit ist das Kongruenztheorem bewiesen.

Unser zweites Problem ist *zu zeigen, dass es zu gegebenen Grundformen, wenn gewisse (in der Folge abgeleitete) Bedingungsrelationen zwischen den Komponenten dieser Grundformen erfüllt sind, stets F_e dieser Grundformen gibt.*

Um das zu zeigen, muss man aus der Theorie totaler Differentialgleichungen mit Nebenbedingungen folgendes wissen:

Unser System (2) stellt die Nebenbedingungen dar, die dem System (1′) der Lösungen des Frenet-Systems auferlegt sind, damit die integrierte F_e die gegebenen Grundformen hat.

Die sogenannten Integrabilitätsbedingungen des Frenet-Systems stellen wieder Gleichungen zwischen den Grössen (1′) dar, also weitere Bedingungen, die, wenn (2) die einzigen Nebenbedingungen sein sollen, eine Folge von(2) sein müssen.

Unter dem *abgeleiteten System* des Systemes (2) verstehen wir jenes, das durch Differentiation des Systemes (2) unter Verwendung des Frenet-Systems entsteht. Da auch das abgeleitete System die Form von Gleichungen zwischen den Grössen (1′) hat, muss es eine Folge des Systemes (2) sein, wenn dieses das alleinige System der Nebenbedingungen ist.

Aus der Theorie totaler Differentialgleichungen mit Nebenbedingungen aber wissen wir, dass wir, wenn sowohl die Integrabilitätsbedingungen als auch das abgeleitete System der Nebenbedingungen eine Folge der Nebenbedingungen sind, das System für solche Anfangswerte lösen können, die den Nebenbedingungen genügen. Die integrierte Lösung erfüllt dann in ihrem ganzen Geltungsbereich die Nebenbedingungen.

Wir haben also zu zeigen, dass wir die E und Γ so berechnen, dass bei Erfülltsein gewisser Bedingungen zwischen den Komponenten der Grundformen, die Integrabilitätsbedingungen und das abgeleitete System aus (2) *eine Folge von* (2) *sind.*

Wir betrachten zuerst das aus (2) abgeleitete System:

Für $k=h\pm 2,\ h\pm 3,\ \cdots$ u. s. w. erhalten wir aus dem Frenet-System §3 (11) und (2)

$$d\left(\underline{\frac{\partial^h x_i}{\partial y_{p_1}\cdots\partial y_{p_h}}}\ \underline{\frac{\partial^k x_i}{\partial y_{q_1}\cdots\partial y_{q_k}}}\right)=0. \tag{3}$$

Das abgeleitete System dieses Teilsystems von (2) ist also von selbst eine Folge von (2).

Dagegen folgt aus

$$
\begin{aligned}
&\frac{\partial}{\partial y_t}\left(\frac{\partial^k x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_k}}}\;\frac{\partial^{k+1} x_i}{\underline{\partial y_{q_1}\cdots\partial y_{q_{k+1}}}}\right)\\
&\qquad = \frac{\partial}{\partial y_t}\left(\frac{\partial^k x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_k}}}\right)\frac{\partial^{k+1} x_i}{\underline{\partial y_{q_1}\cdots\partial y_{q_{k+1}}}}\\
&\qquad\quad + \frac{\partial^k x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_k}}}\;\frac{\partial}{\partial y_t}\left(\frac{\partial^{k+1} x_i}{\underline{\partial y_{q_1}\cdots\partial y_{q_{k+1}}}}\right)
\end{aligned}
\tag{4}
$$

wegen (2) und §3 (11):

$$
0 = E_{p_1\cdots p_k t\,|\,q_1\cdots q_{k+1}} + \Gamma^{r_1\cdots r_k}_{q_1\cdots q_{k+1}t}E_{r_1\cdots r_k\,|\,p_1\cdots p_k}. \tag{5}
$$

Das System (5) ist für eine gegebene F_e erfüllt. Wir werden es zur Berechnung der E and Γ-Grössen zu verwenden haben. Sofern dann die berechneten E und Γ das System nicht *identisch* erfüllen, stellt es *Bedingungsgleichungen für die Grundformen dar*. Ist (5) erfüllt, so ist die aus (2) für $k=h-1$ abgeleitete Gleichung eine Folge von (2).

Wir haben jetzt noch das aus der zweiten Relation des Systemes (2) abgeleitete System zu betrachten. Wir erhalten aus

$$
\begin{aligned}
\frac{\partial}{\partial y_t}E_{p_1\cdots p_k\,|\,q_1\cdots q_k} &= \frac{\partial}{\partial y_t}\left(\frac{\partial^k x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_k}}}\right)\frac{\partial^k x_i}{\underline{\partial y_{q_1}\cdots\partial y_{q_k}}}\\
&\quad + \frac{\partial^k x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_k}}}\;\frac{\partial}{\partial y_t}\left(\frac{\partial^k x_i}{\underline{\partial y_{q_1}\cdots\partial y_{q_k}}}\right)
\end{aligned}
\tag{6}
$$

wegen (2) und §3 (11)

$$
\frac{\partial}{\partial y_t}E_{p_1\cdots p_k\,|\,q_1\cdots q_k} - \Gamma^{r_1\cdots r_k}_{p_1\cdots p_k t}E_{r_1\cdots r_k\,|\,q_1\cdots q_k} - \Gamma^{r_1\cdots r_k}_{q_1\cdots q_k t}E_{r_1\cdots r_k\,|\,p_1\cdots p_k} = 0. \tag{7}
$$

Diese Gleichung ist für eine gegebene F_e erfüllt. (Sie sagt aus, dass die I_h-Ableitung des Masstensors des I_h verschwindet.)

Wir verwenden (7) wie (5) zur Berechnung der E und Γ und es gilt für (7), was wir für (5) schrieben.

Ist (7) erfüllt, so ist die aus der zweiten Relation (2) abgeleitete Gleichung eine Folge von (2).

Unser Resultat lautet:

Gelten die Relationen (5) *und* (7), *so ist das aus* (2) *abgeleitete System eine Folge von* (2).

Die Gleichungen (5) und (7) genügen allein noch nicht zur Berechnung der E und Γ, wohl aber zusammen mit den Integrabilitätsbedingungen des Frenet-Systems.

Die Berechnung geschieht schrittweise, ausgehend von der Integrabilitätsbedingung der ersten der Frenet-Gleichungen §3 (11).

Diese Integrabilitätsbedingung wird unter Verwendung des Frenet-Systems aus

$$\frac{\partial}{\partial y_q}\left(\frac{\partial x_i}{\partial y_p}\right) - \frac{\partial}{\partial y_p}\left(\frac{\partial x_i}{\partial y_q}\right) = 0 \tag{8}$$

gewonnen. Sie lautet

$$\Gamma^r_{pq}\frac{\partial x_i}{\partial y_r} + \frac{\partial^2 x_i}{\underline{\partial y_p \partial y_q}} = \Gamma^r_{qp}\frac{\partial x_i}{\partial y_r} + \frac{\partial^2 x_i}{\underline{\partial y_q \partial y_p}}, \tag{8'}$$

und zerfällt als Folge von (2) in

$$(\Gamma^r_{pq} - \Gamma^r_{qp})\frac{\partial x_i}{\partial y_r} = 0 \tag{9}$$

und

$$\frac{\partial^2 x_i}{\underline{\partial y_p \partial y_q}} - \frac{\partial^2 x_i}{\underline{\partial y_q \partial y_p}} = 0. \tag{9'}$$

Die Relation (9′) ist als Folge von (2) erfüllt. In der Tat ist ihre linke Seite, multipliziert mit $\partial^2 x_i/\underline{\partial y_r \partial y_s}$, wegen (2) und der Symmetrie Eigenschaft der $E_{pq|rs}$ Null. Also verschwindet auch der absolute Betrag des Raum-Vektors der linken Seite von (9′) und somit dieser Vektor selbst als Folge von (2).

Multipliziert man (9) mit $\partial x_i/\partial y_s$ so gibt das nach (2)

$$(\Gamma^r_{pq} - \Gamma^r_{qp})E_{r|s} = (\Gamma^r_{pq} - \Gamma^r_{qp})B_{rs} = 0. \tag{9''}$$

Ist (9″) erfüllt, dann gilt (§2) (9). *Also ist* (9) *eine Folge von* (2), *sobald* (9″) *gilt. Von der ersten Grundform* B_{rs}, *der Massform der* F_s *wird* $|B_{pq}| \neq 0$ *vorausgesetzt.**

Dann ist (9″) äquivalent

$$\Gamma^r_{pq} = \Gamma^r_{qp}. \tag{9'''}$$

* Eine Voraussetzung, die für die gegebene F_s natürlich erfüllt ist.

Diese Relation zusammen mit (7) für $k=1$ gestattet aber (bekanntlich) die Berechnung der Γ^r_{pq}.

In der Tat lautet (7) für $k=1$:

$$\partial E_{p|q}/\partial y_t = \Gamma^r_{pt}E_{r|q} + \Gamma^r_{qt}E_{r|p}. \tag{10}$$

Setzen wir

$$\partial E_{p|q}/\partial y_t = (pqt)$$

so gilt (9‴)

$$\begin{aligned}(pqt) - (tpq) + (qtp) &= (\Gamma^r_{pt} + \Gamma^r_{tp})E_{r|q} + (\Gamma^r_{qt} - \Gamma^r_{tq})E_{r|p} + (\Gamma^r_{pq} - \Gamma^r_{qp})E_{r|t}\\ &= 2\Gamma^r_{pt}E_{r|q}.\end{aligned} \tag{11}$$

Wegen $|E_{r|q}| \neq 0$ aber erhalten wir aus (11) die Γ^r_{pt} *als die bekannten Christoffelgrössen* für den Masstensor

$$B_{pq} = E_{p|q}. \tag{12}$$

Wir haben damit als Resultat:

Aus der Integrabilitätsbedingung der ersten Gleichung des Frenet-Systems und (7) *für* $k=1$ *konnten wir die in der zweiten Gleichung des Frenet-Systems auftretenden* Γ *als Funktion der* $B_{pq}=E_{p|q}$ *allein berechnen.* (Da $|B_{pq}| \neq 0$ angenommen wurde, ist $\|\partial x_i/\partial y_t\|$ vom Range l (§2 (8).) *Wir konstatieren nochmals, dass die bei dieser Berechnung benutzte Integrabilitätsbedingung eine Folge des Systems* (2) *ist.*

Wir könnten bereits hier den allgemeinen (Rekursions)-Schluss durchführen, wollen aber des besseren Verständnis wegen vorerst noch den zweiten Schritt in unserer Schlussfolge tun, d. h. die Integrabilitätsbedingung der zweiten Gleichung des Frenet-Systems zur Berechnung des Masstensors $E_{ab|cd}$ des I_2 und der Γ Koeffizienten der dritten Gleichung des Frenet-Systems heranziehen.

Wir haben also aus §3 (11)

$$\frac{\partial}{\partial y_t}\left[\frac{\partial}{\partial y_q}\left(\frac{\partial x_i}{\partial y_p}\right)\right] - \frac{\partial}{\partial y_q}\left[\frac{\partial}{\partial y_t}\left(\frac{\partial x_i}{\partial y_p}\right)\right] = 0 \tag{13}$$

zu berechnen. Wir erhalten für den ersten Term in (13)

$$\begin{aligned}&\frac{\partial\Gamma^r_{pq}}{\partial y_t}\frac{\partial x_i}{\partial y_r} + \Gamma^r_{pq}\left(\Gamma^s_{rt}\frac{\partial x_i}{\partial y_s} + \underline{\frac{\partial^2 x_i}{\partial y_r\partial y_t}}\right) + \Gamma^r_{pqt}\frac{\partial x_i}{\partial y_r}\\ &\quad + \Gamma^{rs}_{pqt}\underline{\frac{\partial^2 x_i}{\partial y_r\partial y_s}} + \underline{\frac{\partial^3 x_i}{\partial y_p\partial y_q\partial y_t}}.\end{aligned} \tag{14}$$

Bilden wir (13), so zerfällt dieses System wegen (2) in

$$\left(\frac{\partial}{\partial y_t}\Gamma^{r}_{pq} - \frac{\partial}{\partial y_q}\Gamma^{r}_{pt} + \Gamma^{s}_{pq}\Gamma^{r}_{st} - \Gamma^{s}_{pt}\Gamma^{r}_{sq} + \Gamma^{r}_{pqt} - \Gamma^{r}_{ptq}\right)\frac{\partial x_i}{\partial y_r} = 0, \tag{15}$$

$$(\Gamma^{r}_{pq}\delta^{s}_{t} - \Gamma^{r}_{pt}\delta^{s}_{q} + \Gamma^{rs}_{pqt} - \Gamma^{rs}_{ptq})\frac{\partial^2 x_i}{\underline{\partial y_r \partial y_s}} = 0, \tag{16}$$

und

$$\frac{\partial^3 x_i}{\underline{\partial y_p \partial y_q \partial y_t}} - \frac{\partial^3 x_i}{\underline{\partial y_p \partial y_t \partial y_q}} = 0. \tag{17}$$

Die Relation (17) *ist eine Folge von* (2). Der Beweis ist analog dem für die entsprechende Behauptung (9′) betreffend.

Multipliziert man (15) mit $\partial x_i/\partial y_s$, so folgt wegen (5) für $k=1$

$$R^{r}{}_{ptq}E_{r|s} = E_{st|pq} - E_{sq|pt}, \tag{18}$$

wo die uns bereits bekannte linke Seite den ersten (Riemannschen) Krümmungstensor der F_e darstellt.

Die Relation (18) und die §2 (12) für $k=2$

$$\begin{aligned} 3B_{pqrs} &= E_{pq|rs} + E_{pr|qs} + E_{ps|rq} \\ &= 3E_{pq|rs} + (E_{pr|qs} - E_{pq|rs}) + (E_{ps|rq} - E_{pq|rs}) \end{aligned} \tag{19}$$

gestatten die Berechnung von $E_{pq|rs}$ durch die Komponenten der zwei ersten Grundformen B_{pq}, B_{pqrs}.

Haben wir aus (18) und (19) die $E_{pq|rs}$ gewonnen, so liefert die Einsetzung dieser Grössen in (18), (19) entweder Identitäten allein oder auch Bedingungsgleichungen für die Grundformen. (Eine nähere Untersuchung dieser Verhältnisse zeigt aber, dass diese Relationen identisch erfüllt sind, also keine Bedingungsgleichungen liefern.) Da wir von den Relationen (15) durch Multiplikation mit $\partial x_i/\partial y_s$ zu (18) (bei Verwendung von (2)) gelangten, *so ist* (15) *eine Folge von* (2), *sobald* (18) *erfüllt ist.**

Zur Berechnung der Γ^{rs}_{pqt} verwenden wir (16) und (7) für $k=2$. Die letztere Gleichung lautet

$$\frac{\partial E_{pq|rs}}{\partial y_t} = \Gamma^{ab}_{pqt}E_{ab|rs} + \Gamma^{ab}_{rst}E_{ab|pq}. \tag{20}$$

---

* Wir erinnern, dass

$$\frac{\partial^r x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_r}}}\theta^{p_1\cdots p_r} = 0 \quad \text{und} \quad \frac{\partial^r x_i}{\underline{\partial y_{q_1}\cdots\partial y_{q_r}}}\,\frac{\partial^r x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_r}}}\theta^{p_1\cdots p_r} = 0$$

dieselben Lösungen $\theta^{p_1\cdots p_r}$ haben.

Schreiben wir

$$\frac{\partial E_{pq|rs}}{\partial y_t} = (pqrst), \tag{21}$$

so benutzen wir zur Berechnung das folgende Teil-System von (20):

$$\begin{aligned} &(pqrst) - (stpqr) + (qrstp) - (tpqrs) + (rstpq) \\ &= (\Gamma^{ab}_{pqt} + \Gamma^{ab}_{tpq})E_{ab|rs} + (\Gamma^{ab}_{rst} - \Gamma^{ab}_{str})E_{ab|pq} + (\Gamma^{ab}_{qrp} - \Gamma^{ab}_{pqr})E_{ab|st} \\ &\quad + (\Gamma^{ab}_{stp} - \Gamma^{ab}_{tps})E_{ab|qr} + (\Gamma^{ab}_{rsq} - \Gamma^{ab}_{qrs})E_{ab|tp}. \end{aligned} \tag{22}$$

Die Multiplikation von (16) mit

$$\underline{\frac{\partial^2 x_i}{\partial y_a \partial y_b}} \tag{23}$$

führt wegen (2) auf

$$E_{ab|rs}(\Gamma^{r}_{pq}\delta^{s}_{t} - \Gamma^{r}_{pt}\delta^{s}_{q}) = (\Gamma^{rs}_{ptq} - \Gamma^{rs}_{pqt})E_{rs|ab}. \tag{24}$$

Da uns aber die linken Seiten von (22) und (24) bereits bekannt sind, so können wir unter Verwendung der Symmetrieverhältnisse der Γ_{pqt} aus (22) und (24)

$$\Gamma^{ab}_{pqt}E_{ab|rs} \tag{25}$$

als bekannte Grösse in den B_{pq}, B_{pqrs} berechnen.

Aus (25) gewinnen wir endlich (bis auf die notwendige Unbestimmtheit, Nulltensor θ^{ab}) die Γ^{ab}_{pqt} durch die B_{pq}, B_{pqrs} allein ausgedrückt.

Da (24) *aus* (16) *durch Multiplikation mit* (23) *unter Verwendung von* (2) *gewonnen wurde, so ist, wenn* (24) *gilt,* (16) *eine Folge von* (2).

(Auch hier bleibt zu untersuchen, ob die zur Berechnung benutzten Relationen nach Einsetzung der gefundenen Werte identisch gelten, oder als Bedingungsgleichungen für die Grundformen anzusehen sind.)

Zur Berechnung der Γ^{r}_{pqt} verwenden wir wieder das System (5) für $k=1$:

$$E_{pk|qt} + \Gamma^{r}_{qtk}E_{r|p} = 0, \tag{26}$$

und erhalten daraus die Γ^{r}_{qth} in den Grössen B_{pq}, B_{pqrs} ausgedrückt.

Das Resultat unseres zweiten Schritts lautet also:

Aus der Integrabilitätsbedingung der zweiten Gleichung des Systems der Frenet-Gleichungen und den Relationen (7) ($k=2$), (5) ($k=1$) *und* §2 (12) (*für* $k=2$) *konnten wir* $E_{ab|cd}$, *die Massform des* I_2 *und die in der dritten Gleichung des Frenet-System auftretenden* Γ *durch die Grössen* B_{pq}, B_{pqrs} *allein berechnen.*

Die bei der Rechnung benutzte Integrabilitätsbedingung ist dabei eine Folge von (2).

Wir beweisen jetzt das *Haupttheorem*:

Haben wir aus den Komponenten der h ersten Grundformen

$$B_{pq},\ B_{pqrs},\ \cdots,\ B_{p_1\cdots p_{2h}} \tag{27}$$

die Masstensoren der h ersten I_r *Räume,* $r=1, \cdots, h$:

$$E_{p|q},\ E_{pq|rs},\ \cdots,\ E_{p_1\cdots p_h|q_1\cdots q_h} \tag{28}$$

und die Koeffizienten der ($h+1$) *ersten Gleichungen des Frenet-Systems*

$$\Gamma^{r_1}_{pq},\ \Gamma^{r_1}_{pqt},\ \Gamma^{r_1r_2}_{pqt},\ \cdots,\ \Gamma^{r_1\cdots r_{h-1}}_{p_1\cdots p_hp_{h+1}},\ \Gamma^{r_1\cdots r_h}_{p_1\cdots p_hp_{h+1}} \tag{29}$$

berechnet, so können wir aus der Integrabilitätsbedingung der ($h+1$)*ten Gleichung dieses Systems, ferner aus Relation* (7) *für* $k=h+1$, (5) *für* $k=h$ *und* §2 (12) *für* $k=h+1$ *sowohl den Masstensor des* I_{h+1}: $E_{p_1\cdots p_{h+1}|q_1\cdots q_{h+1}}$, *als auch die Koeffizienten*

$$\Gamma^{r_1\cdots r_h}_{p_1\cdots p_{h+1}p_{h+2}},\qquad \Gamma^{r_1\cdots r_{h+1}}_{p_1\cdots p_{h+1}p_{h+2}}$$

der ($h+2$)*ten Gleichung des Frenet-System berechnen und zwar ausgedrückt durch die Grössen* (27) *und* $B_{p_1\cdots p_{2h+2}}$ *allein.*

Der Beweis verläuft ziemlich analog dem des zweiten Schrittes. Wir bilden zuerst

$$\begin{aligned}\frac{\partial}{\partial y_{p_{h+2}}}\left(\frac{\partial}{\partial y_{p_{h+1}}}\frac{\partial^h x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_h}}}\right) &= \frac{\partial}{\partial y_{p_{h+2}}}\Gamma^{r_1\cdots r_{h-1}}_{p_1\cdots p_{h+1}}\cdot\frac{\partial^{h-1}x_i}{\underline{\partial y_{r_1}\cdots\partial y_{r_{h-1}}}} \\ &+ \Gamma^{r_1\cdots r_{h-1}}_{p_1\cdots p_{h+1}}\left(\Gamma^{s_1\cdots s_{h-2}}_{r_1\cdots r_{h-1}p_{h+2}}\frac{\partial^{h-2}x_i}{\underline{\partial y_{s_1}\cdots\partial y_{s_{h-2}}}}\right. \\ &\left.+ \Gamma^{s_1\cdots s_{h-1}}_{r_1\cdots r_{h-1}p_{h+2}}\frac{\partial^{h-1}x_i}{\underline{\partial y_{s_1}\cdots\partial y_{s_{h-1}}}} + \frac{\partial^h x_i}{\underline{\partial y_{r_1}\cdots\partial y_{r_{h-1}}\partial y_{p_{h+2}}}}\right) \\ &+ \frac{\partial}{\partial y_{p_{h+2}}}\Gamma^{r_1\cdots r_h}_{p_1\cdots p_hp_{h+1}}\frac{\partial^h x_i}{\underline{\partial y_{r_1}\cdots\partial y_{r_h}}}\end{aligned} \tag{30}$$

$$+ \Gamma^{r_1\cdots r_h}_{p_1\cdots p_h p_{h+1}}\left(\Gamma^{s_1\cdots s_{h-1}}_{r_1\cdots r_h p_{h+2}} \frac{\partial^{h-1}x_i}{\underline{\partial y_{s_1}\cdots\partial y_{s_h}}} + \Gamma^{s_1\cdots s_h}_{r_1\cdots r_h p_{h+2}} \frac{\partial^h x_i}{\underline{\partial y_{s_1}\cdots\partial y_{s_h}}}\right.$$

$$\left. + \frac{\partial^{h+1}x_i}{\underline{\partial y_{r_1}\cdots\partial y_{r_h}\partial y_{p_{h+2}}}}\right) + \Gamma^{r_1\cdots r_h}_{p_1\cdots p_{h+1}p_{h+2}} \frac{\partial^h x_i}{\underline{\partial y_{r_1}\cdots\partial y_{r_h}}}$$

$$+ \Gamma^{r_1\cdots r_{h+1}}_{p_1\cdots p_{h+1}p_{h+2}} \frac{\partial^{h+1}x_i}{\underline{\partial y_{r_1}\cdots\partial y_{r_{h+1}}}} + \frac{\partial^{h+2}x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_{h+1}}\partial y_{p_{h+2}}}},$$

und erhalten die Integrabilitätsbedingung als Folge von (2) gespalten in

$$(31)\qquad (\Gamma^{r_1\cdots r_{h-1}}_{p_1\cdots p_h p_{h+1}}\Gamma^{s_1\cdots s_{h-2}}_{r_1\cdots r_{h-1}p_{h+2}} - \Gamma^{r_1\cdots r_{h-1}}_{p_1\cdots p_h p_{h+2}}\Gamma^{s_1\cdots s_{h-2}}_{r_1\cdots r_{h-1}p_{h+1}}) \frac{\partial^{h-2}x_i}{\underline{\partial y_{s_1}\cdots\partial y_{s_{h-2}}}} = 0;$$

$$(32)\qquad \begin{aligned} &\left(\frac{\partial\Gamma^{s_1\cdots s_{h-1}}_{p_1\cdots p_h p_{h+1}}}{\partial y_{p_{h+2}}} - \frac{\partial\Gamma^{s_1\cdots s_{h-1}}_{p_1\cdots p_h p_{h+2}}}{\partial y_{p_{h+1}}} + \Gamma^{r_1\cdots r_{h-1}}_{p_1\cdots p_h p_{h+1}}\Gamma^{s_1\cdots s_{h-1}}_{r_1\cdots r_{h-1}p_{h+2}}\right. \\ &\quad - \Gamma^{r_1\cdots r_{h-1}}_{p_1\cdots p_h p_{h+2}}\Gamma^{s_1\cdots s_{h-1}}_{r_1\cdots r_{h-1}p_{h+1}} + \Gamma^{r_1\cdots r_h}_{p_1\cdots p_h p_{h+1}}\Gamma^{s_1\cdots s_{h-1}}_{r_1\cdots r_h p_{h+2}} \\ &\quad \left. - \Gamma^{r_1\cdots r_h}_{p_1\cdots p_h p_{h+2}}\Gamma^{s_1\cdots s_{h-1}}_{r_1\cdots r_h p_{h+1}}\right) \frac{\partial^{h-1}x_i}{\underline{\partial y_{s_1}\cdots\partial y_{s_{h-1}}}} = 0; \end{aligned}$$

$$(33)\qquad \begin{aligned} &\left(\frac{\partial\Gamma^{s_1\cdots s_h}_{p_1\cdots p_h p_{h+1}}}{\partial y_{p_{h+2}}} - \frac{\partial\Gamma^{s_1\cdots s_h}_{p_1\cdots p_h p_{h+2}}}{\partial y_{p_{h+1}}} + \Gamma^{r_1\cdots r_h}_{p_1\cdots p_h p_{h+1}}\Gamma^{s_1\cdots s_h}_{r_1\cdots r_h p_{h+2}}\right. \\ &\quad - \Gamma^{r_1\cdots r_h}_{p_1\cdots p_h p_{h+2}}\Gamma^{s_1\cdots s_h}_{r_1\cdots r_h p_{h+1}} + \Gamma^{s_1\cdots s_{h-1}}_{p_1\cdots p_h p_{h+1}}\delta^{s_h}_{p_{h+2}} - \Gamma^{s_1\cdots s_{h-1}}_{p_1\cdots p_h p_{h+2}}\delta^{s_h}_{p_{h+1}} \\ &\quad \left. + \Gamma^{s_1\cdots s_h}_{p_1\cdots p_h p_{h+1}p_{h+2}} - \Gamma^{s_1\cdots s_h}_{p_1\cdots p_h p_{h+2}p_{h+1}}\right) \frac{\partial^h x_i}{\underline{\partial y_{s_1}\cdots\partial y_{s_h}}} = 0; \end{aligned}$$

$$(34)\qquad \begin{aligned} &\left(\Gamma^{s_1\cdots s_h}_{p_1\cdots p_h p_{h+1}}\delta^{s_{h+1}}_{p_{h+2}} - \Gamma^{s_1\cdots s_h}_{p_1\cdots p_h p_{h+2}}\delta^{s_{h+1}}_{p_{h+1}} + \Gamma^{s_1\cdots s_{h+1}}_{p_1\cdots p_h p_{h+1}p_{h+2}}\right. \\ &\quad \left. - \Gamma^{s_1\cdots s_{h+1}}_{p_1\cdots p_h p_{h+2}p_{h+1}}\right) \frac{\partial^{h+1}x_i}{\underline{\partial y_{s_1}\cdots\partial y_{s_{h+1}}}} = 0; \end{aligned}$$

$$(35)\qquad \frac{\partial^{h+2}x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_h}\partial y_{p_{h+1}}\partial y_{p_{h+2}}}} - \frac{\partial^{h+2}x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_h}\partial y_{p_{h+2}}\partial y_{p_{h+1}}}} = 0.$$

Die Relation (35) ist eine Folge von (2) (vergl. (17) und (9′)). Schreiben wir (31) bis (34) abkürzend

$$(31')\qquad \frac{\partial^{h-2}x_i}{\underline{\partial y_{s_1}\cdots\partial y_{s_{h-2}}}}\,\theta^{s_1\cdots s_{h-2}}_{p_1\cdots p_{h+2}} = 0,$$

$$(32') \qquad \frac{\partial^{h-1} x_i}{\partial y_{s_1} \cdots \partial y_{s_{h-1}}} \theta^{s_1 \cdots s_{h-1}}_{p_1 \cdots p_{h+2}} = 0,$$

$$(33') \qquad \frac{\partial^{h} x_i}{\partial y_{s_1} \cdots \partial y_{s_h}} \theta^{s_1 \cdots s_h}_{p_1 \cdots p_{h+2}} = 0,$$

$$(34') \qquad \frac{\partial^{h+1} x_i}{\partial y_{s_1} \cdots \partial y_{s_{h+1}}} \theta^{s_1 \cdots s_{h+1}}_{p_1 \cdots p_{h+2}} = 0,$$

so erhält man die Gleichungen, die wir zur weiteren Rechnung verwenden durch entsprechende Multiplikation der obigen Relation mit

$$\frac{\partial^k x_i}{\partial y_{a_1} \cdots \partial y_{a_k}}, \qquad k = h - 2, \text{ resp. } h - 1, \text{ resp. } h, \text{ resp. } h + 1,$$

bei Verwendung von (2) in der Gestalt

$$(31'') \qquad E_{a_1 \cdots a_{h-2} | s_1 \cdots s_{h-2}} \theta^{s_1 \cdots s_{h-2}}_{p_1 \cdots p_{h+2}} = 0,$$

$$(32'') \qquad E_{a_1 \cdots a_{h-1} | s_1 \cdots s_{h-1}} \theta^{s_1 \cdots s_{h-1}}_{p_1 \cdots p_{h+2}} = 0,$$

$$(33'') \qquad E_{a_1 \cdots a_h | s_1 \cdots s_h} \theta^{s_1 \cdots s_h}_{p_1 \cdots p_{h+2}} = 0,$$

$$(34'') \qquad E_{a_1 \cdots a_{h+1} | s_1 \cdots s_{h+1}} \theta^{s_1 \cdots s_{h+1}}_{p_1 \cdots p_{h+2}} = 0.$$

Sind (31″) *bis* (34″) *erfüllt, so sind die entsprechenden Integrabilitätsgleichungen* (31) *bis* (34) *eine Folge von* (2).

Die Relationen (31″) und (32″) enthalten nur die bereits berechneten Grössen (28), (29) und stellen also Gleichungen für die Reihe (27) dar, die, wie wir zeigen werden, bereits erledigt sind. Die Relation (33″) wieder hat die Form

$$(35') \qquad E_{a_1 \cdots a_h | s_1 \cdots s_h} (\Gamma^{s_1 \cdots s_h}_{p_1 \cdots p_h p_{h+1} p_{h+2}} - \Gamma^{s_1 \cdots s_h}_{p_1 \cdots p_h p_{h+2} p_{h+1}}) = \text{bekannt in den Grössen der Reihe (27)}.$$

Nach Formel (5), für $k = h$ können wir statt (35) schreiben

$$(35'') \qquad E_{a_1 \cdots a_h p_{h+1} | p_1 \cdots p_h p_{h+2}} - E_{a_1 \cdots a_h p_{h+2} | p_1 \cdots p_h p_{h+1}} = \text{bekannt in den Grössen der Reihe (27)}.$$

Gleichung (12) §2 für $k = h + 1$ lautet

$$(36) \qquad (2h + 2)! B_{p_1 \cdots p_{2h+2}} = \sum E_{c_1 \cdots c_{h+1} | c_{h+2} \cdots c_{2h+2}},$$

wo in der Summe rechts die $c_1, \cdots, c_{2h+2}$ *alle* $(2h+2)!$ Permutationen von $p_1 \cdots p_{2h+2}$ durchlaufen.

Bei Benutzung der Symmetrie-Eigenschaften von $E_{c_1\cdots c_{h+1}|d_1\cdots d_{h+1}}$ kann man aus (35″) und (36) den Masstensor $E_{c_1\cdots c_{h+1}|d_1\cdots d_{h+1}}$ des I_{h+1} eindeutig, und zwar durch die Grössen (27) und $B_{p_1\cdots p_{2h+2}}$ allein ausgedrückt, berechnen.

Aus der eben verwendeten Formel (5) für $k=h$ berechnet man weiter (und zwar aus $E_{a_1\cdots a_h|b_1\cdots b_h}$ und $E_{a_1\cdots a_{h+1}|b_1\cdots b_{h+1}}$, d. h. also) aus den $B_{p_1\cdots p_{2k}}$, $k=1,\cdots,h+1$, die

$$\Gamma^{s_1\cdots s_h}_{p_1\cdots p_h p_{h+1} p_{h+2}}$$

mit der ihnen zukommenden Unbestimmtheit. Um die

$$\Gamma^{r_1\cdots r_{h+1}}_{p_1\cdots p_{h+1} p_{h+2}} \tag{37}$$

zu berechnen, benutzen wir neben der Relation (34″) noch die Relation (7) für $k=h+1$:

$$\begin{aligned}\frac{\partial}{\partial y_t} E_{p_1\cdots p_{h+1}|q_1\cdots q_{h+1}} = {} & \Gamma^{r_1\cdots r_{h+1}}_{p_1\cdots p_{h+1}t} E_{r_1\cdots r_{h+1}|q_1\cdots q_{h+1}} \\ & + \Gamma^{r_1\cdots r_{h+1}}_{q_1\cdots q_{h+1}t} E_{r_1\cdots r_{h+1}|p_1\cdots p_{h+1}}.\end{aligned} \tag{38}$$

Wir bezeichnen

$$\frac{\partial}{\partial y_t} E_{p_1\cdots p_{h+1}|q_1\cdots q_{h+1}} = (p_1\cdots p_{h+1}\,|\,q_1\cdots q_{h+1}\,|\,t) \tag{39}$$

und bilden das aus (38) gebildete System mit bekannter linker Seite:

$$\begin{aligned}
& (p_1\cdots p_{h+1}\,|\,q_1\cdots q_{h+1}\,|\,t) - (q_2\cdots q_{h+1}t\,|\,p_1\cdots p_{h+1}\,|\,q_1) \\
& \quad + (p_2\cdots p_{h+1}q_1\,|\,q_2\cdots q_{h+1}t\,|\,p_1) - (q_3\cdots q_{h+1}tp_1\,|\,p_2\cdots p_{h+1}q_1\,|\,q_2) \\
& \quad + (p_3\cdots p_{h+1}q_1q_2\,|\,q_3\cdots q_{h+1}tp_1\,|\,p_2) \\
& \qquad\qquad - (q_4\cdots q_{h+1}tp_1p_2\,|\,p_3\cdots p_{h+1}q_1q_2\,|\,q_3) \\
& \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& \quad + (p_{h+1}q_1\cdots q_h\,|\,q_{h+1}tp_1\cdots p_{h-1}\,|\,p_h) - (tp_1\cdots p_h\,|\,p_{h+1}q_1\cdots q_h\,|\,q_{h+1}) \\
& \quad + (q_1q_2\cdots q_{h+1}\,|\,tp_1\cdots p_h\,|\,p_{h+1}) \\
& = \Gamma^{r_1\cdots r_{h+1}}_{p_1\cdots p_{h+1}t} E_{r_1\cdots r_{h+1}|q_1\cdots q_{h+1}} + \Gamma^{r_1\cdots r_{h+1}}_{q_1\cdots q_{h+1}t} E_{r_1\cdots r_{h+1}|p_1\cdots p_{h+1}} \\
& \quad - \Gamma^{r_1\cdots r_{h+1}}_{q_2\cdots q_{h+1}tq_1} E_{r_1\cdots r_{h+1}|p_1\cdots p_{h+1}} - \Gamma^{r_1\cdots r_{h+1}}_{p_1\cdots p_{h+1}q_1} E_{r_1\cdots r_{h+1}|q_2\cdots q_{h+1}t} \\
& \quad + \Gamma^{r_1\cdots r_{h+1}}_{p_2\cdots p_{h+1}q_1p_1} E_{r_1\cdots r_{h+1}|q_2\cdots q_{h+1}t} + \Gamma^{r_1\cdots r_{h+1}}_{q_2\cdots q_{h+1}tp_1} E_{r_1\cdots r_{h+1}|p_2\cdots p_{h+1}q_1} \\
& \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& \quad - \Gamma^{r_1\cdots r_{h+1}}_{tp_1\cdots p_hq_{h+1}} E_{r_1\cdots r_{h+1}|p_{h+1}q_1\cdots q_h} - \Gamma^{r_1\cdots r_{h+1}}_{p_{h+1}q_1\cdots q_hq_{h+1}} E_{r_1\cdots r_{h+1}|tp_1\cdots p_h} \\
& \quad + \Gamma^{r_1\cdots r_{h+1}}_{q_1\cdots q_{h+1}p_{h+1}} E_{r_1\cdots r_{h+1}|tp_1\cdots p_h} + \Gamma^{r_1\cdots r_{h+1}}_{tp_1\cdots p_hp_{h+1}} E_{r_1\cdots r_{h+1}|q_1\cdots q_{h+1}}.
\end{aligned} \tag{40}$$

Aus (40) und (34″), welche Gleichung geschrieben werden kann

$$(\Gamma^{s_1\cdots s_{h+1}}_{p_1\cdots p_h p_{h+1} p_{h+2}} - \Gamma^{s_1\cdots s_{h+1}}_{p_1\cdots p_h p_{h+2} p_{h+1}})E_{s_1\cdots s_{h+1}|a_1\cdots a_{h+1}} = \text{bekannt},$$

berechnen wir unter Verwendung der Symmetrie Eigenschaften der Γ

$$\Gamma^{s_1\cdots s_{h+1}}_{p_1\cdots p_h p_{h+1} p_{h+2}} E_{s_1\cdots s_{h+1}|a_1\cdots a_{h+1}} = \text{bekannt}. \tag{41}$$

Daraus wieder erhält man (bis auf die ihnen zukommende Unbestimmtheit) die Grössen (37) ausgedrückt durch die Komponenten der Reihe (27) und $B_{p_1\cdots p_{2h+2}}$. Um schliesslich zu zeigen, dass die Integrabilitätsbedingungen (31″) und (32″) zu keinen neuen Bedingungsgleichungen zwischen den Formenkomponenten führen, bilden wir

$$\begin{aligned}
&\frac{\partial}{\partial y_s}\left[\frac{\partial}{\partial y_t}\left(\frac{\partial^h x_i}{\partial y_{p_1}\cdots\partial y_{p_h}}\right)\right]\frac{\partial^k x_i}{\underline{\partial y_{q_1}\cdots\partial y_{q_k}}} \\
&\quad= \frac{\partial}{\partial y_s}\left[\frac{\partial}{\partial y_t}\left(\frac{\partial^h x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_h}}}\right)\frac{\partial^k x_i}{\underline{\partial y_{q_1}\cdots\partial y_{q_k}}}\right] \\
&\qquad- \frac{\partial}{\partial y_t}\left(\frac{\partial^h x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_h}}}\right)\frac{\partial}{\partial y_s}\left(\frac{\partial^k x_i}{\underline{\partial y_{q_1}\cdots\partial y_{q_k}}}\right).
\end{aligned} \tag{42}$$

Der zweite Term rechts ist aber

$$\begin{aligned}
&= -\frac{\partial}{\partial y_t}\left[\frac{\partial^h x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_h}}}\,\frac{\partial}{\partial y_s}\left(\frac{\partial^k x_i}{\underline{\partial y_{q_1}\cdots\partial y_{q_k}}}\right)\right] \\
&\quad+ \frac{\partial^h x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_h}}}\,\frac{\partial}{\partial y_t}\left[\frac{\partial}{\partial y_s}\left(\frac{\partial^k x_i}{\underline{\partial y_{q_1}\cdots\partial y_{q_k}}}\right)\right].
\end{aligned}$$

Also gilt

$$\begin{aligned}
&\left\{\frac{\partial}{\partial y_s}\left[\frac{\partial}{\partial y_t}\left(\frac{\partial^h x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_h}}}\right)\right]\right. \\
&\qquad\left.- \frac{\partial}{\partial y_t}\left[\frac{\partial}{\partial y_s}\left(\frac{\partial^h x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_h}}}\right)\right]\right\}\frac{\partial^k x_i}{\underline{\partial y_{q_1}\cdots\partial y_{q_k}}} \\
&\quad= -\left\{\frac{\partial}{\partial y_s}\left[\frac{\partial}{\partial y_t}\left(\frac{\partial^k x_i}{\underline{\partial y_{q_1}\cdots\partial y_{q_k}}}\right)\right]\right. \\
&\qquad\left.- \frac{\partial}{\partial y_t}\left[\frac{\partial}{\partial y_s}\left(\frac{\partial^k x_i}{\underline{\partial y_{q_1}\cdots\partial y_{q_k}}}\right)\right]\right\}\frac{\partial^h x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_h}}}.
\end{aligned} \tag{43}$$

(Denn die Differenz der beiden Seiten von (43) ist nach (42)

$$\frac{\partial}{\partial y_s}\left[\frac{\partial}{\partial y_t}\left(\underline{\frac{\partial^h x_i}{\partial y_{p_1}\cdots\partial y_{p_h}}}\;\underline{\frac{\partial^k x_i}{\partial y_{q_1}\cdots\partial y_{q_k}}}\right)\right]$$

$$-\frac{\partial}{\partial y_t}\left[\frac{\partial}{\partial y_s}\left(\underline{\frac{\partial^h x_i}{\partial y_{p_1}\cdots\partial y_{p_h}}}\;\underline{\frac{\partial^k x_i}{\partial y_{q}\cdots\partial y_{q_k}}}\right)\right],$$

welcher Ausdruck für $h\neq k$ verschwindet, aber auch für $h=k$, die zweimalige stetige Differenzierbarkeit der Masstensoren vorausgesetzt.) Die linke Seite der Gleichung (43) stimmt mit der linken Seite (31″) für $k=h-2$ und mit der linken Seite (32″) für $k=h-1$ überein. Wir haben somit das Resultat:

Die Gleichung (31″), *die erste aus der Integrabilitätsbedingung der* $(h+1)$*ten Gleichung des Frenet-Systems hergeleitete, ist identisch mit der letzten Gleichung, die aus der Integrabilitätsbedingung der* $(h-1)$*ten Gleichung des Frenet-Systems entspringt. Und ebenso ist* (32″), *die zweite aus der Integrabilitätsbedingung der* $(h+1)$*ten Gleichung des Frenet-Systems hergeleitete, mit der vorletzten Gleichung identisch, die aus der Integrabilitätsbedingung der hten Gleichung des Frenet-Systems folgt. Als* (*mögliche*) *Bedingungsgleichungen für die Formen verbleiben somit die Relationen* (5) *und* (7), *weiter* (33″) *und* (34″) *und die Relationen* (36).

Da die integrierte F_e aber *reell* sein soll, eine stillschweigende Annahme, die in unseren Überlegungen wesentlich verwendet wurde, so müssen die E-Tensoren *positiv halbdefinit* sein, damit die Nebenbedingungen (2) durch ein *reelles* Basis-Bein erfüllbar sind.

Das gibt für die Grund-Tensoren aber Bedingungen, die anscheinend in eine einfache Gestalt nicht gebracht werden können.

Dagegen ist es nicht schwer, für die *Masstensoren* $E_{p_1\cdots p_h|q_1\cdots q_h}$ als vollständiges "Invarianten"-System die notwendigen und hinreichenden Bedingungsrelationen anzugeben:

Ausser der oben erwähnten Eigenschaft *positiv halbdefinit* zu sein, müssen sie die Relationen (5), (7), (33″), (34″) erfüllen.

Im folgenden Paragraphen leiten wir einige geometrische Tatsachen ab, die aus dem Haupttheorem sofort folgen.

5. Einige geometrische Folgerungen

I. Die Einbettungszahl der F_e

Wir definieren: Lässt sich eine F_e in eine ϵ-dimensionale Hyperebene E_ϵ, aber nicht in eine $(\epsilon-1)$-dimensionale $E_{\epsilon-1}$ einbetten, so nennen wir ϵ die *Einbettungszahl der* F_e.

Es gilt dann der Satz:

Bezeichnet l_σ die Dimension des I_σ, und ist I_m der letzte Normalvektorraum der F_e, so ist

$$\epsilon = \sum_{\sigma=1}^{m} l_\sigma. \tag{1}$$

D. h. die Einbettungszahl der F_e ist gleich der Dimension des grössten Schmieg-Vektorraums.

Wir gehen, um den Satz zu beweisen, auf das Frenet-System (§3 (11)) und die Nebenbedingungen (§4 (2)) für dasselbe zurück.

Für die gegebene F_e sind die dort auftretenden E und Γ-Grössen so gegeben, dass sowohl die Integrabilitätsbedingungen des Frenet-Systems als auch das aus den Nebenbedingungen abgeleitete System eine Folge dieser Nebenbedingungen sind. (Die Relationen (5), (7), (33″), (34″) und (36) des §4 sind erfüllt.)

Wenn wir daher die Anfangswerte für die Reihe

$$x_i, \quad \frac{\partial x_i}{\partial y_{p_1}}, \cdots, \quad \frac{\partial^m x_i}{\partial y_{p_1} \cdots \partial y_{p_m}} \tag{2}$$

so wählen können, dass die Nebenbedingungen (2) erfüllt sind, erhalten wir eine $\tilde{F}_e$ mit demselben Formensystem wie die F_e.

Das können wir aber in einem R_ϵ (d. h. einem Raum der Dimension des letzten Schmieg-Vektorraums der F_e) und in keinem R_ν, $\nu<\epsilon$. Wählen wir als den R_ϵ jene durch

$$x_{\epsilon+1} = 0, \quad x_{\epsilon+2} = 0, \cdots, x_n = 0 \tag{3}$$

gegebene ϵ-dimensionale Hyperebene des R_n, so liegt die integrierte $\tilde{F}_e$ ganz in diesem R_ϵ, und da sie der gegebene F_e kongruent ist, liegt auch diese ganz in einer ϵ-dimensionalen Hyperebene des R_n. Wie die $\tilde{F}_e$ selbst kann sie (schon vermöge der Nebenbedingungen (2) §4) in keiner Hyperebene niedrigerer Dimension liegen.

II. Über die Krümmungstensoren der F_e

Der Tensor

$$E_{p_1 \cdots p_h p_{h+1} | q_1 \cdots q_h q_{h+1}} - E_{p_1 \cdots p_h q_{h+1} | q_1 \cdots q_h p_{h+1}} \tag{4}$$

ist (§4 (35′)) ausdrückbar durch die Komponenten der ersten h Grundformen der F_e

$$B_{p_1 \cdots p_{2k}} \qquad (k = 1, \cdots, h) \tag{5}$$

resp. deren Ableitungen. *Wir nennen ihn den $h$ten Krümmungstensor der F_e.*

(Der erste Krümmungstensor ist der Riemannsche Krümmungstensor der F_e.)

Es gilt der Satz:

Verschwindet für eine F_e der hte Krümmungstensor, so gibt es immer eine $\tilde{F}_e$, deren vollständiges Formensystem aus den h ersten Grundformen der F_e besteht.

Wenn wir im Frenet-System (§3 (11)) und im System der Nebenbedingungen (§4 (2)) alle

$$\frac{\partial^t x_i}{\partial y_{r_1} \cdots \partial y_{r_t}}, \qquad E_{r_1 \cdots r_t | a_1 \cdots a_t} \tag{6}$$

für $t > h$ und ausserdem $\Gamma^{r_1 \cdots r_h}_{p_1 \cdots p_{h+1} p_{h+2}}$ Null setzen, so erhalten wir ein analog gebautes System totaler Differentialgleichungen mit Nebenbedingungen.

Können wir zeigen, dass auch für dieses die Integrabilitätsbedingungen, sowie das abgeleitete System der Nebenbedingungen *eine Folge der Nebenbedingungen sind*, so ist der Satz offenbar bewiesen.

Für das abgeleitete System der Nebenbedingungen gilt das, da die Relationen ((5) und (7), §4) wegen $\Gamma^{r_1 \cdots r_h}_{p_1 \cdots p_h p_{h+1}} = 0$ erfüllt sind.

Was die Integrabilitätsbedingungen betrifft, so könnte sie nur die für die $(h+1)$te Gleichung des neuen Frenet-Systems nicht erfüllt sein, da nur diese Gleichung von der entsprechenden des ursprünglichen Systems abweicht.

Die fragliche Integrabilitätsbedingung aber ist aus der der ursprünglichen $(h+1)$ten Gleichung, wie wir sie §4 (31), (32), (33) und (34) anschrieben, sofort ableitbar.

Die Gleichungen (31) und (32) bleiben unverändert, (34) fällt weg, und nur die (33) entsprechenden Relationen unterscheiden sich um die Differenz

$$(\Gamma^{s_1 \cdots s_h}_{p_1 \cdots p_h p_{h+1} p_{h+2}} - \Gamma^{s_1 \cdots s_h}_{p_1 \cdots p_h p_{h+2} p_{h+1}}) \frac{\partial^h x_i}{\partial y_{s_1} \cdots \partial y_{s_h}} \tag{7}$$

(wo die Γ natürlich die der gegebenen F_e sind). Aber (7) verschwindet, wenn

$$(\Gamma^{s_1 \cdots s_h}_{p_1 \cdots p_h p_{h+1} p_{h+2}} - \Gamma^{s_1 \cdots s_h}_{p_1 \cdots p_h p_{h+2} p_{h+1}}) E_{s_1 \cdots s_h | a_1 \cdots a_h} = 0 \tag{8}$$

ist. Das aber trifft gerade zu wegen §4 (5) und der Voraussetzung unseres Satzes.

Damit ist der Beweis geliefert.

Bemerkung. Für $h = 1$ hat die $\tilde{F}_e$ nur die erste Grundform. Sie hat also nur einen invarianten Vektorraum, den I_1. Ihre Einbettungszahl ist also gleich der Dimensionszahl des I_1, d. h. gleich e.

Die $\tilde{F}_s$, die ganz in einer E_s liegt, ist somit diese E_s selbst. *Damit ist die Abwickelbarkeit der F_s in eine E_s nachgewiesen, sobald der erste Krümmungstensor verschwindet.*

III. Untermannigfaltigkeiten F_r der F_s

Ein Hauptergebnis des vorhergehenden Paragraphen bringen wir in Erinnerung:

Die Masstensoren $E_{p_1\cdots p_k|q_1\cdots q_k}$ wie die

$$\Gamma^{r_1\cdots r_{k-1}}_{p_1\cdots p_k p_{k+1}},\ \Gamma^{r_1\cdots r_k}_{p_1\cdots p_k p_{k+1}}\ \textit{für}\ k = 1, \cdots, h$$

sind durch die h ersten Grundformen $B_{p_1\cdots p_{2k}}$, $k=1, \cdots, h$, bestimmt.

Wir betrachten eine in der F_s eingebettete F_r

$$y_p = y_p(z_1, \cdots, z_r) \qquad (p = 1, \cdots, l) \tag{9}$$

die wir als im euklidischen R_n liegend in der selben Art wie die F_s behandeln.

Der Tangential-Vektorraum der F_r wird durch die r Vektoren

$$\frac{\partial x_i}{\partial z_a} = \frac{\partial x_i}{\partial y_p}\frac{\partial y_p}{\partial z_a} \qquad (a = 1, \cdots, r) \tag{10}$$

aufgespannt und liegt ganz im I_1 der F_s.

Jeder Schmieg-Vektorraum der F_r liegt ebenso im entsprechenden Schmieg-Vektorraum der F_s.

Um die Projektionen der die Schmieg-Vektorräume aufspannenden Raumvektoren der F_r von den entsprechenden der F_s zu unterscheiden bezeichne

$$\underline{\underline{\frac{\partial^h x_i}{\partial z_{a_1}\cdots\partial z_{a_h}}}}$$

die Projektion von $\partial^h x_i/\partial z_{a_1}\cdots\partial z_{a_h}$ in den I_h der F_r. Die Massvektoren der I_h der F_r bezeichnen wir $F_{a_1\cdots a_h|b_1\cdots b_h}$ und die Grundtensoren $C_{a_1\cdots a_{2h}}$.

Für den Masstensor $F_{a|b}$ des I_1 der F_r erhalten wir aus (10)

$$F_{a|b} = E_{p|q}\frac{\partial y_p}{\partial z_a}\frac{\partial y_q}{\partial z_b} \qquad (a, b = 1, \cdots, r). \tag{11}$$

Wir gehen nun über zum zweiten Schmieg-Vektorraum. Durch Differentiation von (10) nach z_b erhalten wir aus (10):

$$\frac{\partial^2 x_i}{\partial z_a\partial z_b} = \frac{\partial^2 x_i}{\partial y_p\partial y_q}\frac{\partial y_p}{\partial z_a}\frac{\partial y_q}{\partial z_b} + \frac{\partial x_i}{\partial y_p}\frac{\partial^2 y_p}{\partial z_a\partial z_b}. \tag{12}$$

Bezeichnen wir die den Γ der F_e entsprechenden Grössen der F_r mit P, so geben die zweiten Relationen im System der Frenet-Gleichungen aus (12)

$$P^{c}_{ab}\frac{\partial x_i}{\partial z_c}+\underline{\underline{\frac{\partial^2 x_i}{\partial z_a \partial z_b}}}=\left(\Gamma^{r}_{pq}\,\frac{\partial x_i}{\partial y_r}+\underline{\frac{\partial^2 x_i}{\partial y_p \partial y_q}}\right)\frac{\partial y_p}{\partial z_a}\,\frac{\partial y_q}{\partial z_b}+\frac{\partial x_i}{\partial y_p}\,\frac{\partial^2 y_p}{\partial z_a \partial z_b}. \tag{13}$$

Daraus folgt

$$\underline{\underline{\frac{\partial^2 x_i}{\partial z_a \partial z_b}}}=\underline{\frac{\partial^2 x_i}{\partial y_p \partial y_q}}\,\frac{\partial y_p}{\partial z_a}\,\frac{\partial y_q}{\partial z_b}+\frac{\partial x_i}{\partial y_r}\left[\frac{\partial^2 y_r}{\partial z_a \partial z_b}+\Gamma^{r}_{pq}\,\frac{\partial y_p}{\partial z_a}\,\frac{\partial y_q}{\partial z_b}-P^{c}_{ab}\,\frac{\partial y_r}{\partial z_c}\right], \tag{14}$$

wo der Faktor von $\partial x_i/\partial y_r$ die "verallgemeinerte" invariante Ableitung

$$\frac{D^2 y_r}{Dz_b Dz_a}=\frac{\partial^2 y_r}{\partial z_a \partial z_b}+\Gamma^{r}_{pq}\,\frac{\partial y_p}{\partial z_a}\,\frac{\partial y_q}{\partial z_b}-P^{c}_{ab}\,\frac{\partial y_r}{\partial z_c} \tag{15}$$

von $\partial y_r/\partial z_a$ nach z_b ist. (Ist $r=l$, so verschwindet die invariante Ableitung und (15) stellt das Transformationsgesetz der Christoffelklammer dar.)

Wir haben in (14) die Projektions-Vektoren (in den I_2 der F_r):

$$\underline{\underline{\frac{\partial^2 x_i}{\partial z_a \partial z_b}}} \tag{16}$$

linear dargestellt durch die Basis des I_{12} der F_e mit Koeffizienten, die von

$$B_{pq} \text{ und Ableitungen der } y \text{ nach den } z \tag{17}$$

allein abhängen.

In der Tat ist P^{c}_{ab} ebenfalls durch diese Grössen ausdrückbar, da P^{c}_{ab} durch $F_{a|b}$ bestimmt ist und (11) gilt. *Es besteht allgemein die Darstellung*

$$\begin{aligned}\underline{\underline{\frac{\partial^k x_i}{\partial z_{a_1}\cdots\partial z_{a_k}}}}&=\underline{\frac{\partial^k x_i}{\partial y_{p_1}\cdots\partial y_{p_k}}}\,\frac{\partial y_{p_1}}{\partial z_{a_1}}\cdots\frac{\partial y_{p_k}}{\partial z_{a_k}}\\&\quad+\sum_{m=1}^{k-1}\underline{\frac{\partial^m x_i}{\partial y_{p_1}\cdots\partial y_{p_m}}}\,T^{p_1\cdots p_m}_{a_1\cdots a_k},\end{aligned} \tag{18}$$

wo die Tensoren $T^{p_1\cdots p_m}_{a_1\cdots a_k}$, $m=1, 2, \cdots, k-1$, *durch die Reihe*

$$B_{pq}, \cdots, B_{p_1\cdots p_{2k-2}}, \tag{18'}$$

und Ableitungen der y *nach den* z *allein bestimmt sind.*

Die Behauptung ist richtig für $k=1$ und 2. Wenn wir zeigen, dass sie für $k+1$ gilt, wenn sie für $1, 2, \cdots, k$ besteht, ist sie allgemein bewiesen.

Aus (18) erhalten wir für die Masstensoren die Beziehung

$$
\begin{aligned}
F_{a_1\cdots a_k|b_1\cdots b_k} &= E_{p_1\cdots p_k|q_1\cdots q_k}\frac{\partial y_{p_1}}{\partial z_{a_1}}\cdots\frac{\partial y_{p_k}}{\partial z_{a_k}}\frac{\partial y_{q_1}}{\partial z_{b_1}}\cdots\frac{\partial y_{q_k}}{\partial z_{b_k}} \\
&\quad+\sum_{m=1}^{k-1}E_{p_1\cdots p_m|q_1\cdots q_m}T^{p_1\cdots p_m}_{a_1\cdots a_k}T^{q_1\cdots q_m}_{b_1\cdots b_k}.
\end{aligned}\tag{19}
$$

Also ist auch $F_{a_1\cdots a_k|b_1\cdots b_k}$ durch die Grössen (18′) und $B_{p_1\cdots p_{2k}}$ ausgedrückt.

Durch Differentiation von (18) nach z_{a+1} erhalten wir unter Verwendung der Frenet-Formeln §3 (11):

$$
\begin{aligned}
&P^{c_1\cdots c_{k-1}}_{a_1\cdots a_k a_{k+1}}\frac{\partial^{k-1}x_i}{\underline{\underline{\partial z_{c_1}\cdots\partial z_{c_{k-1}}}}}+P^{c_1\cdots c_k}_{a_1\cdots a_k a_{k+1}}\frac{\partial^k x_i}{\underline{\underline{\partial z_{c_1}\cdots\partial z_{c_k}}}}+\frac{\partial^{k+1}x_i}{\underline{\underline{\partial z_{a_1}\cdots\partial z_{a_k}\partial z_{a_{k+1}}}}} \\
&=\frac{\partial^{k+1}x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_k}\partial y_{p_{k+1}}}}\frac{\partial y_{p_1}}{\partial z_{a_1}}\cdots\frac{\partial y_{p_k}}{\partial z_{a_k}}\frac{\partial y_{p_{k+1}}}{\partial z_{a_{k+1}}} \\
&\quad+\sum_{m=1}^{k}\frac{\partial^m x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_m}}}U^{p_1\cdots p_m}_{a_1\cdots a_{k+1}},
\end{aligned}\tag{20}
$$

wo die U-Tensoren jetzt neben den Grössen (18′) noch $B_{p_1\cdots p_{2k}}$ enthalten. Aber die P-Grössen der linken Seite von (20) sind durch $C_{a_1\cdots a_{2t}}$, $t=1,\cdots,k$, ausdrückbar, also (19) durch die Grössen (18′) und $B_{p_1\cdots p_{2k}}$. Da für $1, 2, \cdots, k$ die Behauptung voraussetzungsgemäss gilt, so folgt damit aus (20) ihre Richtigkeit für $k+1$.

Eine Folge des Tatbestandes (18) ist der Satz:

Haben zwei F_s die Grundformen bis zur $(2k)$ten Stufe gleich:

$$B_{p_1p_2},\ \cdots,\ B_{p_1\cdots p_{2k}},$$

so haben entsprechend zugeordnete Untermannigfaltigkeiten (d. h. durch gleiche Parameterdarstellung (9) zugeordnete) ebenfalls die ersten k Grundformen gleich.

Aber auch die Umkehrung gilt:

Wenn zwei F_s so zugeordnet werden können, dass entsprechende F_r (r fixiert) gleiche Grundformen bis zur $(2k)$ten Stufe besitzen, so haben die beiden F_s ebenfalls identische Grundformen bis zur $(2k)$ten Stufe.

Beweis der Umkehrung. Multipliziert man (11) mit $dz_a dz_b$, so folgt, dass $E_{p|q}dy_p dy_q$ für beide F_s gleich ist.

Aber mit der Willkür der F_r ist auch

$$dy_p = \frac{\partial y_p}{\partial z_a} dz_a$$

willkürlich. Also ist

$$E_{p|q} dy_p dy_q$$

für beliebige dy_p für beide F_s gleich, woraus aber die Gleichheit der ersten Grundformen folgt.

Wir denken uns nun die Gleichheit der $(h-1)$ ersten Grundformen bewiesen, dann folgt aus (19) für $k=h$ und der Voraussetzung des Satzes, dass

$$E_{p_1\cdots p_h|q_1\cdots q_h} dy_{p_1} \cdots dy_{p_h} dy_{q_1} \cdots dy_{q_h}$$

für beliebige dy_p für beide F_s gleich ist. Also gilt die Behauptung auch für die $h$te Grundform, und da sie für $h=1$ gilt, ist die Umkehrung bewiesen.

Besonders interessant wird der Satz für die $F_1(r=1)$, d. h. für Kurven der F_e. Der Zusammenhang der Formen der Kurve mit ihren Krümmungen wird durch

$$(21) \qquad \underline{\frac{d^k x_i}{ds^k}}\, \underline{\frac{d^k x_i}{ds^k}} = \frac{1}{(\rho_1\rho_2 \cdots \rho_{k-1})^2}, \qquad ds^2 = B_{11} dy^2,$$

gegeben. *Also sind durch die ersten k "Grundformen" die Bogenlänge und die* $(\rho-1)$ *ersten Krümmungen gegeben und umgekehrt.*

Wir verweisen für den Satz, der den Spezialfall des soeben bewiesenen für $r=1$ darstellt, auf das *Lehrbuch* (XI §9, p. 226).

Bemerkungen zu III

(A) *Wir können von den in die Relation* (18) *eintretenden Tensoren* $T^{p_1\cdots p_m}_{a_1\cdots a_k}$ *nachweisen, dass sie nur Ableitungen der y nach z bis zur kten Ordnung inklusive enthalten.*

Verfolgt man nämlich die Bildung der Γ-Grössen, so sieht man, dass

$$(22) \qquad \Gamma^{s_1\cdots s_{h+1}}_{p_1\cdots p_{h+1}p_{h+2}}$$

allein aus $E_{p_1\cdots p_{h+1}|q_1\cdots q_{h+1}}$, deren ersten Ableitungen und aus $\Gamma^{s_1\cdots s_h}_{p_1\cdots p_h p_{h+1}}$ (unabgeleitet) gebildet ist. *Daraus folgt, dass* (22) *nur aus den* $E_{p_1|q_1}, \cdots,$ $E_{p_1\cdots p_{h+1}|q_1\cdots q_{h+1}}$ *und deren ersten Ableitungen aufgebaut ist.*

Der Tensor

$$\Gamma^{s_1\cdots s_h}_{p_1\cdots p_{h+1}p_{h+2}}$$

wieder hängt nur ab von den nichtdifferenzierten $E_{p_1\cdots p_h|q_1\cdots q_h}$ und $E_{p_1\cdots p_{h+1}|q_1\cdots q_{h+1}}$.

Diese Tatsachen aber reichen hin, um die angegebene Eigenschaft der Tensoren T in (18) nachzuweisen.

(B) Aus der Formel (19) gewinnt man sofort Relationen zwischen den Krümmungen der F_r und der F_e und den "Relativkrümmungen" der F_r in bezug auf die F_e.

(Man hat dazu nur die Differenz $F_{a_1\cdots a_{k-1}a_k|b_1\cdots b_{k-1}b_k} - F_{a_1\cdots a_{k-1}b_k|b_1\cdots b_{k-1}a_k}$ aus (19) zu bilden.)

6. Der $I_{12\cdots h}$-Schmieg-Vektorraum, seine Metrik und Parallelverschiebung*

Ist

$$\lambda_i = \sum_{k=1}^{h} \frac{\partial^k x_i}{\underline{\partial y_{p_1} \cdots \partial y_{p_k}}} l^{p_1\cdots p_k} \tag{1}$$

ein Vektor des $I_{12\cdots h}$, so nennen wir die in (1) auftretenden symmetrischen Flächentensoren

$$l^{p_1\cdots p_k} \qquad (k = 1, \cdots, h) \tag{2}$$

die *kontravariante Darstellung* des Vektors λ_i.

Die kontravariante Darstellung ist bis auf die Nulltensoren $\theta^{p_1\cdots p_k}$ gegeben, d. h. bis auf die Lösungen von

$$\frac{\partial^k x_i}{\underline{\partial y_{p_1} \cdots \partial y_{p_k}}} \theta^{p_1\cdots p_k} = 0 \qquad (k = 2, \cdots, h), \tag{3}$$

resp.

$$E_{q_1\cdots q_k|p_1\cdots p_k}\theta^{p_1\cdots p_k} = 0 \qquad (k = 2, \cdots, h). \tag{3'}$$

Als *kovariante Darstellung* des Raumvektors λ_i bezeichnen wir die Gesamtheit der symmetrischen Flächentensoren

$$l_{p_1\cdots p_k} = \lambda_i \frac{\partial^k x_i}{\underline{\partial y_{p_1} \cdots \partial y_{p_k}}}. \tag{4}$$

Während die kontravarianten Darstellungstensoren (2) keiner Einschränkung unterliegen, gilt für die kovarianten (notwendig und hinreichend):

* Dazu siehe: *Beitrag zur Differentialgeometrie* u. s. w., Sitzungsberichte der Preussischen Akademie, 1931. *Zum Tensorkalkül in Vektorräumen Riemannscher Mannigfaltigkeiten*, Monatshefte für Mathematik und Physik, 1933.

$$l_{p_1\cdots p_k}\theta^{p_1\cdots p_k} = 0, \tag{5}$$

für jeden Nulltensor $\theta^{p_1\cdots p_k}$.

Das innere Produkt eines kovarianten Darstellungstensors kter Stufe mit einem Nulltensor kter Stufe verschwindet.

Zwischen der kontra- und kovarianten Darstellung (2) und (4) besteht die Beziehung

$$l_{p_1\cdots p_k} = E_{p_1\cdots p_k|q_1\cdots q_k}l^{q_1\cdots q_k}, \tag{6}$$

die man erhält, wenn man in (4) für λ_i den Ausdruck (1) substituiert. Ferner folgt aus (1)

$$\lambda_i\lambda_i = \sum_{k=1}^{h} E_{p_1\cdots p_k|q_1\cdots q_k}l^{p_1\cdots p_k}l^{q_1\cdots q_k}. \tag{7}$$

Führt man den Tensor

$$E_{p_1\cdots p_k|q_1\cdots q_r} = \frac{\partial^k x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_k}}}\;\frac{\partial^r x_i}{\underline{\partial y_{q_1}\cdots\partial y_{q_r}}} \tag{8}$$

ein, so können wir (6) und (7) auch schreiben

$$l_{p_1\cdots p_k} = \sum_{r=1}^{h} E_{p_1\cdots p_k|q_1\cdots q_r}l^{q_1\cdots q_r} \tag{6'}$$

resp.

$$\lambda_i\lambda_i = \sum_{r,s=1}^{h} E_{p_1\cdots p_r|q_1\cdots q_s}l^{p_1\cdots p_r}l^{q_1\cdots q_s}. \tag{7'}$$

Den Tensor (8) (für $r, k=1, \cdots, h$) nennen wir den *metrischen Tensor des* $I_{12\cdots h}$.

Wir nennen nun den Raumvektor λ_i (1) *des* $I_{12\cdots h}$, *der längs eines Kurvenstückes C der* F_e *definiert ist,* $I_{12\cdots h}$*-parallel, wenn sein Raumdifferential* $d\lambda_i$ *stets normal ist zum* $I_{12\cdots h}$.

Aus der Definition folgt ohne weiteres, dass die $I_{12\cdots h}$-Parallelverschiebung (von Vektoren des $I_{12\cdots h}$) die Längen und Winkel unverändert lässt.

Sind nämlich λ_i und μ_i jetzt $I_{12\cdots h}$-parallelverschobene Vektoren des $I_{12\cdots h}$, so ist

$$d(\lambda_i\mu_i) = d\lambda_i\mu_i + \lambda_i d\mu_i = 0,$$

da λ_i, μ_i im $I_{12\cdots h}$ und $d\lambda_i$, $d\mu_i$ normal zum $I_{12\cdots h}$ liegen.

Ist λ_i längs C ein $I_{12\cdots h}$-parallelverschobener Vektor, so folgt aus (4) und der Definition der $I_{12\cdots h}$-Parallelverschiebung

$$dl_{p_1\cdots p_k} = \lambda_i \frac{\partial}{\partial y_t}\left(\frac{\partial^k x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_k}}}\right) dy_t. \tag{9}$$

Unter Verwendung der Frenet-Gleichungen §3 (11) schreiben wir (9)

$$\begin{aligned} dl_{p_1\cdots p_k} = \lambda_i \Bigg(& \Gamma^{r_1\cdots r_{k-1}}_{p_1\cdots p_k t} \frac{\partial^{k-1} x_i}{\underline{\partial y_{r_1}\cdots\partial y_{r_{k-1}}}} + \Gamma^{r_1\cdots r_k}_{p_1\cdots p_k t} \frac{\partial^k x_i}{\underline{\partial y_{r_1}\cdots\partial y_{r_k}}} \\ & + \frac{\partial^{k+1} x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_k}\partial y_t}}\Bigg) dy_t, \end{aligned} \tag{9'}$$

also wegen (4)

$$dl_{p_1\cdots p_k} = (\Gamma^{r_1\cdots r_{k-1}}_{p_1\cdots p_k t} l_{r_1\cdots r_{k-1}} + \Gamma^{r_1\cdots r_k}_{p_1\cdots p_k t} l_{r_1\cdots r_k} + l_{p_1\cdots p_k t}) dy_t \qquad (k = 1, \cdots, h) \tag{10}$$

wo

$$l_{p_1\cdots p_h t} = 0 \tag{10'}$$

ist (4).

Das System (10), (10′) *beschreibt die $I_{12\cdots h}$-Parallelverschiebung durch die kovarianten* (*Flächen*)-*Darstellungstensoren des Raumvektors* λ_i *des* $I_{12\cdots h}$.*

Bezeichnet D (besser $D_{12\cdots h}$) das *absolute* $I_{12\cdots h}$-Differential

$$Dl_{p_1\cdots p_k} = dl_{p_1\cdots p_k} - (\Gamma^{r_1\cdots r_{k-1}}_{p_1\cdots p_k t} l_{r_1\cdots r_{k-1}} + \Gamma^{r_1\cdots r_k}_{p_1\cdots p_k t} l_{r_1\cdots r_k} + l_{p_1\cdots p_k t}) dy_t, \tag{11}$$

so folgt durch einfache Rechnung für den metrischen Tensor (§4, (5) und (7))

$$DE_{p_1\cdots p_r | q_1\cdots q_s} = 0 \qquad (r, s = 1, \cdots, h). \tag{12}$$

Wir hätten (12) auch aus der Tatsache ableiten können, dass die $I_{12\cdots h}$-Parallelverschiebung Längen und Winkel von Raumvektoren invariant lässt. Das System Differentialgleichungen für die kovariante $I_{12\cdots h}$-Parallelverschiebung (10) ist völlig gleichgebaut jenem Teilsystem der Frenet-Gleichungen §3 (11), das man durch Streichung der ersten sowie der $(h+2)$ten, $(h+3)$ten, $\cdots$ bis letzten Gleichungen erhält, wenn man ausserdem in der $(h+1)$ten Gleichung noch (10′) entsprechend

---

* Ist (10) erfüllt, so folgt

$$d\lambda_i \frac{\partial^k x_i}{\underline{\partial y_{p_1}\cdots\partial y_{p_k}}} = 0 \qquad (k = 1, 2, \cdots, h)$$

also $d\lambda_i$ normal zum $T_{12\cdots h}$.

$$(13) \qquad \frac{\partial^{h+1} x_i}{\partial y_{p_1} \cdots \partial y_{p_h} \partial y_{p_{h+1}}}$$

weglässt.

Die Nebenbedingungen für die Lösungen von (10) sind aber die Relationen (5).

Wir stellen nun die Frage nach der vollständigen Integrabilität der $I_{12\cdots h}$-*Parallelverschiebung, also nach der des Systems* (10) *mit den Nebenbedingungen* (5).

Was die Integrabilitätsbedingung der $k$ten Gleichung (10) betrifft, wenn $k=1, 2, \cdots, h-1$, so ist ihre Form

$$(14) \qquad \sum l_{p_1 \cdots p_r} A^{p_1 \cdots p_r} = 0,$$

wenn die entsprechende der $(k+1)$ten Gleichung des Frenet-Systems

$$(15) \qquad \sum \frac{\partial^r x_i}{\partial y_{p_1} \cdots \partial y_{p_r}} A^{p_1 \cdots p_r} = 0$$

lautet. Da (15) für die gegebene F_e natürlich erfüllt ist und mit dem System

$$(16) \qquad \frac{\partial^r x_i}{\partial y_{p_1} \cdots \partial y_{p_r}} A^{p_1 \cdots p_r} = 0$$

äquivalent ist, so bedeutet das, dass die Koeffizienten $A^{p_1 \cdots p_r}$ Nulltensoren sind. *Damit ist* (14) *als Folge der Nebenbedingungen* (5) *erfüllt.*

Die Integrabilitätsbedingung der $h$ten Gleichung des Systems (10) dagegen ist nur dann eine Folge der Nebenbedingungen (5), wenn (§4 (33″)):

$$(17) \qquad (\Gamma^{s_1 \cdots s_h}_{p_1 \cdots p_h p_{h+1} p_{h+2}} - \Gamma^{s_1 \cdots s_h}_{p_1 \cdots p_h p_{h+2} p_{h+1}}) E_{s_1 \cdots s_h | r_1 \cdots r_h} = 0$$

ist, d. h. wenn (§4 (5), §5) der $h$te Krümmungstensor verschwindet. Denn auch die Integrabilitätsbedingungen der $h$ten Gleichung (10) resp. der $(h+1)$ten Gleichung des Frenet-Systems, die wir ebenfalls in der Form (14) resp. (15) schreiben können, haben dieselben Koeffizienten A mit Ausnahme des $A^{s_1 \cdots s_h}$ (soweit die $l_{p_1 \cdots p_r}$ auftreten).

Diese beiden A aber unterscheiden sich §4 (33) gerade um die Differenz

$$(18) \qquad \Gamma^{s_1 \cdots s_h}_{p_1 \cdots p_{h+1} p_{h+2}} - \Gamma^{s_1 \cdots s_h}_{p_1 \cdots p_{h+2} p_{h+1}}.$$

Da aber der entsprechende Koeffizient A in der aus dem Frenet-System abgeleiteten Integrabilitätsbedingung offenbar ein Nulltensor ist, so kann der aus (10) hergeleitete nur dann einer sein, wenn (18) einer ist.

Wir haben somit:

Die Integrabilitätsbedingungen des Systems (10) *sind dann und nur dann eine Folge der Nebenbedingungen* (5), *wenn der hte Krümmungstensor der* F_e *verschwindet.*

Sei jetzt

$$M = l_{p_1 \cdots p_k} \theta^{p_1 \cdots p_k} = 0, \tag{19}$$

$\theta^{p_1 \cdots p_k}$ Nulltensor, eine Nebenbedingung des Systems (10). Wir bilden die "abgeleitete" Gleichung

$$\begin{aligned} dM &= (\Gamma^{r_1 \cdots r_{k-1}}_{p_1 \cdots p_k t} l_{r_1 \cdots r_{k-1}} + \Gamma^{r_1 \cdots r_k}_{p_1 \cdots p_k t} l_{r_1 \cdots r_k} + l_{p_1 \cdots p_k t}) \theta^{p_1 \cdots p_k} dy_t \\ &\quad + l_{p_1 \cdots p_k} d\theta^{p_1 \cdots p_k} \\ &= \Gamma^{r_1 \cdots r_{k-1}}_{p_1 \cdots p_k t} \theta^{p_1 \cdots p_k} l_{r_1 \cdots r_{k-1}} dy_t \\ &\quad + (d\theta^{r_1 \cdots r_k} + \Gamma^{r_1 \cdots r_k}_{p_1 \cdots p_k t} \theta^{p_1 \cdots p_k} dy_t) l_{r_1 \cdots r_k} + l_{p_1 \cdots p_k t} \theta^{p_1 \cdots p_k} dy_t = 0, \end{aligned} \tag{20}$$

die aber (§3 (19), (20) und (21))

$$dM = \tilde{\theta}^{r_1 \cdots r_{k-1}} l_{r_1 \cdots r_{k-1}} + \tilde{\tilde{\theta}}^{r_1 \cdots r_k} l_{r_1 \cdots r_k} + \tilde{\tilde{\tilde{\theta}}}^{p_1 \cdots p_k t} l_{p_1 \cdots p_k t} \tag{21}$$

geschrieben werden kann. D. h.: das abgeleitete System der Nebenbedingungen (5) selbst ist eine Folge von (5). Somit gilt der Satz (nur für den euklidischen R_n):

Notwendig und hinreichend für die vollständige Integrabilität der $I_{12 \cdots h}$-*Parallelverschiebung ist das Verschwinden des hten Krümmungstensors der* F_e.

Erste Bemerkung. Ist der $I_{12 \cdots m}$ der grösste Schmieg-Vektorraum, so ist die $I_{12 \cdots m}$-Parallelverschiebung total integrabel.

Dieser Satz ist eine triviale Folge des vorhergehenden. Wir können ihn aber ohne Mühe direkt ableiten. Die Einbettungszahl ϵ der F_e fällt ja mit der Dimensionszahl des $I_{12 \cdots m}$ zusammen, d. h. die F_e liegt ganz in einem E_ϵ so, dass *jeder* Raumvektor λ_i des E_ϵ, der in einem Punkt der F_e definiert ist, durch (seine) Flächenkomponenten (4) beschreibbar ist.

Weiter ist die $I_{12 \cdots m}$-Parallelverschiebung hier identisch mit der Parallelverschiebung der E_ϵ, *also vollständig integrabel.* (Da es keinen Normalvektorraum zum $I_{12 \cdots m}$ in der E_ϵ gibt, ist $d\lambda_i = 0$ die Gleichung der $I_{12 \cdots m}$-Parallelverschiebung.)

Zweite Bemerkung. Die $I_{12 \cdots h}$-Parallelverschiebung gibt Anlass zur Definition der "$I_{12 \cdots h}$-*Geodätischen*" als jenen Kurven, die enstehen, wenn ein Vektor λ_i des $I_{12 \cdots h}$ stets in der Richtung $I_{12 \cdots h}$-parallel verschoben wird, die durch seine Projektion in den Tangentialraum I_1 gegeben ist.

7. Die F_e im R_n konstanter Krümmung

Da wir jetzt nicht mehr den Vorteil haben, kartesische Koordinaten verwenden zu können, müssen wir für die folgende Betrachtung das *absolute Raumdifferential* von Vektoren resp. Tensoren des R_n verwenden statt des gewöhnlichen Differential wie bisher.

Ist also λ^i ein (kontravarianter) Raumvektor, so bezeichne $\vartheta\lambda^i$

$$\vartheta\lambda^i = d\lambda^i + \left\{ {i \atop jk} \right\} \lambda^j dx_k \qquad (i, j, k = 1, \cdots, n) \tag{1}$$

das absolute Differential dieses Vektors.

Für den Masstensor g_{ik} des R_n gilt $\vartheta g_{ik} = 0$.

Bezeichnung. Für Raum-Indizes seien die Buchstaben $a, b, \cdots, k$ verwendet, für Flächen-Indizes die übrigen: $l, m, n, \cdots$. (Also laufen a, b bis k von 1 bis n, und l, m bis z von 1 bis l.) Wenn wir von *absoluter Ableitung* sprechen, so denken wir vorerst nur an die Transformationen der Raumkoordinaten. Wir denken uns also in der in Parameterform gegebenen F_e

$$x_i = x_i(y_1, \cdots, y_e) \qquad (i = 1, \cdots, n) \tag{2}$$

die y fest. (Wie sich dann die definierten Grössen gegen Parametertransformation verhalten, werden wir genau wie im euklidischen Fall ohne Mühe konstatieren.)

Was die Schmieg-Vektorräume $I_{12\cdots h}$ betrifft, so müssen wir für ihre Definition die absoluten Ableitungen verwenden. Die Vektoren

$$\frac{\partial x_i}{\partial y_p} \qquad (p = 1, \cdots, l) \tag{3}$$

spannen den I_1 auf. Durch absolute Differentiation der Vektoren (3) gewinnen wir die Raum-Vektoren

$$\frac{\vartheta^2 x_i}{\vartheta y_q \vartheta y_p} = \frac{\vartheta}{\vartheta y_q}\left(\frac{\partial x_i}{\partial y_p}\right) = \frac{\partial^2 x_i}{\partial y_p \partial y_q} + \left\{ {i \atop jk} \right\} \frac{\partial x_j}{\partial y_p} \frac{\partial x_k}{\partial y_q}, \tag{4}$$

die mit den Vektoren (3) zusammen den I_{12} aufspannen. Eine Parametertransformation (§ (1), (4)) führt zu

$$\frac{\partial x_i}{\partial \bar{y}_p} = \frac{\partial x_i}{\partial y_r} \frac{\partial y_r}{\partial \bar{y}_p}. \tag{5}$$

Da für die absolute Ableitung die Rechenregeln wie für die gewöhnliche gelten, folgt aus (5)

$$\frac{\vartheta^2 x_i}{\vartheta \bar{y}_q \vartheta \bar{y}_p} = \frac{\vartheta^2 x_i}{\vartheta y_r \vartheta y_s} \frac{\partial y_r}{\partial \bar{y}_p} \frac{\partial y_s}{\partial \bar{y}_q} + \frac{\partial x_i}{\partial y_r} \frac{\partial^2 y_r}{\partial \bar{y}_p \partial \bar{y}_q}, \tag{6}$$

wo $\partial^2 y_r/\partial\bar{y}_p\partial\bar{y}_q$ die gewöhnliche zweite Ableitung von y_r nach $\bar{y}_p$, $\bar{y}_q$ ist. (Die absolute Ableitung betrifft ja nur Raum-Indizes.)

Aus (5) und (6) folgt die Invarianz der I_1, I_{12} gegen Parameter-Änderung. Wir definieren nun genau wie zuvor den I_2 als grössten Untervektorraum des I_{12} normal zum I_1. Wenn dann wieder

$$\underline{\frac{\vartheta^2 x_i}{\vartheta y_p \vartheta y_q}} \tag{7}$$

die Projektion der zweiten absoluten Ableitung $\vartheta^2 x_i/\vartheta y_p \vartheta y_q$ in den I_2 bezeichnet, so folgt wie früher aus (6) der Flächentensorcharakter der Grössen (7), die den I_2 aufspannen.

Die Definition der $I_{12\ldots k}$- resp. I_k-Räume und der Nachweis der Invarianz gegen Parametertransformationen, sowie des Flächen-Tensorcharakters der Grössen

$$\underline{\frac{\vartheta^k x_i}{\vartheta y_{p_1} \cdots \vartheta y_{p_k}}} \tag{8}$$

geschieht wie im §1 und bietet keine Schwierigkeiten.

Wir haben noch nicht benutzt, dass der R_n von konstanter Krümmung ist. Diese Voraussetzung wollen wir nun zur Herleitung einer wichtigen Relation verwenden:

Wir gehen dabei von der Relation aus

$$\begin{aligned} &\frac{\vartheta}{\vartheta y_r}\left[\frac{\vartheta}{\vartheta y_q}\left(\underline{\frac{\vartheta^t x_i}{\vartheta y_{p_1} \cdots \vartheta y_{p_t}}}\right)\right] - \frac{\vartheta}{\vartheta y_q}\left[\frac{\vartheta}{\vartheta y_r}\left(\underline{\frac{\vartheta^t x_i}{\vartheta y_{p_1} \cdots \vartheta y_{p_t}}}\right)\right] \\ &= -R^{i}_{\cdot jab}\, \underline{\frac{\vartheta^t x_j}{\vartheta y_{p_1} \cdots \vartheta y_{p_t}}}\, \frac{\partial x_a}{\partial y_q} \frac{\partial x_b}{\partial y_r}, \end{aligned} \tag{9}$$

die man aus (1) sofort ableitet, und in der

$$R_{ijab} = k(g_{ia}g_{jb} - g_{ib}g_{ja}) \tag{10}$$

der Krümmungstensor des R_n der konstanten Krümmung k ist. Setzt man (10) in (9) ein, so wird die rechte Seite gleich

$$-k\frac{\partial x_i}{\partial y_q}\left(g_{jb}\, \underline{\frac{\vartheta^t x_j}{\vartheta y_{p_1} \cdots \vartheta y_{p_t}}}\, \frac{\partial x_b}{\partial y_r}\right) + k\frac{\partial x_i}{\partial y_r}\left(g_{ja}\, \underline{\frac{\vartheta^t x_j}{\vartheta y_{p_1} \cdots \vartheta y_{p_t}}}\, \frac{\partial x_a}{\partial y_q}\right),$$

also Null für $t \neq 1$. Für $t = 1$ wird sie gleich

$$- k \frac{\partial x_i}{\partial y_q}\left(g_{jb}\frac{\partial x_j}{\partial y_{p_1}}\frac{\partial x_b}{\partial y_r}\right) + k\frac{\partial x_i}{\partial y_r}\left(g_{ja}\frac{\partial x_j}{\partial y_{p_1}}\frac{\partial x_a}{\partial y_q}\right)$$

$$= - k\frac{\partial x_i}{\partial y_s}(\delta_q^s B_{p_1 r} - \delta_r^s B_{p_1 q}),$$

wo

$$B_{pq} = g_{ik}\frac{\partial x_i}{\partial y_p}\frac{\partial x_k}{\partial y_q} \tag{11}$$

der Masstensor der F_e (des I_1) ist. Wir erhalten somit

$$\begin{aligned}&\frac{\vartheta}{\vartheta y_r}\left[\frac{\vartheta}{\vartheta y_q}\left(\underline{\frac{\vartheta^t x_i}{\vartheta y_{p_1}\cdots\vartheta y_{p_t}}}\right)\right] - \frac{\vartheta}{\vartheta y_q}\left[\frac{\vartheta}{\vartheta y_r}\left(\underline{\frac{\vartheta^t x_i}{\vartheta y_{p_1}\cdots\vartheta y_{p_t}}}\right)\right]\\ &= \begin{cases} 0, & \text{für } t \neq 1,\\ - k\dfrac{\partial x_i}{\partial y_s}(\delta_q^s B_{p_1 r} - \delta_r^s B_{p_1 q}), & \text{für } t = 1.\end{cases}\end{aligned} \tag{12}$$

Wir sind jetzt in der Lage die Symmetrie der Grössen (8) in bezug auf die Indizes $p_1, \cdots, p_k$ nachzuweisen. Aus der Definition dieser Grössen als Projektion von

$$\frac{\vartheta^k x_i}{\vartheta y_{p_1}\cdots\vartheta y_{p_k}}$$

in den I_k folgt

$$\frac{\vartheta^t x_i}{\vartheta y_{p_1}\cdots\vartheta y_{p_t}} = \underline{\frac{\vartheta^t x_i}{\vartheta y_{p_1}\cdots\vartheta y_{p_t}}} + \text{Vektor des } I_{12\cdots t-1}. \tag{13}$$

Durch absolute Differentiation nach y_q erhalten wir daraus

$$\begin{aligned}\frac{\vartheta}{\vartheta y_q}\left(\frac{\vartheta^t x_i}{\vartheta y_{p_1}\cdots\vartheta y_{p_t}}\right) = \frac{\vartheta^{t+1} x_i}{\vartheta y_q \vartheta y_{p_1}\cdots\vartheta y_{p_t}} &= \frac{\vartheta}{\vartheta y_q}\left(\underline{\frac{\vartheta^t x_i}{\vartheta y_{p_1}\cdots\vartheta y_{p_t}}}\right)\\ &\quad + \text{Vektor des } I_{12\cdots t}.\end{aligned} \tag{14}$$

Wir schreiben (14) anders

$$\frac{\vartheta}{\vartheta y_q}\left(\underline{\frac{\vartheta^t x_i}{\vartheta y_{p_1}\cdots\vartheta y_{p_t}}}\right) = \frac{\vartheta^{t+1} x_i}{\vartheta y_q \vartheta y_{p_1}\cdots\vartheta y_{p_t}} + \text{Vektor des } I_{12\cdots t}. \tag{14'}$$

Daraus folgt durch Differentiation nach y_r

$$\begin{aligned}\frac{\vartheta}{\vartheta y_r}\left[\frac{\vartheta}{\vartheta y_q}\left(\underline{\frac{\vartheta^t x_i}{\vartheta y_{p_1}\cdots\vartheta y_{p_t}}}\right)\right] &= \frac{\vartheta^{t+2} x_i}{\vartheta y_r \vartheta y_q \vartheta y_{p_1}\cdots\vartheta y_{p_t}}\\ &\quad + \text{Vektor des } I_{12\cdots t+1},\end{aligned} \tag{15}$$

und weiter durch Projektion in den I_{t+2}

$$(16)\qquad \underline{\frac{\vartheta}{\vartheta y_r}\left[\frac{\vartheta}{\vartheta y_q}\left(\frac{\vartheta^t x_i}{\underline{\vartheta y_{p_1}\cdots\vartheta y_{p_t}}}\right)\right]}_{\,t+2} = \frac{\vartheta^{t+2} x_i}{\underline{\vartheta y_r \vartheta y_q \vartheta y_{p_1}\cdots\vartheta y_{p_t}}}.$$

Aus (12) und (16) folgt *die Symmetrie der linken also auch der rechten Seite von* (16) *in r und q.*

Wir erhalten weiter aus (14′), bei Benutzung von (13) (für $t+1$ statt t)

$$(17)\qquad \frac{\vartheta}{\vartheta y_q}\left(\frac{\vartheta^t x_i}{\underline{\vartheta y_{p_1}\cdots\vartheta y_{p_t}}}\right) = \frac{\vartheta^{t+1} x_i}{\underline{\vartheta y_q \vartheta y_{p_1}\cdots\vartheta y_{p_t}}} + \text{Vektor des } I_{12\cdots t},$$

was durch Differentiation und darauf folgender Projektion in den I_{t+2}

$$(18)\qquad \underline{\frac{\vartheta}{\vartheta y_r}\left[\frac{\vartheta}{\vartheta y_q}\left(\frac{\vartheta^t x_s}{\underline{\vartheta y_{p_1}\cdots\vartheta y_{p_t}}}\right)\right]}_{\,t+2} = \underline{\frac{\vartheta}{\vartheta y_r}\left(\frac{\vartheta^{t+1} x_i}{\underline{\vartheta y_q \vartheta y_{p_1}\cdots\vartheta y_{p_t}}}\right)}_{\,t+2}$$

gibt, also (16),

$$(19)\qquad \underline{\frac{\vartheta}{\vartheta y_r}\left(\frac{\vartheta^{t+1} x_i}{\underline{\vartheta y_q \vartheta y_{p_1}\cdots\vartheta y_{p_t}}}\right)}_{\,t+2} = \frac{\vartheta^{t+2} x_i}{\underline{\vartheta y_r \vartheta y_q \vartheta y_{p_1}\cdots\vartheta y_{p_t}}}.$$

Haben wir gezeigt, dass

$$\frac{\vartheta^{t+1} x_i}{\underline{\vartheta y_q \vartheta y_{p_1}\cdots\vartheta y_{p_t}}}$$

in den unteren Indizes symmetrisch ist, so folgt die Symmetrie der rechten Seite von (19) in den Indizes $q, p_1, \cdots, p_t$. Aber wir zeigten soeben auch die Symmetrie in q und r. Das heisst also, da $(t=1)$ $\vartheta^2 x_i/\vartheta y_q \vartheta y_p$ (offenbar) symmetrisch ist, *dass die Grössen* (8) *symmetrische Flächentensoren darstellen.*

Wir haben damit alles hergeleitet, was nötig ist, um die Verhältnisse aus dem euklidischen R_n auf unseren Fall zu übertragen. Die Masstensoren der I_k-Räume sind wie dort

$$(20)\qquad E_{p_1\cdots p_k|q_1\cdots q_k} = g_{ij}\,\frac{\vartheta^k x_i}{\underline{\vartheta y_{p_1}\cdots\vartheta y_{p_k}}}\,\frac{\vartheta^k x_j}{\underline{\vartheta y_{q_1}\cdots\vartheta y_{q_k}}},$$

und aus ihnen werden die Grundtensoren wie dort durch symmetriesieren gewonnen. Die Frenet-Gleichungen §3 (11) ersetzt man durch ein völlig gleichgebautes System, nur dass für das gewöhnliche Differential auf der linken Seite das absolute Differential zu setzen ist. Dasselbe gilt für die Nebenbedingungen §4 (2), und für die wichtigen Formeln (5) und (7) des §4 erhalten wir dieselben Ausdrücke.

Was die Integrabilitätsbedingungen des neuen Systems der Frenet-Gleichungen betrifft, so lauten sie bis auf die zweite (nach (12)) genau wie die für den euklidischen Fall abgeleiteten.

Nur die der Gleichung ((15), §4) entsprechende lautet jetzt etwas anders, da der der Gleichung (§4 (13)) entsprechende Ausdruck für absolute Ableitungen jetzt nicht Null ist, sondern gleich ist der rechten Seite von (12). Aber an den Überlegungen, die dort zum Ziele führten, ist nichts zu ändern. Sie liefern genau wie dort den Beweis der Theoreme.

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ON THE HIGHER DERIVATIVES AT THE BOUNDARY IN CONFORMAL MAPPING†

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Introduction

Let R be a region bounded by a closed Jordan curve C, and $w=f(z)$ a function which maps the unit circle $|z|<1$ conformally on R. As we know, $f(z)$ is then continuous over the circle $|z|\leq 1$. In an earlier paper‡ we have investigated the conditions under which $f(z)$ is differentiable at a boundary point z_1 $(w_1=f(z_1))$, that is,

$$\lim_{z\to z_1}\frac{f(z)-f(z_1)}{z-z_1}=f'(z_1)$$

exists for unrestricted approach in $|z|\leq 1$, $z\neq z_1$, and in addition the conditions under which $f'(z)$ is continuous at each point of an arc of $|z|=1$. In the present paper we consider the corresponding questions for the higher derivatives and obtain results of similar nature, of which the following are the principal ones. Let $\Theta(s)$ be an angle from the direction of the positive axis of reals to the tangent line, where s denotes arc length. Let

$$\kappa^{(n)}(s)=d^n\Theta(s)/ds^n$$

be called the curvature of order n. If further

$$\lim\frac{\kappa^{(n-1)}(s)-\kappa^{(n-1)}(s')}{s-s'}$$

exists when s and s' $(s\neq s')$ approach s_1 simultaneously, we say that C has an L-curvature of order n at s_1.§

I. *If C has an L-curvature of order $(n-1)$ at w_1: $s=s_1$, and if*

$$(*)\qquad \int_0^a\left|\kappa^{(n-2)}(s+t)+\kappa^{(n-2)}(s-t)-2\kappa^{(n-2)}(s)\right|\frac{dt}{t^2}$$

converges for $s=s_1$, and if further $w_1=f(z_1)$, then $f^{(n-1)}(z)$ assumes continuous boundary values in a neighborhood of $z=z_1$, and is differentiable at z_1. (Theorem IV.)

† Presented to the Society, October 27, 1934; received by the editors November 8, 1934.

‡ Mathematische Zeitschrift, vol. 35 (1932), pp. 321–456. We refer to this as WR.

§ The idea of the L-curvature generalizes that of the L-tangent. The idea of the L-tangent was introduced by E. Lindelöf, the name "L-tangent" by A. Ostrowski. Cf. p. 312.

II. *If C has continuous curvature of order* $(n-1)$ *along an open arc c, and* $(*)$ *approaches zero uniformly with a on every closed subarc of c, then* $f^{(n)}(z)$ *assumes continuous boundary values on the arc* γ *which corresponds to c.* (Theorem III (b).)

An extension of II in the case $c \equiv C$ shows how the *modulus of continuity*† of $f^{(n)}(z)$ on $|z|=1$ depends on the given function $\kappa^{(n-1)}(s)$ and on some other simple properties of C (Theorem III (c)). Thus we obtain a result about the *equicontinuity* of the nth derivatives of the mapping functions at the boundary, for a family of curves which satisfy certain common conditions.

As is well known, the mapping function $f(z)$ varies continuously in $|z| \leqq 1$ under a suitable continuous deformation of C. R. Courant,‡ T. Radó§ and the author‖ have given conditions under which this is true. By means of the above-mentioned extension of II we prove an *analogous result for the derivatives* (Theorem V).

Earlier results on the higher derivatives of the mapping function were obtained by P. Painlevé,¶ O. D. Kellogg,†† and W. Seidel.‡‡ In all these cases the hypotheses involve an entire arc of the complete curve, whereas in I above we impose conditions merely at one point. The result of Painlevé, which infers the existence and continuity of $f^{(n)}(z)$ at $|z|=1$ from the continuity of $\kappa^{(n+1)}(s)$ on C, is a corollary of II. Kellogg's first paper yields the result that if $\kappa^{(n-1)}(s)$ exists and satisfies a Hölder condition, then $f^{(n)}(z)$ satisfies such a condition with the same exponent at the boundary. This result does not imply any of our results, nor is it implied by any of them. However, it can also be obtained by a modification of our method of proving Theorem III.§§ On the other hand the results of his second paper are easily seen to entail a special case of II.‖‖ Seidel proves that, if $\kappa^{(n-2)}(s)$ is absolutely continuous on C and $|\kappa^{(n-1)}(s)|^p$ $(p>1)$ is L-integrable, then $f^{(n-1)}(z)$ assumes absolutely continuous boundary values on $|z|=1$, $f^{(n)}(z)$ has radial boundary values, $f^{(n)}(e^{i\theta})$, almost everywhere on $|z|=1$, and $|f^{(n)}(e^{i\theta})|^p$ is L-integrable. This result neither contains any of our theorems nor is it contained in any of them.

† If $\phi(x)$ is defined and continuous on a closed interval I so that for every $\epsilon>0$ there is a $\delta(\epsilon)>0$ such that $|\phi(x)-\phi(x')| \leqq \epsilon$ if $|x-x'| \leqq \delta(\epsilon)$, for x, x' in I, then we call the function $\delta(\epsilon)$ a *modulus of continuity* of $\phi(x)$ (in the interval I).

‡ Göttinger Nachrichten, 1914, pp. 101–109, and 1922, pp. 69–70.

§ Acta Societatis Litterarum ac Scientiarum, Szeged, vol. 1 (1923), pp. 180–186.

‖ Göttinger Nachrichten, 1930, pp. 344–369 (Theorem III).

¶ Comptes Rendus, vol. 112 (1891), pp. 653–657.

†† These Transactions, vol. 13 (1912), pp. 109–132, and vol. 33 (1931), pp. 486–510.

‡‡ Mathematische Annalen, vol. 104 (1931), pp. 182–243 (Theorems 21, 22, 23).

§§ Cf. Göttinger Nachrichten, 1932, pp. 73–86.

‖‖ See p. 326.

If, however, one is interested only in the existence at a given point, or only in the existence and continuity on an arc, of the nth derivative at the boundary, then the results of the present paper are the less restrictive.

I. Preliminary theorems

1. The converse of a theorem of Lindelöf. If a Jordan arc c has a tangent at a point P and if every cord P_1P_2 of c ($P_1 \neq P_2$) approaches the tangent at P as P_1 and P_2 approach P simultaneously, we say that c has an L-tangent at P. This idea was introduced by Lindelöf† in the statement of the following theorem, due to him: Let $w=f(z)$ be regular in the circle $|z|<1$ and let $f(z)$ map a neighborhood $\{|z-1|<r,\ |z|<1\}$ of $z=1$ conformally on a region bounded by a closed Jordan curve which has an L-tangent at the point $w_1=f(1)$. Then $\lim_{z\to 1} \operatorname{arc} f'(z)$ exists for unrestricted approach in $|z|<1$.

We shall need the following converse of Lindelöf's theorem:

Theorem I. *Let $f(z)$ be regular in $|z|<1$ and let any branch of* arc $f'(z)$ *be harmonic in the region $\mathfrak{N}_0$ $\{|z-1|<r_0<1,\ |z|<1\}$. Let* $\lim_{z\to 1} \operatorname{arc} f'(z)$ *exist for unrestricted approach in $|z|<1$. Then we have*

(1) *$f(z)$ assumes continuous boundary values $f(e^{i\theta})$ on an arc γ of $|z|=1$‡ with mid-point $z=1$. Furthermore,*

$$\lim_{r\uparrow 1} f'(re^{i\theta}) = f'(e^{i\theta})\S$$

exists almost everywhere on γ and

$$f(e^{i\theta}) - f(1) = \int_0^\theta f'(e^{it})ie^{it}dt,$$

$|f'(e^{it})|$ being integrable on γ in the sense of Lebesgue.

(2) *For some $r>0$, $w=f(z)$ is univalent in $\mathfrak{N}$: $\{|z-1|<r,\ |z|\leqq 1\}$; thus the boundary of $\mathfrak{N}$ is mapped on a closed Jordan curve Γ.*

(3) *Γ has an L-tangent at the point $w_1=f(1)$.*

The following lemma will be used in the proof.

Lemma 1. *If $f(z)$ is regular in a convex region D and if $\mathfrak{R}(f'(z))=u(z)>0$ in D, then $f(z)$ is univalent in D.*‖

† *Compte Rendu du 4ième Congrès des Mathématiciens Scandinaves à Stockholm* (1916), pp. 89–91. The term "L-tangent" was introduced by A. Ostrowski, Acta Mathematica, vol. 64 (1934), pp. 81–185, see p. 93.

‡ We say that a function $f(z)$, regular in $|z|<1$, assumes continuous boundary values on an arc γ of $|z|=1$ if there is a function $f(e^{i\theta})$ continuous on γ such that $\lim_{z\to e^{i\theta}} f(z) = f(e^{i\theta})$ for unrestricted approach.

§ The symbol $\alpha \uparrow a$, which was introduced by A. Ostrowski, means that α approaches a monotonically from below.

‖ See J. Wolff, Comptes Rendus, vol. 198 (1934), pp. 1209–1210.

For, we have first

$$f(z_2) - f(z_1) = \int_{z_1}^{z_2} f'(\zeta)d\zeta,$$

the integral being taken along the straight line from z_1 to z_2 in D. Set $z_2 = z_1 + le^{i\alpha}$, $\zeta = z_1 + \lambda e^{i\alpha}$, $0 \leqq \lambda \leqq l$. Then

$$|f(z_2) - f(z_1)| = \left|\int_0^l f'(\zeta)e^{i\alpha}d\lambda\right| = \left|\int_0^l f'(\zeta)d\lambda\right| \geqq \int_0^l u(\zeta)d\lambda > 0,$$

the last integral being zero if and only if $z_1 = z_2$.

Proof of the Theorem I. (1) We may assume, without loss of generality, that $\lim_{z\to 1} \operatorname{arc} f'(z) = 0$. Let $r_1 > 0$ be chosen so that $|\operatorname{arc} f'(z)| \leqq \eta$ in $\mathfrak{N}_1$: $\{|z-1| < r_1, |z| < 1\}$, where η denotes any positive number $< (\log 2)/(4e)$. We choose a subarc γ of the part of $|z| = 1$ belonging to the boundary of $\mathfrak{N}_1$ with mid-point $z = 1$, and join its end points by another Jordan arc γ' within $\mathfrak{N}_1$ in such a manner that γ and γ' form a closed Jordan curve C with continuous curvature. Let $z = \phi(\zeta)$ map the circle $|\zeta| < 1$ on the interior of C with $\zeta = 1$ corresponding to $z = 1$. The function $\phi(\zeta)$ has continuous boundary values on $|\zeta| = 1$ and continuous non-vanishing first derivative in $|\zeta| \leqq 1$.†

Since $|\operatorname{arc} f'(\phi(\zeta))| \leqq \eta$, to every positive $p < (\log 2)/(2e\eta)$ there corresponds a constant K, depending only on p and η, such that

$$(1.1) \qquad \int_0^{2\pi} |f'(\phi(\rho e^{i\tau}))|^{\pm p} d\tau \leqq K|f'(\phi(0))|^{\pm p}, \text{ if } 0 \leqq \rho < 1.‡$$

† See, for example, W. Seidel, Mathematische Annalen, vol. 104 (1931), p. 217, Theorem 18, and p. 226, Theorem 20, or WR, p. 433, Theorem 10, or the theorem of O. D. Kellogg quoted in the introduction.

‡ We use the following theorem (see Göttinger Nachrichten, 1930, p. 356, Lemma 1): Let $F(z) = U(z) + iV(z)$, with $U(0) = 0$, be regular in $|z| = |re^{i\theta}| < 1$ and let $|V(z)| \leqq \eta$, $\eta > 0$. Then for any $p < (\log 2)/(2e\eta)$ there is a constant $K(\eta, p) \geqq 1$ depending only on η and p, such that

$$\int_0^{2\pi} \exp\,[p|F(re^{i\theta})|]d\theta \leqq K(\eta, p),$$

if $0 \leqq r < 1$. We apply this to $F(\zeta) = \log f'(\phi(\zeta)) - \log |f'(\phi(0))|$ and obtain

$$\int_0^{2\pi} \exp\left[p\left|\log\frac{f'(\phi(\rho e^{i\tau}))}{|f'(\phi(0))|}\right|\right]d\tau \leqq K(\eta, p) \qquad (p > 0),$$

and since for any α: $\exp[\mathfrak{R}(\alpha)] \leqq \exp[|\alpha|]$, we have

$$\int_0^{2\pi} \left|\frac{f'(\phi(\rho e^{i\tau}))}{f'(\phi(0))}\right|^{\pm p} d\tau = \int_0^{2\pi} \exp\left[\pm p \log\left|\frac{f'(\phi(\rho e^{i\tau}))}{f'(\phi(0))}\right|\right]d\tau$$
$$\leqq \int_0^{2\pi} \exp\left[p\left|\log\frac{f'(\phi(\rho e^{i\tau}))}{|f'(\phi(0))|}\right|\right]d\tau \leqq K(\eta, p), \text{ which gives (1.1).}$$

As $(\log 2)/(2e\eta)>2$, we may take $p=1$ in (1.1). Since $|\phi'(\zeta)|$ is bounded in $|\zeta|\leqq 1$: $|\phi'(\zeta)|\leqq M$, we infer from (1.1) that

$$\int_0^{2\pi} |f'(\phi(\rho e^{i\tau}))|\,|\phi'(\rho e^{i\tau})|\,d\tau \leqq M\int_0^{2\pi} |f'(\phi(\rho e^{i\tau}))|\,d\tau \leqq \text{constant}$$

for every ρ in $0\leqq\rho<1$. Therefore, as is well known,†

$$\lim_{\zeta\to e^{i\tau}} f'(\phi(\zeta))\phi'(\zeta) = g(\tau)$$

exists for almost every τ, when ζ approaches $e^{i\tau}$ in any angle lying in $|\zeta|<1$ with vertex at $\zeta=e^{i\tau}$. Furthermore,

$$f(\phi(e^{i\tau})) - f(\phi(1)) = \int_0^{\tau} g(\tau)ie^{i\tau}d\tau,$$

$|g(\tau)|$ being integrable on $|\zeta|=1$. Since $\phi'(\zeta)$ is continuous in $|\zeta|\leqq 1$ and not zero, part (1) of the conclusion then follows.

Remark. If, for any branch of $\log f'(z)$ in $\mathfrak{N}_0$, we set

$$f^*(z) = \int_{z_0}^{z} \frac{\log f'(u)du}{u},$$

where the integral is taken along the straight line $z_0\,z$ in $\mathfrak{N}_0$ from a fixed point z_0, then $f^*(z)$ also assumes absolutely continuous boundary values $f^*(e^{i\theta})$ on the arc γ of $|z|=1$ and

$$f^*(e^{i\theta}) - f^*(1) = \int_0^{\theta} \log f'(e^{it})i\,dt,$$

where $|\log f'(e^{it})|$ is also integrable.

For since, for any z in the interior of the curve C mentioned above,

$$|\log f'(z)| \leqq |\log|f'(z)|| + |\operatorname{arc} f'(z)| \leqq |f'(z)| + \frac{1}{|f'(z)|} + \text{constant},$$

it follows from (1.1) for $p=1$ that the integral

$$\int_0^{2\pi} |\log f'(\phi(\rho e^{i\tau}))|\,|\phi'(\rho e^{i\tau})| \frac{d\tau}{|\phi(\rho e^{i\tau})|}$$

is bounded for $0\leqq\rho<1$, from which the conclusion follows.

(2) Part (2) of the conclusion of our theorem is an immediate consequence of Lemma 1 since we have in $\mathfrak{N}_1$ $|\operatorname{arc} f'(z)|\leqq\eta<\pi/2$ and therefore $\Re(f'(z))>0$.

† See, for example, F. Riesz, Mathematische Zeitschrift, vol. 18 (1923), pp. 87–95.

(3) The existence of the L-tangent at $w_1=f(1)$ evidently follows from the following fact. Let $\epsilon>0$ be an arbitrary number. Then, for any two points $z_1=e^{i\theta_1}$, $z_2=e^{i\theta_2}$, $\theta_1<\theta_2$, in a sufficiently small neighborhood of $z=1$, we have, for a suitable branch of the argument,

$$\left|\operatorname{arc}\,(f(z_2)-f(z_1))-\frac{\pi}{2}\right|\leqq\epsilon. \tag{1.2}$$

In order to prove this, we first note that we can choose $\delta=\delta(\epsilon)<\epsilon/2$ such that, for z belonging to T: $\{\,|\operatorname{arc} z|\leqq\delta(\epsilon),\ 0\leqq 1-|z|\leqq\delta(\epsilon)\}$

$$|\operatorname{arc} f'(z)|\leqq\frac{\epsilon}{2}\cdot \tag{1.3}$$

Let $z_1=e^{i\theta_1}$, $z_2=e^{i\theta_2}$, $\theta_1<\theta_2$, be two fixed points in T. According to Rolle's theorem, to each r with $1-\delta(\epsilon)<r<1$, there corresponds a point $z_0=re^{i\theta_0}$, $\theta_1<\theta_0<\theta_2$, such that for a suitable branch of arc $(f(rz_2)-f(rz_1))$

$$\operatorname{arc}\,(f(rz_2)-f(rz_1))=\operatorname{arc} f'(z_0)+\theta_0+\frac{\pi}{2}\cdot$$

Since $e^{i\theta_1}$ and $e^{i\theta_2}$ lie in T, $|\theta_0|\leqq\delta(\epsilon)<\epsilon/2$. Therefore, since r satisfies $1-\delta(\epsilon)<r<1$, it follows from (1.3) that

$$\left|\operatorname{arc}\,(f(rz_2)-f(rz_1))-\frac{\pi}{2}\right|\leqq\epsilon.$$

Since z_2, z_1 are fixed, we may let r approach 1. Hence (1.2) is valid.

2. **On a property of certain functions.** We prove the following theorem:

THEOREM II. *Let $h(z)=u(z)+iv(z)$ be regular in the circle $|z|<1$, $h(0)=0$, and let $\lim_{z\to 1}v(z)=v_0$ exist for unrestricted approach in $|z|<1$. Suppose that $\lim_{r\uparrow 1}h(r)=h_0$ exists. Then the functions*

$$F(z)=\int_0^z e^{h(u)}du \quad and \quad G(z)=\frac{1}{i}\int_0^z\frac{h(u)}{u}\,du,$$

regular in $|z|<1$, assume (absolutely) continuous boundary values on an arc γ of $|z|=1$ with mid-point $z=1$, and have the derivatives $F'(1)=e^{h_0}$, $G'(1)=h_0/i$ at $z=1$.†

† A function $f(z)$ which is regular in $|z|<1$ and defined on $|z|=1$ in a neighborhood of a point $z=z_1$ of $|z|=1$ is said to be differentiable at z_1 if

$$\lim_{z\to z_1}\frac{f(z)-f(z_1)}{z-z_1}$$

exists for unrestricted approach in $|z|\leqq 1$, $z\neq z_1$.

First we see, by applying Theorem I to $F(z)$, that $F(z)$ maps the interior of a certain region $\mathfrak{N}$: $\{|z-1|<r,\ |z|<1\}$ on the interior of a closed Jordan curve Γ which has an L-tangent at $w_1=F(1)$. Let $z=\phi(\zeta)$ be a function which maps the circle $|\zeta|<1$ on $\mathfrak{N}$ in such a manner that $\phi(1)=1$ and that the segment $-1\leqq\zeta\leqq 1$ corresponds to the segment $1-r\leqq z\leqq 1$. Then $\phi(\zeta)$ is analytic also in a neighborhood of $\zeta=1$, and $\phi'(1)\neq 0$. The function $e^{h(\phi(\zeta))}\phi'(\zeta)$ tends to $e^{h_0}\phi'(1)$ as ζ approaches 1 along the radius at $\zeta=1$. Therefore, $F(\phi(\zeta))$ satisfies the hypothesis of a theorem of the writer,† according to which

$$\left[\frac{dF(\phi(\zeta))}{d\zeta}\right]_{\zeta=1}$$

exists and is equal to $e^{h_0}\phi'(1)$.

Hence also, for unrestricted approach in $|z|\leqq 1, z\neq 1$,

$$\lim_{z\to 1}\frac{F(z)-F(1)}{z-1}=e^{h_0}. \tag{2.1}$$

We shall now use (2.1) to prove the conclusion regarding $G(z)$.

According to the Remark above, $G(z)$ is continuous on an arc γ of $|z|=1$ with mid-point $z=1$, and we have, for $e^{i\theta}$ on γ,

$$G(e^{i\theta})-G(1)=\int_0^\theta h(e^{it})dt,$$

.where

$$h(e^{it})=u(e^{it})+iv(e^{it})=\lim_{r\uparrow 1}h(re^{it})$$

exists for almost every e^{it} on γ and $|h(e^{it})|$ is integrable along γ. It is sufficient to prove the conclusion, namely

$$\lim_{z\to 1}\frac{G(z)-G(1)}{z-1}=\frac{1}{i}h_0,$$

only for the case that z approaches 1 along $|z|=1$ $(z\neq 1)$, that is, to prove that

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_0^\theta[h(e^{it})-h_0]dt=0. \tag{2.2}$$

---

† If $w=f(z)$ maps the circle $|z|<1$ on the interior of a closed Jordan curve C in such a manner that $z=1$ corresponds to w_1 on C, if C has an L-tangent in w_1 and if $\lim_{r\uparrow 1}f'(r)$ exists, then $f(z)$ is differentiable at $z=1$ and $f'(1)=\lim_{r\uparrow 1}f'(r)$. This is a special case of a theorem in WR, p. 376 (Theorem 3). Compare also the paper of the writer in Compositio Mathematica, vol. 1 (1935), p. 320.

For, since (2.2) implies that $\Phi(z)\equiv(G(z)-G(1))/(z-1)$ is bounded on γ, and since $\Phi(z)=O(1/|1-z|)$ in a region $\mathfrak{N}_1\{|z-1|\leqq r_1, |z|<1\}$ it follows from a well known theorem of Phragmén-Lindelöf that $\Phi(z)$ is bounded in $\mathfrak{N}_1$. Hence, according to a theorem of Lindelöf, it follows from (2.2) that $\lim_{z\to 1}\Phi(z)=h_0/i$ for unrestricted approach in $|z|<1$.

As $v(e^{i\theta})$, defined for almost every $e^{i\theta}$ on γ, is continuous at $\theta=0$ we have

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_0^\theta [v(e^{it}) - v_0]dt = 0.$$

Therefore, (2.2) is equivalent to

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_0^\theta [u(e^{it}) - u_0]dt = 0. \tag{2.3}$$

In order to prove (2.3), we first note that we may infer from (2.1), with the help of the relation

$$\int_{e^{i\theta}}^1 [e^{h(\zeta)} - e^{h_0}]d\zeta = - ie^{h_0}\int_0^\theta [e^{h(e^{it})-h_0} - 1]e^{it}dt \qquad (|\zeta| = 1),$$

that

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_0^\theta e^{[u(e^{it})-u_0]+i[v(e^{it})-v_0]}e^{it}dt = 1. \tag{2.4}$$

Since, according to Theorem I, (1), $|e^{h(e^{it})}|=e^{u(e^{it})}$ and hence $e^{u(e^{it})-u_0}$ is L-integrable along γ, we have, $|\theta|<\pi$,

$$\left|\frac{1}{\theta}\int_0^\theta e^{[u(e^{it})-u_0]+i[v(e^{it})-v_0]}(e^{it} - 1)dt\right| \leqq \frac{1}{\theta}\int_0^\theta e^{u(e^{it})-u_0}|e^{it} - 1|\,dt$$
$$\leqq \frac{2|\sin(\theta/2)|}{\theta}\int_0^\theta e^{u(e^{it})-u_0}dt,$$

which approaches 0 with θ. Therefore it follows from (2.4) that

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_0^\theta e^{[u(e^{it})-u_0]+i[v(e^{it})-v_0]}dt = 1. \tag{2.5}$$

Because of the continuity of $v(e^{i\theta})$ at $\theta=0$ it follows from (2.5) that

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_0^\theta [e^{u(e^{it})-u_0} - 1]dt = 0. \tag{2.6}$$

Now, since $-h(z)$ satisfies the hypotheses of the theorem, if $h(z)$ satisfies them, (2.1) remains true when we replace $F(z)$ by $F_1(z)=\int_0^z e^{-h(u)}du$ and e^{h_0} by e^{-h_0}. Therefore we obtain, by the method used to establish (2.6),

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_0^\theta [e^{-(u(e^{it})-u_0)}-1]dt = 0. \tag{2.7}$$

With the aid of the inequality

$$\alpha^2 \leqq e^\alpha + e^{-\alpha} - 2 = e^\alpha - 1 + e^{-\alpha} - 1,$$

which holds for every real α, we obtain from (2.6) and (2.7):

$$\frac{1}{\theta}\int_0^\theta [u(e^{it}) - u_0]^2 dt \leqq \frac{1}{\theta}\int_0^\theta [e^{(u(e^{it})-u_0)} - 1]dt$$
$$+ \frac{1}{\theta}\int_0^\theta [e^{-(u(e^{it})-u_0)} - 1]dt.$$

Hence

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_0^\theta [u(e^{it}) - u_0]^2 dt = 0.$$

From Schwarz's inequality it then follows that

$$\left(\frac{1}{\theta}\int_0^\theta [u(e^{it}) - u_0]dt\right)^2 \leqq \frac{1}{\theta}\int_0^\theta [u(e^{it}) - u_0]^2 dt$$

which proves (2.3). Therefore the theorem is true.

II. Existence and continuity of the nth derivative of the mapping function on the boundary

3. Lemmas. Let $V(s)$ be defined and continuous in the interval I: $A \leqq s \leqq B$, and, for an s in I and an $a>0$, let the integral

$$\int_0^a | V(s+t) + V(s-t) - 2V(s) | \frac{dt}{t^2} \tag{3.1}$$

exist. We shall make a few remarks about this integral which we shall use later.

(1) If $V'(s)$ exists and is continuous in I, and if

$$\int_0^a | V'(s+t) - V'(s-t) | \frac{dt}{t}$$

exists for an s in I, then also (3.1) exists and we have

$$\begin{aligned} &\int_0^a | V(s+t) + V(s-t) - 2V(s) | \frac{dt}{t^2} \\ &\leqq \int_0^a | V'(s+t) - V'(s-t) | \frac{dt}{t}\cdot \end{aligned} \tag{3.2}$$

For, if we set $\psi(\sigma)=V'(s+\sigma)-V'(s-\sigma)$ and if $0<\epsilon<a$, we have

$$\int_\epsilon^a |V(s+t)+V(s-t)-2V(s)|\frac{dt}{t^2} \leqq \int_\epsilon^a \frac{dt}{t^2}\int_0^t |\psi(\sigma)|\,d\sigma$$
$$= \iint_{T_\epsilon} \frac{|\psi(\sigma)|}{t^2}\,d\sigma dt,$$

the double integral being taken over the quadrilateral T_ϵ with vertices $\sigma=0,\ t=\epsilon;\ \sigma=\epsilon,\ t=\epsilon;\ \sigma=a,\ t=a;\ \sigma=0,\ t=a$. Now we observe that

$$\iint_{T_\epsilon}\frac{|\psi(\sigma)|}{t^2}\,d\sigma dt = \int_\epsilon^a |\psi(\sigma)|\int_\sigma^a \frac{dt}{t^2}\,d\sigma + \int_0^\epsilon |\psi(\sigma)|\int_\epsilon^a \frac{dt}{t^2}\,d\sigma$$
$$\leqq \int_0^a \frac{|\psi(\sigma)|}{\sigma}\,d\sigma,$$

and letting ϵ approach zero, we see that the result is true.

(1a) If $V'(s)$ satisfies a Hölder condition at $s=s_1$:

$$|V'(s_1+t)-V'(s_1)| \leqq H|t|^\beta, \qquad 0<\beta\leqq 1,$$

then, because of (3.2),

$$\int_0^a |V(s_1+t)+V(s_1-t)-2V(s_1)|\frac{dt}{t^2} \leqq \frac{2^\beta H}{\beta}a^\beta. \tag{3.3}$$

(2) *Sum and product.* We shall sometimes use the notation

$$\Delta_t^{(1)}V(s) \equiv V(s+t)-V(s), \qquad \Delta_t^{(2)}V(s)\equiv V(s+t)+V(s-t)-2V(s).$$

If $V_1(s)$ and $V_2(s)$ are defined and continuous in I, we have

$$\begin{aligned}\int_0^a |\Delta_t^{(2)}[V_1(s)+V_2(s)]|\frac{dt}{t^2} &\leqq \int_0^a |\Delta_t^{(2)}V_1(s)|\frac{dt}{t^2}\\ &\quad + \int_0^a |\Delta_t^{(2)}V_2(s)|\frac{dt}{t^2},\end{aligned} \tag{3.4}$$

provided both integrals on the right exist.

If, throughout I, $|V_1(s)|\leqq M_1$, $|V_2(s)|\leqq M_2$, if, for a certain value $s=s_0$ in $A<s<B$, (3.1) exists for $V(s)=V_1(s)$, $V(s)=V_2(s)$, and if

$$|\Delta_t^{(1)}V_1(s_0)|\leqq K_1|t|, \qquad |\Delta_t^{(1)}V_2(s_0)|\leqq K_2|t|$$

(M_1, M_2, K_1, K_2 being constants), we have

$$
\begin{aligned}
(3.5)\qquad \int_0^a \left| \Delta_t^{(2)} [V_1(s_0)V_2(s_0)] \right| \frac{dt}{t^2} \leqq M_1 \int_0^a \left| \Delta_t^{(2)} V_2(s_0) \right| \frac{dt}{t^2} \\
+ M_2 \int_0^a \left| \Delta_t^{(2)} V_1(s_0) \right| \frac{dt}{t^2} + 2K_1K_2a.
\end{aligned}
$$

For

$$
\begin{aligned}
\Delta_t^{(2)} [V_1(s_0)V_2(s_0)] &= V_1(s_0 + t)V_2(s_0 + t) + V_1(s_0 - t)V_2(s_0 - t) \\
&\quad - 2V_1(s_0)V_2(s_0) \\
&= V_1(s_0 + t)[V_2(s_0 + t) + V_2(s_0 - t) - 2V_2(s_0)] \\
&\quad + V_2(s_0 - t)[V_1(s_0 + t) + V_1(s_0 - t) - 2V_1(s_0)] \\
&\quad - 2V_1(s_0 + t)V_2(s_0 - t) + 2V_2(s_0)V_1(s_0 + t) \\
&\quad + 2V_1(s_0)V_2(s_0 - t) - 2V_1(s_0)V_2(s_0) \\
&= V_1(s_0 + t)\Delta_t^{(2)} V_2(s_0) + V_2(s_0 + t)\Delta_t^{(2)} V_1(s_0) \\
&\quad - 2[V_1(s_0 + t) - V_1(s_0)][V_2(s_0 - t) - V_2(s_0)].
\end{aligned}
$$

By multiplying this by $1/t^2$ and integrating over $0 \cdots a$, we obtain a relation from which (3.5) follows at once.

(3) *Change of variables.* Let (3.1) converge at $s = s_1$, $A < s_1 < B$, and let

$$
(3.6)\qquad | V(s') - V(s'') | \leqq K | s' - s'' |
$$

for s', s'' in I. Let $s = s(\theta)$, $A \leqq s(\theta) \leqq B$, be defined in the interval I^*, with continuous positive first derivative. Thus there are two constants μ_1, μ_2 such that for θ' and θ'' in I^*,

$$
(3.7)\qquad 0 < \mu_1 \leqq \frac{s(\theta') - s(\theta'')}{\theta' - \theta''} \leqq \mu_2.
$$

Suppose that $s_1 = s(\theta_1)$ and that for an $\alpha > 0$

$$
\int_0^\alpha | s(\theta_1 + \tau) + s(\theta_1 - \tau) - 2s(\theta_1) | \frac{d\tau}{\tau^2}
$$

exists. Then, if we denote $V(s(\theta))$ by $V^*(\theta)$,

$$
\int_0^\alpha | \Delta_\tau^{(2)} V^*(\theta_1) | \frac{d\tau}{\tau^2}
$$

also exists, and we have, for $a = s(\theta_1 + \alpha) - s(\theta_1)$,

$$
(3.8)\qquad \int_0^\alpha | \Delta_\tau^{(2)} V^*(\theta_1) | \frac{d\tau}{\tau^2} \leqq \frac{\mu_2^2}{\mu_1} \int_0^a | \Delta_t^{(2)} V(s_1) | \frac{dt}{t^2} + K \int_0^\alpha | \Delta_\tau^{(2)} s(\theta_1) | \frac{d\tau}{\tau^2}.
$$

For if, for sufficiently small $|\tau|$, we set $s(\theta_1+\tau)=s_1+t$, $s(\theta_1-\tau)=s_1-t'$, we have

$$\Delta_\tau^{(2)}V^*(\theta_1) = \Delta_t^{(2)}V(s_1) + V(s_1 - t') - V(s_1 - t).$$

It follows from (3.6) that

$$| V(s_1 - t') - V(s_1 - t) | \leqq K | t - t' | = K | s(\theta_1 + \tau) + s(\theta_1 - \tau) - 2s(\theta_1) |.$$

If we denote by $\theta(s)$ the inverse function of $s(\theta)$, we obtain

$$\int_0^\alpha |\Delta_\tau^{(2)} V^*(\theta_1)| \frac{d\tau}{\tau^2} \leqq \int_0^a |\Delta_t^{(2)} V(s_1)| \left(\frac{t}{\tau}\right)^2 \frac{d\theta(s_1+t)}{dt} \frac{dt}{t^2}$$
$$+ K \int_0^\alpha | \Delta_\tau^{(2)} s(\theta_1) | \frac{d\tau}{\tau^2},$$

from which (3.8) follows with the help of (3.7).

We shall also have to use the following simple lemmas.

LEMMA 2. *Let $f(t)$ be continuous for $A \leqq t \leqq B$, $0 < \lambda \leqq B - A \leqq \Lambda$, and let $\delta(\epsilon)$ be a modulus of continuity of $f(t)$. Suppose $|\int_A^B f(t)dt| \leqq m$, $m \geqq 0$. Then there is a number $M \geqq 1$, which depends only on m, λ, Λ and the function $\delta(\epsilon)$, but not otherwise on $f(t)$, such that in $A \leqq t \leqq B$: $|f(t)| \leqq M$.*

We may omit the simple proof of this lemma. We shall need the following corollary.

LEMMA 3. *Let $f(t)$ have in $A \leqq t \leqq B$, with $0 < \lambda \leqq B - A \leqq \Lambda$, a continuous nth derivative and let $\delta(\epsilon)$ be a modulus of continuity of $f^{(n)}(t)$. Suppose that $|\int_A^B f^{(\nu)}(t)dt| \leqq m$, $m \geqq 0$, $\nu = 1, 2, \cdots, n$. Then there exists a $K \geqq 1$, depending only on λ, Λ, m, and the function $\delta(\epsilon)$, such that, in $A \leqq t \leqq B$, $|f^{(\nu)}(t)| \leqq K$, $\nu = 1, 2, \cdots, n$.*

First, according to Lemma 2, there is an $M_n \geqq 1$, depending only on λ, Λ, m and $\delta(\epsilon)$, such that $|f^{(n)}(t)| \leqq M_n$ in $A \leqq t \leqq B$. Hence

$$| f^{(n-1)}(t_1) - f^{(n-1)}(t_2) | \leqq M_n | t_1 - t_2 |.$$

Therefore, we may apply Lemma 2 to $f^{(n-1)}(t)$ with the modulus of continuity $\delta^*(\epsilon) = \epsilon/M_n$. Thus we obtain an M_{n-1}, which also depends only on λ, Λ, m, and $\delta(\epsilon)$, such that $|f^{(n-1)}(t)| \leqq M_{n-1}$ in $A \leqq t \leqq B$. By applying the same method to $f^{(n-2)}(t)$ and $f^{(n-1)}(t)$, $f^{(n-3)}(t)$ and $f^{(n-2)}(t)$, and so on, the proof is easily completed.

4. Radial boundary values of the derivatives of analytic functions. We prove

LEMMA 4. *Let $F(z)=u(z)+iv(z)$ be regular in $|z|<1$ and let $v(z)$ be continuous in $|z|\leqq 1$. Set $v(e^{i\theta})=V(\theta)$.*

(a) *If, for a certain value of θ, $V'(\theta)$ and the integral*

$$(4.1)\qquad \int_0^a |V(\theta+\tau)+V(\theta-\tau)-2V(\theta)|\frac{d\tau}{\sin^2(\tau/2)},\qquad 0<a<\pi,$$

exist, then also $\lim_{r\uparrow 1}F'(re^{i\theta})$ exists and

$$(4.2)\qquad \begin{aligned} &ie^{i\theta}\lim_{r\uparrow 1}F'(re^{i\theta})\\ &\quad=\frac{1}{4\pi}\int_0^\pi\{V(\theta+\tau)+V(\theta-\tau)-2V(\theta)\}\frac{d\tau}{\sin^2(\tau/2)}+iV'(\theta).\dagger\end{aligned}$$

(b) *If $V'(\theta)$ exists and is continuous on the arc γ: $\theta_1\leqq\theta\leqq\theta_2(0<\theta_2-\theta_1\leqq 2\pi)$ of $|z|=1$ and if* (4.1) *exists for every θ on γ and approaches zero uniformly with a on γ, then $F'(z)$ assumes continuous boundary values on the open arc γ.*

(c) *Suppose that the hypotheses of* (b) *are satisfied for $\gamma\equiv\{0\leqq\theta\leqq 2\pi\}$ and that $V'(0)=V'(2\pi)$. Suppose $\delta(\xi)$ is a modulus of continuity of $V'(\theta)$, and $\eta(\xi)$ is a modulus of convergence‡ of* (4.1). *Then also $F'(z)$ assumes continuous boundary values $F'(e^{i\theta})$ on $|z|=1$ and the modulus of continuity of $F'(e^{i\theta})$, $\Delta(\epsilon)$, depends only on ϵ, and the functions $\delta(\xi)$ and $\eta(\xi)$. Furthermore, there is an upper bound for $|F'(e^{i\theta})|$, also depending only on the functions $\delta(\xi)$ and $\eta(\xi)$.*

By using Poisson's formula

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† Part (a) of this lemma has already been used by the author in WR, pp. 407 and 424. A theorem of similar nature was obtained by A. Plessner, *Zur Theorie der konjugierten trigonometrischen Reihen*, Dissertation, Giessen, 1923, p. 2. The corresponding generalization of condition (4.1) for functions harmonic within a sphere has been used by O. D. Kellogg, in the second paper mentioned in footnote †† on p. 311.

‡ We call a function $\eta(\xi)>0$ defined for $\xi>0$ a *modulus of convergence* of the integral (4.1) if for every $\xi>0$, over all of $|z|=1$, the integral (4.1) is $\leqq\xi$, provided that $0<a\leqq\eta(\xi)$.

Since $\sin(\tau/2)\leqq\tau/2$ and, for $0\leqq\tau\leqq\pi$, $\sin(\tau/2)\geqq\tau/\pi$, we have

$$\pi^2\int_0^a|\Delta_\tau^{(2)}V(\theta)|\frac{d\tau}{\tau^2}\geqq\int_0^a|\Delta_\tau^{(2)}V(\theta)|\frac{d\tau}{\sin^2(\tau/2)}\geqq 4\int_0^a|\Delta_\tau^{(2)}V(\theta)|\frac{d\tau}{\tau^2}.$$

Hence, if $\eta(\xi)$ is a modulus of convergence for

$$\int_0^a|\Delta_\tau^{(2)}V(\theta)|\frac{d\tau}{\sin^2(\tau/2)}$$

then $\eta(\xi)$ is also one for

$$\int_0^a|\Delta_\tau^{(2)}V(\theta)|\frac{d\tau}{\tau^2}.$$

Conversely, if $\eta(\xi)$ is a modulus of convergence for the second integral, $\eta(\xi/\pi^2)$ is one for the first one.

$$v(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(\phi)K(r, \phi - \theta)d\phi = \frac{1}{2\pi}\int_{\theta-\pi}^{\theta+\pi} V(\phi)K(r, \phi - \theta)d\phi$$

$$\left(K(r, \alpha) = \frac{1 - r^2}{1 + r^2 - 2r\cos\alpha} = \frac{1 - r^2}{(1 - r)^2 + 4r\sin^2(\alpha/2)}\right),$$

which holds for $0 \leqq r < 1$, we obtain by setting $\phi - \theta = \tau$

$$v(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(\theta + \tau)K(r, \tau)d\tau = \frac{1}{2\pi}\int_{0}^{\pi} \{V(\theta + \tau) + V(\theta - \tau)\}K(r, \tau)d\tau$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} \{V(\theta + \tau) + V(\theta - \tau) - 2V(\theta)\}K(r, \tau)d\tau + V(\theta).$$

Hence

$$\frac{\partial v(re^{i\theta})}{\partial r} = \frac{1}{\pi}\int_{0}^{\pi} \{V(\theta + \tau) + V(\theta - \tau) - 2V(\theta)\} \cdot \frac{(1 - r)^2 - 2(1 + r^2)\sin^2(\tau/2)}{[(1 - r)^2 + 4r\sin^2(\tau/2)]^2} d\tau.$$

The expression

$$\frac{(1 - r)^2 - 2(1 + r^2)\sin^2(\tau/2)}{[(1 - r)^2 + 4r\sin^2(\tau/2)]^2}\sin^2(\tau/2)$$

is uniformly bounded for $\frac{1}{2} \leqq r < 1$ and every real τ. Furthermore, it tends to $-\frac{1}{4}$ uniformly in the interval $\delta \leqq \tau \leqq \pi$ for any fixed positive δ as r approaches 1. Since the integral (4.1) approaches zero with a, we therefore see that

$$r\frac{\partial v(re^{i\theta})}{\partial r} = -\frac{\partial u(re^{i\theta})}{\partial \theta} \to -\frac{1}{4\pi}\int_{0}^{\pi} \frac{\{V(\theta + \tau) + V(\theta - \tau) - 2V(\theta)\}}{\sin^2(\tau/2)} d\tau$$

as r approaches 1, and also that this convergence is uniform in any interval $\theta_1 \leqq \theta \leqq \theta_2$ in which (4.1) tends uniformly to zero with a. Furthermore, according to a theorem of Fatou,

$$\lim_{r \uparrow 1} \frac{\partial v(re^{i\theta})}{\partial \theta} = V'(\theta)$$

for every θ for which

$$V'(\theta) = \frac{dV(\theta)}{d\theta}$$

exists, and the convergence is uniform in $\theta_1 \leqq \theta \leqq \theta_2$, if $V'(\theta)$ is continuous there. This establishes parts (a) and (b) of the theorem.

It follows immediately from (b) that, under the hypothesis of part (c), $F'(z)$ assumes continuous boundary values on $|z|=1$. To prove the remaining conclusion in (c), it is evidently sufficient to prove the result for the function $ie^{i\theta}F'(e^{i\theta})$ instead of $F'(e^{i\theta})$.

As far as the modulus of continuity of $ie^{i\theta}F'(e^{i\theta})$ is concerned, it remains only to show that the modulus of continuity of its real part depends only on ϵ and on the functions $\delta(\xi)$ and $\eta(\xi)$. In order to demonstrate this we first note that, since $\int_0^{2\pi}V'(\theta)d\theta=0$, according to Lemma 2 there is an $M\geqq 1$, depending only on $\delta(\xi)$, such that $|V'(\theta)|\leqq M$. Therefore $|V(\theta_1)-V(\theta_2)|\leqq M|\theta_1-\theta_2|$. Let $\epsilon>0$ be an arbitrarily small number. Then we have

$$\left|\int_0^{\pi}\frac{\{\Delta_\tau^{(2)}V(\theta_1)-\Delta_\tau^{(2)}V(\theta_2)\}d\tau}{\sin^2(\tau/2)}\right|\leqq\int_0^{\eta(\epsilon/3)}\frac{|\Delta_\tau^{(2)}V(\theta_1)|d\tau}{\sin^2(\tau/2)}$$

$$+\int_0^{\eta(\epsilon/3)}\frac{|\Delta_\tau^{(2)}V(\theta_2)|d\tau}{\sin^2(\tau/2)}+\int_{\eta(\epsilon/3)}^{\pi}\frac{|\Delta_\tau^{(2)}\{V(\theta_1)-V(\theta_2)\}|d\tau}{\sin^2(\tau/2)}$$

$$\leqq\frac{2\epsilon}{3}+\frac{4M\pi|\theta_1-\theta_2|}{\sin^2(\frac{1}{2}\eta(\epsilon/3))}\leqq\epsilon,$$

provided that

$$|\theta_1-\theta_2|\leqq\frac{\epsilon}{12M\pi}\sin^2(\tfrac{1}{2}\eta(\epsilon/3)).$$

Since

$$\int_0^{2\pi}\frac{dF(e^{i\theta})}{d\theta}d\theta=\int_0^{2\pi}ie^{i\theta}F'(e^{i\theta})d\theta=0,$$

it follows from Lemma 2 that there is an $N\geqq 1$, depending only on the modulus of continuity of $ie^{i\theta}F'(e^{i\theta})$, such that $|ie^{i\theta}F'(e^{i\theta})|=|F'(e^{i\theta})|\leqq N$ for all θ. This completes the proof.

The following lemma generalizes Lemma 4 (a) and (b).

LEMMA 5. *Let $F(z)=u(z)+iv(z)$ be regular in $|z|<1$ and let $v(z)$ assume continuous boundary values $V(\theta)$ on the arc γ: $\theta_1\leqq\theta\leqq\theta_2$ of $|z|=1$.*

(a) *If at $\theta=\theta_0=(\theta_1+\theta_2)/2$, $V'(\theta)$ exists and the integral* (4.1) *converges for an $a>0$, then $\lim_{r\uparrow 1}F'(re^{i\theta})$ exists.*

(b) *If $V'(\theta)$ is continuous over all of γ and if* (4.1) *exists at each point of a subarc γ': $\theta_1<\theta_1'\leqq\theta\leqq\theta_2'<\theta_2$ of γ and approaches zero uniformly with a on γ', then $F'(z)$ assumes continuous boundary values on the open arc γ'.*

Let Γ denote the closed Jordan curve formed by γ and the part within $|z|\leqq 1$ of the circle with center $z=e^{i\theta_0}$ and through the end points of γ. Γ is symmetric in the diameter of the circle $|z|=1$ through $z=e^{i\theta_0}$. Let $z=g(\zeta)$ map the circle $|\zeta|<1$ on the interior of Γ in such a manner that $g(1)=e^{i\theta_0}$ and that the segment $-1<\zeta<1$ corresponds to the part within Γ of the diameter of $|z|=1$ through $e^{i\theta_0}$. Let γ^* be the arc of $|\zeta|=1$ to which γ corresponds. The function $g(\zeta)$ is also regular on the open arc γ^* and $g'(\zeta)$ is not zero there.

Part (a) of our lemma follows immediately from Lemma 4 (a) applied to $F(g(\zeta))$. The function $F(g(\zeta))$ satisfies its hypotheses: for, first

$$\left(\frac{\partial v(g(e^{i\phi}))}{\partial\phi}\right)_{\phi=0} = V'(\theta_0)\,|\,g'(1)\,|\ \text{ exists,}$$

and secondly, because of the symmetry of Γ mentioned above, we have, if we set $g(e^{i\psi})=e^{i(\theta_0+\tau)}$ for sufficiently small $\psi>0$, $g(e^{-i\psi})=e^{i(\theta_0-\tau)}$, and hence,

$$\begin{aligned}\int_0^\alpha &\left|\, v(g(e^{i\psi})) + v(g(e^{-i\psi})) - 2v(g(1))\,\right| \frac{d\psi}{\psi^2}\\ &= \int_0^a \left|\, V(\theta_0+\tau)+V(\theta_0-\tau)-2V(\theta_0)\,\right| \left(\frac{\tau}{\psi}\right)^2 \frac{1}{|\,g'(e^{i\psi})\,|}\,\frac{d\tau}{\tau^2}\\ &\leqq \text{constant}\cdot\int_0^a \left|\, V(\theta_0+\tau)+V(\theta_0-\tau)-2V(\theta_0)\,\right| \frac{d\tau}{\tau^2},\end{aligned}$$

where α is defined by $g(e^{i\alpha})=e^{i(\theta_0+a)}$.

Part (b) follows from Lemma 4 (b) applied to $F(g(\zeta))$, since in this case $\partial v(g(e^{i\phi}))/\partial\phi = V'(\theta)\,|\,g'(e^{i\phi})\,|$ is continuous on the open arc γ^*, and since also the second hypothesis of Lemma 4 (b) is satisfied, as may be easily seen by use of §3, (3).

5. Boundary values of the *n*th derivative of the mapping function. Let c be a rectifiable Jordan arc with continuously turning tangent and let $\Theta(s)$ be an angle from the direction of the positive axis of reals to the tangent line, where s denotes the arc length. We define $\Theta(s)$ first at an arbitrary point of c and then at the other points so that $\Theta(s)$ varies continuously with s. If $\kappa^{(n)}(s)=d^n\Theta(s)/ds^n$ exists at $s=s_1$ we call $\kappa^{(n)}(s_1)$ the *curvature of order n of c at s_1*; $\kappa^{(0)}(s)$ is understood to be $\Theta(s)$. If $\kappa^{(n-1)}(s)$ exists in a neighborhood of $s=s_1$ and if

$$\lim \frac{\kappa^{(n-1)}(s)-\kappa^{(n-1)}(s')}{s-s'}$$

exists when s and s' $(s\neq s')$ approach s_1 simultaneously, we say that c has *an L-curvature of order n at s_1*.

THEOREM III. *Let R be a simply connected region. Let the boundary of R contain a free Jordan arc c.*† *Suppose $w=f(z)$ is a function which maps the circle $|z|<1$ on R.*

(a) *If in a neighborhood of a point $w_1(s=s_1)$ $\kappa^{(n-2)}(s)$ exists and has bounded difference quotient, if, for $s=s_1$, $\kappa^{(n-1)}(s)$ and*

$$\int_0^a \left| \kappa^{(n-2)}(s+t)+\kappa^{(n-2)}(s-t)-2\kappa^{(n-2)}(s)\right| \frac{dt}{t^2} \tag{5.1}$$

exist and if $w_1=f(1)$, then $\lim_{r\uparrow 1} f^{(n)}(r)$ exists.

(b) *If $\kappa^{(n-1)}(s)$ exists over all of c and is continuous, and if* (5.1) *exists at every point of a closed subarc c′ of the open arc c and approaches zero there uniformly with a, then $f^{(n)}(z)$ assumes continuous boundary values on the open arc γ': $\theta_1<\theta<\theta_2$ corresponding to c′. Furthermore, if $s(\theta)$ denotes the arc length along c, measured from a fixed point, $s^{(n)}(\theta)$ exists on γ' and is continuous.*‡

(c) *Suppose R is the interior of a closed Jordan curve C with continuous curvature of order $(n-1)$. Suppose that* (5.1) *exists at each point of C and approaches zero with a uniformly over all of C. Let $\delta(\xi)$ be a modulus of continuity of $\kappa^{(n-1)}(s)$ and $\eta(\xi)$ a modulus of convergence of* (5.1). *Furthermore, let D be a number such that the diameter of C is $\leqq D$. Let $\rho>0$ be the radius of a circle with the center $f(0)=w_0$ lying entirely in R, and let $d>0$ denote a constant such that $r/\sigma \geqq d$ where r is the distance between any two points of C and σ is the length of the shortest arc of C joining the two points.*

Then $f^{(n)}(z)$ has continuous boundary values over all of $|z|=1$ and $s^{(n)}(\theta)$ exists and is continuous there. Furthermore, there is a modulus of continuity of $f^{(n)}(e^{i\theta})$ and $s^{(n)}(\theta)$, $\Delta(\epsilon)$, which depends only on ϵ, D, ρ, d, and on the functions $\delta(\xi)$ and $\eta(\xi)$, and there is an upper bound for $|f^{(n)}(z)|$ and $|s^{(n)}(\theta)|$ which also depends only on D, ρ, d, and the functions $\delta(\xi)$ and $\eta(\xi)$.

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† That is, a Jordan arc the end points of which can be joined by another Jordan arc b lying in R except for the end points, such that b and c form a closed Jordan curve which bounds a region belonging to R. This idea is due to C. Carathéodory; see, for example, C. Carathéodory, ***Conformal Representation***, Cambridge University Press, 1932, p. 86.

‡ Theorem III (b) contains as a special case a result which can be obtained by applying to the logarithmic potential, and in particular to Green's function of a plane region, the method given by O. D. Kellogg in his investigations of the derivatives at the boundary, of harmonic functions in *space* (see footnote †† on p. 311, second paper). The special case which is obtained in this way is the following (see Theorem III, p. 491, loc. cit.): Let R, c, and $w=f(z)$ have the same meanings as in Theorem III of our paper. If $\kappa^{(n-1)}(s)$ exists on c and if there is a non-decreasing function $D(t)$, defined for $t>0$, for which $\int_0^a (D(t)/t^2)dt$ exists, such that

$$\left| \kappa^{(n-1)}(s+t)-\kappa^{(n-1)}(s)\right| \leqq D(t), \qquad t>0,$$

then $f^{(n)}(z)$ assumes continuous boundary values on the open arc γ corresponding to c.

Our proof of Theorem III (b) makes use of methods of conformal mapping and therefore admits no obvious generalization to the case of potential functions in space.

The following theorem† will be used in the proof.

THEOREM III*. *Let R, c, and $f(z)$ have the same meaning as in Theorem* III.

(a) *If c has a continuously turning tangent and if the angle $\Theta(s)$, defined above, satisfies a Hölder condition on c:*

$$(5.2)\qquad |\Theta(s)-\Theta(s')| \leqq K|s-s'|^{\beta},\quad 0<\beta<1,$$

then, on any fixed closed subarc γ' of the open arc γ corresponding to c, $f'(z)$ and $s'(\theta)=|f'(e^{i\theta})|$ exist, are not zero, and satisfy a Hölder condition with the same exponent:

$$(5.3)\qquad |s'(\theta)-s'(\theta')| \leqq |f'(e^{i\theta})-f'(e^{i\theta'})| \leqq H|\theta-\theta'|^{\beta}.$$

(b) *Suppose that R is the interior of a closed Jordan curve C, on which* (5.2) *is satisfied. Then the constant H in* (5.3) *depends only on β, K, and the three constants D, ρ, d introduced in Theorem* III (c). *Furthermore, there are two numbers μ_1 and μ_2, also depending only on those five constants, such that, in $|z|\leqq 1$: $0<\mu_1\leqq|f'(z)|\leqq\mu_2$.*

Since we shall prove Theorem III by induction, the following simple remarks will be of help to us:

If, in a neighborhood of a point s_1 of a Jordan arc c, $\kappa^{(m-1)}(s)$ exists and has bounded difference quotient, then also $\kappa^{(m-2)}(s)$, $\kappa^{(m-3)}(s)$, $\cdots$, $\kappa^{(1)}(s)$, $\Theta(s)$ have bounded difference quotients there. Hence, according to §3, (1a), each integral

$$(5.4)\qquad \int_0^a |\kappa^{(\nu)}(s+t)+\kappa^{(\nu)}(s-t)-2\kappa^{(\nu)}(s)|\frac{dt}{t^2}\qquad (\nu=1,2,\cdots,m-2)$$

exists and approaches zero uniformly with a in a neighborhood of $s=s_1$. Thus we have

Remark 1. If the hypotheses of part (a) of Theorem III are satisfied for the order $n=m+1$, then those of part (b) are satisfied for every smaller order $n=2,3,\cdots,m$, for a certain neighborhood of $s=s_1$. If the hypotheses of III (b) are fulfilled for the order $n=m+1$, they also are fulfilled for any smaller

† Part (a) of Theorem III* is a theorem of Kellogg from the first paper mentioned in footnote †† on p. 311 and is also proved in WR, p. 447. The statement of part (b) as to the dependence of H on β, K, D, ρ, d only, is proved in WR, pp. 451–452. This statement, as it is given in WR, says that H depends also on a lower bound l for the total length of C. But since we may take $l=2\rho$, the statement about H in III* (b) is true. The assertion in III* (b) concerning the existence of constants μ_1 and μ_2 which depend only on β, K, D, ρ, d is covered by the theorem on p. 440 (equation (10.16)) in WR which says that μ_1 and μ_2 depend only on D, ρ, d, a lower bound $l_1(=2\rho)$ for the total length of C, and the modulus of convergence $\delta'(\eta)$ of the integrals (10.5) on p. 440 in WR. But since $|x'(s+t)-x'(s)|=|\cos\Theta(s+t)-\cos\Theta(s)|\leqq|\Theta(s+t)-\Theta(s)|\leqq K|t|^{\beta}$, $|y'(s+t)-y'(s)|\leqq K|t|^{\beta}$, it is easily seen that $\delta'(\eta)$ can be chosen $(\beta\eta/K)^{1/\beta}$ and therefore depends only on β and K.

order and for any subarc of c as c'. In particular, we may infer from Theorem III*, using it for $\beta=\frac{1}{2}$, that under the hypotheses of part (a) or (b) or (c) of Theorem III for any $n\geqq 2$, we have relations of the form

$$(5.5)\qquad 0<\mu_1\leqq |f'(e^{i\theta})|=|s'(\theta)|\leqq \mu_2,$$

$$(5.6)\qquad |s'(\theta)-s'(\theta')|\leqq |f'(e^{i\theta})-f'(e^{i\theta'})|\leqq H|\theta-\theta'|^{1/2}$$

holding in a neighborhood of $\theta=0$ on $|z|=1$ in case (a)†, on any closed subarc of the open arc γ in case (b), and over all of $|z|=1$ in case (c).

Furthermore, it follows from §3, (1a), and from (5.6) that

$$(5.7)\qquad \int_0^\alpha |s(\theta+\tau)+s(\theta-\tau)-2s(\theta)|\frac{d\tau}{\tau^2}\leqq 4H\alpha^{1/2}\qquad(\alpha>0),$$

if $\theta\pm\alpha$ is inside any interval for which (5.6) is true.

Another remark will be of use in proving part (c):

Suppose the hypotheses of part (c) are satisfied for $n=m+1$. Denote by S the total length of C; evidently $2\rho\leqq S\leqq 2D/d$.‡ Since

$$\int_0^S \kappa^{(\nu)}(s)ds=\begin{cases}0 & \text{for } \nu>1,\\ 2\pi & \text{for } \nu=1,\end{cases}\qquad \nu=1,2,\cdots,m,$$

there is, according to Lemma 3, a constant K, depending only on ρ, $2D/d$, and the modulus of continuity $\delta(\xi)$ of $\kappa^{(m)}(s)$, such that over all of C

$$(5.8)\qquad |\kappa^{(\nu)}(s)|\leqq K\qquad(\nu=1,2,\cdots,m).$$

Hence, for any s_1, s_2,

$$(5.9)\qquad |\kappa^{(\nu)}(s_1)-\kappa^{(\nu)}(s_2)|\leqq K|s_1-s_2|\qquad(\nu=1,2,\cdots,m-1),$$

and for $0\leqq s_1\leqq s_2\leqq S$,

$$(5.10)\qquad |\Theta(s_1)-\Theta(s_2)|\leqq K|s_1-s_2|\leqq k|s_1-s_2|^{1/2},\quad k=K(2D/d)^{1/2}.$$

Thus we obtain

Remark 2. If the hypotheses of Theorem III (c) are satisfied for $n=m+1$, (5.8) holds, and every $\kappa^{(\nu)}(s)$, $\nu=1,2,\cdots,m-1$, has a modulus of continuity which depends only on ϵ, D, ρ, d, and on the modulus of continuity $\delta(\xi)$ of $\kappa^{(m)}(s)$. Furthermore, because of (5.9), each integral (5.4) which approaches 0 with a uniformly on C, has a modulus of convergence also depending only on ϵ, D, ρ, d, and on the function $\delta(\xi)$. Because of (5.10) it follows from Theorem

† It should be noticed that (5.5) is true on a closed neighborhood of $\theta=0$ under hypothesis (a) of Theorem III* because $f'(z)$ is continuous and different from 0 there.

‡ In order to prove $S\leqq 2D/d$ we need only apply the inequality $r/\sigma\geqq d$ to two points on C distant $\sigma=\frac{1}{2}S$ from each other *along* C.

III* (b) that μ_1, μ_2 in (5.5) and H in (5.6) and (5.7) depend only on D, ρ, d, and on $\delta(\xi)$.

6. Proof of Theorem III. We shall first show that if the theorem is true for the orders $n=2, 3, \cdots, m$, it is also true for $n=m+1$. In the proof of each of the three parts we thus assume that all three parts have been proved for $n=2, 3, \cdots, m$ and the hypotheses of the part in question are satisfied for $n=m+1$.

(1) Since $f'(z)$ is not zero in $|z|<1$ every branch of $\log f'(z)$ is regular there. We choose a branch of $\log f'(z)$ and note first that our proof will be completed if we prove, in each of the three cases, that the function $\partial^m \log f'(z)/\partial\theta^m$, $z=re^{i\theta}$, satisfies the result stated in the conclusion about $f^{(m+1)}(z)$. For we have, first,

$$\frac{\partial^m \log f'(z)}{\partial\theta^m} = \frac{d^m \log f'(z)}{dz^m}\left(\frac{\partial z}{\partial\theta}\right)^m + P_m(z),$$

where $P_m(z)$ is a polynomial in the derivatives of $\log f'(z)$ with respect to z of order $\leqq m-1$ and in derivatives of z with respect to θ. By replacing each expression $d^\mu \log f'(z)/dz^\mu$ by

$$\frac{d^{\mu-1}}{dz^{\mu-1}}\left[\frac{f''(z)}{f'(z)}\right] = \sum_{\beta=0}^{\mu-1}\binom{\mu-1}{\beta}\frac{d^\beta f''(z)}{dz^\beta}\,\frac{d^{\mu-\beta-1}}{dz^{\mu-\beta-1}}\left[\frac{1}{f'(z)}\right],$$

we obtain, by use of $\partial^\mu z/\partial\theta^\mu = i^\mu z$,

$$\frac{\partial^m \log f'(z)}{\partial\theta^m} = (iz)^m\frac{f^{(m+1)}(z)}{f'(z)} + R_m(z), \tag{6.1}$$

where $R_m(z)$ is a polynomial in the derivatives of $f(z)$ of order $\leqq m$, in z and $1/f'(z)$.

We supposed that the hypotheses of part (a), (b), or (c) as the case may be are satisfied for $n=m+1$. According to Remarks 1 and 2 above, we may therefore in each of the three cases apply the corresponding part of our theorem about $f''(z), f'''(z), \cdots, f^{(m)}(z)$, which is supposed to be proved, and also the results about $f'(z)$ stated in these remarks. Since this shows that $R_m(z)$ has the properties which are to be established for $f^{(m+1)}(z)$, it follows from (6.1) that $f^{(m+1)}(z)$ will have these properties, if $\partial^m \log f'(z)/\partial\theta^m$ has them.

In order to infer also the result of part (b) or (c) about $s^{(m+1)}(\theta)$ from this property of $\partial^m \log f'(z)/\partial\theta^m$, we first observe that, if $\partial^m \log f'(re^{i\theta})/\partial\theta^m$ $(0<r<1)$ has continuous boundary values on an arc $\theta_1<\theta<\theta_2$ of $|z|=1$, also $d^m \log f'(e^{i\theta})/d\theta^m$ exists on this arc and is equal to $\lim_{r\uparrow 1}\partial^m \log f'(re^{i\theta})/\partial\theta^m$. Therefore, it follows from

$$\frac{d^m \log |f'(e^{i\theta})|}{d\theta^m} = \frac{d^m \log s'(\theta)}{d\theta^m} = \frac{d}{d\theta}\left[\frac{d^{m-2}}{d\theta^{m-2}}\left(\frac{s''(\theta)}{s'(\theta)}\right)\right]$$
$$= \frac{d}{d\theta}\left[s^{(m)}(\theta)\frac{1}{s'(\theta)} + \cdots + s''(\theta)\frac{d^{m-2}}{d\theta^{m-2}}\frac{1}{s'(\theta)}\right]$$

that $s^{(m+1)}(\theta)$ exists on γ' or over all of $|z|=1$ respectively, since $s'(\theta)\neq 0$ there. Hence, we have

$$\frac{d^m \log s'(\theta)}{d\theta^m} = s^{(m+1)}(\theta)\frac{1}{s'(\theta)} + \cdots + s''(\theta)\frac{d^{m-1}}{d\theta^{m-1}}\frac{1}{s'(\theta)}, \tag{6.2}$$

from which we easily infer the result about the continuity of $s^{(m+1)}(\theta)$ in case (b) and the modulus of continuity and the bound of $s^{(m+1)}(\theta)$ in case (c).

(2) We now prove the result concerning $\partial^m \log f'(z)/\partial\theta^m$. Let us first assume that the hypotheses of part (a) are satisfied for $n=m+1$. According to Remark 1 above, the hypotheses of part (b) of our theorem are then also satisfied in a neighborhood of $s=s_1$ for $n=2, 3, \cdots, m$. Hence, if we set $\Theta^*(\theta)=\Theta(s(\theta))$, we see that

$$V(\theta) \equiv \frac{d^{m-1}}{d\theta^{m-1}}\Theta^*(\theta) = \kappa^{(m-1)}(s)(s'(\theta))^{m-1} + \cdots + \kappa^{(1)}(s)s^{(m-1)}(\theta) \tag{6.3}$$

exists in a neighborhood of $\theta=0$. Furthermore, since $\kappa^{(m)}(s)$ exists for $s=s_1$ and $s^{(m)}(\theta)$ at $\theta=0$, also

$$V'(\theta) = \frac{d^m}{d\theta^m}\Theta^*(\theta) = \kappa^{(m)}(s)(s'(\theta))^m + \cdots + \kappa^{(1)}(s)s^{(m)}(\theta)$$

exists at $\theta=0$.

The functions $\kappa^{(\nu)}(s)$ and $s^{(\nu)}(\theta)$, $\nu=1, 2, \cdots, m-1$, have bounded difference quotients in a neighborhood of $s=s_1$ and $\theta=0$ respectively. Hence $V(\theta)$ has this property in an interval $-\delta\leqq\theta\leqq\delta$ $(0<\delta\leqq\pi)$:

$$|V(\theta_1) - V(\theta_2)| \leqq k|\theta_1-\theta_2| \leqq k(2\pi)^{1/2}|\theta_1-\theta_2|^{1/2}. \tag{6.4}$$

By using (6.1) with m replaced by $m-1$, we see that $\partial^{m-1}\log f'(z)/\partial\theta^{m-1}$ assumes continuous boundary values on $|z|=1$ in a neighborhood of $z=1$. Since there, for a suitable branch of arc $f'(z)$, arc $f'(e^{i\theta})=\Theta^*(\theta)-\theta-\pi/2$, it is easily seen that the boundary function of the imaginary part of $\partial^{m-1}\log f'(z)/\partial\theta^{m-1}$ is $V(\theta)$, if $m>2$, and $V(\theta)-1$, if $m=2$. According to a theorem of Privaloff,† it follows from the fact that $V(\theta)$ satisfies a Hölder condition with the exponent $\frac{1}{2}$ for $-\delta\leqq\theta\leqq\delta$, that also the real part of the boundary function of $\partial^{m-1}\log f'(z)/\partial\theta^{m-1}$, that is, $d^{m-1}\log s'(\theta)/d\theta^{m-1}$, satis-

† I. Privaloff, Bulletin de la Société Mathématique de France, vol. 44 (1916), pp. 100–103.

fies a Hölder condition with the same exponent in any fixed interval $-\delta' \leqq \theta \leqq \delta'$ with $0<\delta'<\delta$:

$$(6.5) \qquad \left| \left[\frac{d^{m-1}\log s'(\theta)}{d\theta^{m-1}}\right]_{\theta=\theta_1} - \left[\frac{d^{m-1}\log s'(\theta)}{d\theta^{m-1}}\right]_{\theta=\theta_2} \right| \leqq h\,|\,\theta_1-\theta_2\,|^{1/2}.$$

For the proof of part (c) of our theorem it is important to notice that, if $V(\theta)$ satisfies the Hölder condition in (6.4) over all of $|z|=1$, the proof of Privaloff's theorem implies that also (6.5) is true over all of $|z|=1$ and that the constant h in (6.5) depends only on the constant k in (6.4).

From (6.5) we infer with the help of (6.2), used for $m-1$ instead of m, that, in the neighborhood of $\theta=0$, also

$$(6.6) \qquad |\,s^{(m)}(\theta_1) - s^{(m)}(\theta_2)\,| \leqq g\,|\,\theta_1-\theta_2\,|^{1/2} \qquad (g \text{ constant}).$$

Now we note that we may write (6.3) in the form

$$V(\theta) = \kappa^{(m-1)}(s)(s'(\theta))^{m-1} + \Pi_m(\theta)$$

where $\Pi_m(\theta)$ denotes a polynomial in $\kappa^{(\nu)}(s(\theta)), \nu=1, 2, \cdots, m-2$, and $s^{(\nu)}(\theta)$, $\nu=1, 2, \cdots, m-1$. Since by Remark 1 each integral (5.4) exists at the point $s=s_1$, it follows from §3, (3), and (5.5) and (5.7) that also the integrals

$$\int_0^\alpha |\,\kappa^{(\nu)}(s(\theta+\tau)) + \kappa^{(\nu)}(s(\theta-\tau)) - 2\kappa^{(\nu)}(s)\,|\,\frac{d\tau}{\tau^2} \qquad (\nu = 1, 2, \cdots, m-2)$$

exist at $\theta=0$ for an $\alpha>0$. Furthermore, since in a neighborhood of $\theta=0$ $s^{(\nu)}(\theta)$, $\nu=2, 3, \cdots, m-1$, have bounded difference quotients and $s^{(m)}(\theta)$ satisfies the Hölder condition (6.6), it follows from §3, (1 a), (3.3), that the integrals

$$\int_0^\alpha |\,s^{(\nu)}(\theta+\tau) + s^{(\nu)}(\theta-\tau) - 2s^{(\nu)}(\theta)\,|\,\frac{d\tau}{\tau^2} \qquad (\nu = 1, 2, \cdots, m-1)$$

exist at $\theta=0$ for an $\alpha>0$. According to §3, (2), therefore

$$(6.7) \qquad \int_0^\alpha |\,\Pi_m(\theta+\tau) + \Pi_m(\theta-\tau) - 2\Pi_m(\theta)\,|\,\frac{d\tau}{\tau^2}$$

exists at $\theta=0$.

Now we recall that by hypothesis the integral

$$(6.8) \qquad \int_0^a |\,\kappa^{(m-1)}(s+t) + \kappa^{(m-1)}(s-t) - 2\kappa^{(m-1)}(s)\,|\,\frac{dt}{t^2}$$

exists at $s=s_1$. Because of (5.5) and (5.7) we may apply §3, (3), once more and we see that also

$$\int_0^\alpha \left| \kappa^{(m-1)}(s(\theta+\tau)) + \kappa^{(m-1)}(s(\theta-\tau)) - 2\kappa^{(m-1)}(s(\theta)) \right| \frac{d\tau}{\tau^2}$$

exists at $\theta=0$ for an $\alpha>0$. Set $\Omega_m(\theta)=\kappa^{(m-1)}(s(\theta))(s'(\theta))^{m-1}$. According to §3, (2), (3.5), therefore

$$\int_0^\alpha \left| \Omega_m(\theta+\tau) + \Omega_m(\theta-\tau) - 2\Omega_m(\theta) \right| \frac{d\tau}{\tau^2} \tag{6.9}$$

exists at $\theta=0$ for an $\alpha>0$. From (6.7) and (6.9) we finally infer, with the help of (3.4), that

$$\int_0^\alpha \left| V(\theta+\tau) + V(\theta-\tau) - 2V(\theta) \right| \frac{d\tau}{\tau^2} \tag{6.10}$$

exists at $\theta=0$.

Let us now make a few remarks concerning the proof of part (b) and (c) of our theorem. If the hypotheses of (b) and (c) are satisfied, we see by examining part (2) of our proof above once more and by using Remarks 1 and 2:

First, that $V'(\theta)$ is continuous on the open arc γ or over all of $|z|=1$ respectively, and that in the latter case the modulus of continuity, $\delta^*(\epsilon)$, of $V'(\theta)$ depends only on ϵ, D, ρ, d, and the modulus of continuity, $\delta(\xi)$, of $\kappa^{(m)}(s)$. Furthermore, in this case k in (6.4), hence g in (6.6), also depend only on D, ρ, d and $\delta(\xi)$.

Secondly, we see that the integral (6.7) approaches zero uniformly with α on every closed subarc of the open arc γ or over all of $|z|=1$ respectively, and that in the latter case the modulus of convergence of (6.7) depends only on the same things as $\delta^*(\epsilon)$.

Thirdly, we see that (6.9) also, hence (6.10), approach zero uniformly with α on the arc γ' or over all of $|z|=1$ respectively, and that in case (c) the modulus of convergence $\eta^*(\epsilon)$ of (6.10) depends only on ϵ, D, ρ, d, and the modulus of continuity $\delta(\xi)$ of $\kappa^{(m)}(s)$, and the modulus of convergence $\eta(\xi)$ of (6.8).

Now we can complete the proof of all three parts simultaneously.

The function $V(\theta)-d^{m-1}\theta/d\theta^{m-1}$ is the boundary function of the imaginary part of

$$F(z) \equiv \frac{\partial^{m-1}\log f'(z)}{\partial\theta^{m-1}}, \quad z = re^{i\theta},$$

in a neighborhood of $\theta=0$ on $|z|=1$ in case (a), on the open arc γ in case (b), and on $|z|=1$ in case (c). Since, under the hypotheses of part (a), $V'(\theta)$ and (6.10) exist at $\theta=0$, the result which we wish to prove follows immediately from Lemma 5 (a). According to the three remarks which we have just made

concerning parts (b) and (c), these parts follow from Lemmas 5 (b) and 4 (c) respectively.

To complete the induction we have to show now that Theorem III is true for $n=2$. We obtain this proof for $n=2$ immediately from the preceding proof by setting $m+1=2$, since all properties of $f'(z)$ which are used in this proof are based on Theorem III*. This completes the proof of Theorem III.

7. **Existence of the nth derivative of the mapping function at a boundary point.** We prove the following theorem:

THEOREM IV. *Let R be a simply connected region. Let the boundary of R contain a free Jordan arc c. Suppose that the arc c has an L-curvature of order $n-1$ at an interior point $w_1(s=s_1)$ and that*

$$\int_0^a \left| \kappa^{(n-2)}(s_1+t)+\kappa^{(n-2)}(s_1-t)-2\kappa^{(n-2)}(s_1)\right| \frac{dt}{t^2} \tag{7.1}$$

exists for an $a>0$. If $w=f(z)$ is a function which maps the circle $|z|<1$ on R in such a manner that $z=1$ corresponds to w_1, then $f^{(n-1)}(z)$ assumes continuous boundary values on $|z|=1$ in a neighborhood of $z=1$ and is differentiable at $z=1$.

We shall use one part of the following lemma in the proof.

LEMMA 6. *Let $v(z)$ be harmonic in the circle $|z|<1$ and assume continuous boundary values $V(\theta)$ on $|z|=1$. A necessary and sufficient condition that*

$$\lim_{z\to 1} \frac{\partial v(re^{i\phi})}{\partial \phi}, \quad z = re^{i\phi},$$

exist for unrestricted approach in $|z|<1$ is that

$$\lim \frac{V(\phi)-V(\phi')}{\phi-\phi'} = V'(0) \tag{7.2}$$

exist when ϕ and ϕ' approach zero simultaneously. The limits $\lim_{z\to 1}\partial v(re^{i\phi})/\partial\phi$ and (7.2) are equal if one of them exists.

Proof of Lemma 6. (1) If (7.2) exists, then, for almost every α of a suitable interval $-\delta\leqq\alpha\leqq\delta$, $V'(\alpha)$ exists and is continuous at $\alpha=0$. Furthermore, for every α of this interval, $V(\alpha)-V(0)=\int_0^\alpha V'(\phi)d\phi$, the integral being taken in the sense of Lebesgue. From Poisson's formula

$$v(re^{i\phi}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(\alpha)K(r,\alpha-\phi)d\alpha, \qquad K(r,\psi) \equiv \frac{1-r^2}{1+r^2-2r\cos\psi},$$

which holds for $r<1$, we obtain

$$\frac{\partial v(re^{i\phi})}{\partial\phi} = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(\alpha)\,\frac{\partial K(r,\alpha-\phi)}{\partial\phi}\,d\alpha = -\frac{1}{2\pi}\int_{-\pi}^{\pi} V(\alpha)\,\frac{\partial K(r,\alpha-\phi)}{\partial\alpha}\,d\alpha$$

$$= -\frac{1}{2\pi}\left\{\int_{-\delta}^{\delta} + \int_{\delta}^{\pi} + \int_{-\pi}^{-\delta}\right\}.$$

By integrating the integral $\int_{-\delta}^{\delta} V(\alpha)(\partial K/\partial\alpha)d\alpha$ by parts, we thus get

$$\frac{\partial v(re^{i\phi})}{\partial\phi} = \frac{1}{2\pi}\int_{-\delta}^{\delta} V'(\alpha)K(r,\alpha-\phi)d\alpha + V(-\delta)K(r,\delta+\phi) - V(\delta)K(r,\delta-\phi)$$

$$-\frac{1}{2\pi}\left\{\int_{\delta}^{\pi} + \int_{-\pi}^{-\delta}\right\}.$$

Since $\lim_{\phi\to 0}V'(\phi) = V'(0)$ when $\phi\to 0$ over the point set on which $V'(\phi)$ is defined, it follows by a well known procedure that $\lim_{z\to 1}\partial v(re^{i\phi})/\partial\phi$ exists for unrestricted approach in $|z|<1$ and $=V'(0)$.

(2) If, conversely, $\lim_{z\to 1}\partial v(re^{i\phi})/\partial\phi$, $z=re^{i\phi}$, exists, then there is a $\delta>0$ such that $\partial v/\partial\phi$ is bounded in the sector $\{-\delta\leqq\phi\leqq\delta,\ 0<r<1\}$. According to Fatou's theorem, $\partial v(re^{i\phi})/\partial\phi$ therefore has radial boundary values $h(\phi)$ for almost every ϕ in $-\delta\leqq\phi\leqq\delta$. Evidently $h(\phi)$ is continuous at $\phi=0$. Hence, according to Lebesgue's integral theorem, we have

$$V(\phi) - V(0) = \lim_{r\uparrow 1}\int_0^{\phi}\frac{\partial v(re^{i\alpha})}{\partial\alpha}d\alpha = \int_0^{\phi}\lim_{r\uparrow 1}\frac{\partial v(re^{i\alpha})}{\partial\alpha}d\alpha = \int_0^{\phi} h(\alpha)d\alpha,$$

from which (7.2) with $V'(0)=h(0)$ is easily inferred.

Proof of Theorem IV. According to the Theorems III* and III (b), $f'(z)$, $f''(z), \cdots, f^{(n-1)}(z)$ assume continuous boundary values on a certain arc γ^*: $-\delta\leqq\theta\leqq\delta$ of $|z|=1$, and according to III (a) $\lim_{r\uparrow 1}f^{(n)}(r)$ exists. If we denote, as before, by $s(\theta)$ the arc length along c, then also $s'(\theta), \cdots, s^{(n-1)}(\theta)$ are defined and continuous in $-\delta\leqq\theta\leqq\delta$.

We choose a fixed branch of $\log f'(z)$ and consider $G^*(z)\equiv\partial^{n-2}\log f'(z)/\partial\theta^{n-2}$, $|z|=|re^{i\theta}|<1$. With the help of (6.1), used for $m=n-2$, it follows from what we have just noted about $f'(z), \cdots, f^{(n-1)}(z)$, and $f^{(n)}(z)$ that $G^*(z)$ assumes continuous boundary values on γ^* and that $\lim_{r\uparrow 1}G^{*\prime}(r)$ exists.

Let $\Theta(s)$ denote the angle defined as in the beginning of §5. If $\Theta^*(\theta) = \Theta(s(\theta))$, then $\Im(\log f'(e^{i\theta})) = \Theta^*(\theta)-\theta+C$ on γ^*, C being a constant which depends on the branch of $\log f'(z)$ chosen by us, and therefore

$$V^*(\theta) \equiv \frac{d^{n-2}}{d\theta^{n-2}}[\Theta^*(\theta)-\theta+C] = \kappa^{(n-2)}(s)(s'(\theta))^{n-2} + \cdots + \kappa^{(1)}(s)s^{(n-2)}(\theta)$$

$$-\frac{d^{n-2}}{d\theta^{n-2}}[\theta - C]$$

is the boundary function of the imaginary part of $G^*(z)$. Furthermore, it follows from our hypothesis that

$$\lim \frac{V^*(\theta) - V^*(\theta')}{\theta - \theta'} = V^{*\prime}(0)$$

exists when θ and θ' approach zero simultaneously.

Let Γ denote the closed Jordan curve formed by γ^* and the part within $|z| < 1$ of the circle with center $z = 1$ through $z = e^{i\delta}$ (and $e^{-i\delta}$). Γ is symmetric in the axis of reals. Let $z = g(\zeta)$ be the function which maps the circle $|\zeta| < 1$ on the interior of Γ in such a manner that $g(1) = 1$ and that the segment $-1 < \zeta < 1$ corresponds to the part of the real axis which lies in the interior of Γ. The function $z = g(\zeta)$ is also analytic on the open arc γ of $|\zeta| = 1$ which corresponds to γ^*, and $g'(\zeta) \neq 0$ here. $G(\zeta) = G^*(g(\zeta))$ is regular in $|\zeta| < 1$ and assumes continuous boundary values on $|\zeta| = 1$. The boundary function of its imaginary part on γ is $V(\phi) = V^*(\operatorname{arc} g(e^{i\phi}))$. Evidently

$$\lim \left(V(\phi) - V(\phi')\right)/(\phi - \phi')$$

exists as ϕ and ϕ' approach zero simultaneously. Hence, according to Lemma 6,

$$\lim_{\zeta \to 1} \Im\left(\frac{\partial G(\zeta)}{\partial \phi}\right), \qquad \zeta = \rho e^{i\phi},$$

exists for unrestricted approach in $|\zeta| < 1$. Furthermore,

$$\lim_{\rho \uparrow 1} G'(\rho) = \lim_{r \uparrow 1} G^{*\prime}(r) \cdot \lim_{\rho \uparrow 1} g'(\rho)$$

exists. Finally, we have, for $0 < \rho < 1$:

$$\left[\frac{\partial G(\zeta)}{\partial \phi}\right]_{\zeta = 0} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial G(\rho e^{i\phi})}{\partial \phi}\, d\phi = 0.$$

Therefore we may apply Theorem II with $h(\zeta) = \partial G(\zeta)/\partial \phi$, according to which

$$G(\zeta) - G(0) = \int_0^{\zeta} G'(u)du = \frac{1}{i} \int_0^{\zeta} \frac{h(u)}{u}\, du$$

has a derivative at $\zeta = 1$. Hence $G^*(z)$ has a derivative at $z = 1$. With the help of (6.1), used for $m = n - 2$, we easily infer from this that $f^{(n-1)}(z)$ is also differentiable at $z = 1$.

III. A CONVERGENCE THEOREM

8. Statement of Theorem V and a preliminary remark. The theorem in question is as follows:

THEOREM V. *Let C_m $(m=1, 2, \cdots)$ and C be closed Jordan curves, which are represented by the following functions, differentiable for $0 \leqq t \leqq T$:*

$$w = W_m(t) = U_m(t) + iV_m(t), \quad W_m'(t) \neq 0 \quad (m = 1, 2, \cdots);$$
$$w = W(t), \quad W'(t) \neq 0.\dagger$$

Suppose

(1) *that $W_m(t) \to W(t)$ for each t in $0 \leqq t \leqq T$ as $m \to \infty$,*

(2) *that $W_m(t)$ has a continuous nth derivative $W_m^{(n)}(t)$, $n \geqq 1$, which converges uniformly in $0 \leqq t \leqq T$ as $m \to \infty$,*

(3) *that the integrals*

$$\int_0^a \left| W_m'(t \pm u) - W_m'(t) \right| \frac{du}{u} \quad \text{if } n = 1,$$

$$\int_0^a \left| W_m^{(n-1)}(t + u) + W_m^{(n-1)}(t - u) - 2W_m^{(n-1)}(t) \right| \frac{du}{u^2} \quad \text{if } n > 1,$$

exist and approach zero with a uniformly for all t in $0 \leqq t \leqq T$ and all $m=1, 2, \cdots$.‡

Suppose, further, that there is a point w_0 in the interior R_m of C_m for every m and in the interior R of C. Let $f_m(z)$ and $f(z)$ map the circle $|z| < 1$ on R_m or R respectively in such a manner that $f_m(0) = f(0) = w_0$ and $f_m'(0) > 0$, $f'(0) > 0$. Then the functions $f_m^{(n)}(z)$ and $f^{(n)}(z)$ assume continuous boundary values on $|z| = 1$, and $f_m^{(n)}(z)$ converges uniformly in $|z| \leqq 1$ toward $f^{(n)}(z)$ as $m \to \infty$.

Before we prove this theorem we shall discuss a few simple consequences of the hypotheses (1) and (2) in the following

Remark. As is well known, hypothesis (2) implies that the family of functions $W_m^{(n)}(t)$, $m=1, 2, \cdots$, is equicontinuous§ for $0 \leqq t \leqq T$ and all

† Of course, $W_m(t)$ and $W(t)$ are defined for all real t, by the equations $W_m(t+T) = W_m(t)$, $W(t+T) = W(t)$.

‡ The difference in the nature of the two hypotheses (for $n=1$ and for $n>1$) in (3) is due to the difference between the types of conditions which we have obtained for existence and continuity of the derivatives at the boundary in these cases.

As a corollary, Theorem V is true if hypotheses (2) and (3) are replaced by the condition that there be a non-decreasing function $D(t)$ for which $\int_0^a (D(t)/t)dt$ converges, such that, for all t and m, $|W_m^{(n)}(t+u) - W_m^{(n)}(t)| \leqq D(u)$, $u > 0$.

§ A family of functions $\phi_n(x)$, $n=1, 2, \cdots$, defined for $a \leqq x \leqq b$, is said to be *equicontinuous*, if for every $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that $|\phi_n(x) - \phi_n(x')| \leqq \epsilon$ if $|x - x'| \leqq \delta(\epsilon)$, uniformly for all x, x' in $a \leqq x \leqq b$ and all $n = 1, 2, \cdots$.

$m=1, 2, \cdots$. Since $\int_0^T W_m^{(\nu)}(t)dt=0$, $\nu=1, 2, \cdots, n$, it follows from Lemma 3 that there is a constant k_1 such that for $0\leqq t\leqq T$ and $m=1, 2, \cdots$,

$$(8.1) \qquad | W_m^{(\nu)}(t) | \leqq k_1 \qquad (\nu = 1, 2, \cdots, n).$$

With the help of (8.1) and the hypotheses (1) and (2) we shall prove that also $W(t)$ *has a continuous nth derivative* $W^{(n)}(t)$ *and that* $W_m^{(\nu)}(t)$ *approaches* $W^{(\nu)}(t)$ $(\nu=0, 1, 2, \cdots, n)$ *uniformly for* $0\leqq t\leqq T$ *and thus for all real t.*

First we note that because of (8.1) we can choose from each subsequence of $[W_m(t)]$ a subsequence $[W_{m_k}(t)]$ such that $\lim_{k\to\infty} W_{m_k}^{(\nu)}(0)=c_\nu$ $(\nu=1, 2, \cdots, n-1)$ exists. Let $\Phi^{(n)}(t)$ denote the (continuous) function toward which $W_m^{(n)}(t)$ converges. Set

$$\Phi^{(\nu-1)}(t) = \int_0^t \Phi^{(\nu)}(\tau)d\tau + c_\nu \qquad (\nu = 1, 2, \cdots, n).$$

Then it follows from

$$W_m^{(\nu-1)}(t) = \int_0^t W_m^{(\nu)}(\tau)d\tau + W_m^{(\nu-1)}(0)$$

that $W_{m_k}^{(\nu)}(t)$ approaches $\Phi^{(\nu)}(t)$ uniformly $(\nu=0, 1, 2, \cdots, n-1)$ in $0\leqq t\leqq T$ when $k\to\infty$. Since by hypothesis (1) $\lim_{m\to\infty} W_m(t)=W(t)$, we have $\Phi^{(0)}(t)\equiv W(t)$. Therefore $W(t)$ has a continuous nth derivative and $W_{m_k}^{(\nu)}(t)$ approaches $W^{(\nu)}(t)$ uniformly $(\nu=1, 2, \cdots, n)$. Since every subsequence of $[W_m(t)]$ contains a subsequence $[W_{m_k}(t)]$, for which $W_{m_k}^{(\nu)}(t)$ *always* approaches uniformly *the same limit function* $W^{(\nu)}(t)$, it follows that the sequences $W_m^{(\nu)}(t)$ themselves approach $W^{(\nu)}(t)$ uniformly, $\nu=0, 1, \cdots, n$.

We shall need the following two corollaries of the fact that $W_m'(t)$ approaches $W'(t)$ uniformly.

a. *Since* $W_m'(t)\neq 0$ *and* $W'(t)\neq 0$, *there is evidently a constant* k_0 *such that*

$$(8.2) \qquad | W_m'(t) | \geqq k_0 \quad (m = 1, 2, \cdots), \qquad | W'(t) | \geqq k_0 \quad (0 \leqq t \leqq T).$$

b. *There is a constant* $d>0$, *independent of m, such that*

$$(8.3) \qquad \frac{r}{\sigma} \geqq d > 0,$$

where r is the distance between any two points P_1 *and* P_2 *on* $C_m (m=1, 2, \cdots)$ *and* σ *is the length of the shortest arc of* C_m *joining the two points.*

To prove (8.3), we note first that we can always assign to P_1 and P_2 values t_1 and t_2 of the parameter t such that $|t_1-t_2|\leqq T/2$ and $0\leqq t_1, t_2\leqq 2T$. According to the mean-value theorem, we have

$$\frac{r^2}{\sigma^2} \geqq \frac{|W_m(t_1) - W_m(t_2)|^2}{(\int_{t_2}^{t_1} |W_m'(t)| dt)^2} = \frac{U_m'^2(\tau_1) + V_m'^2(\tau_2)}{|W_m'(\tau_3)|^2}$$

where τ_1, τ_2, τ_3 are numbers between t_1 and t_2. Since the sequence $W_m'(t)$ is equicontinuous for $0 \leqq t \leqq 2T$, $m = 1, 2, \cdots$, there is a $\delta > 0$ such that

$$|W_m'(t') - W_m'(t'')| \leqq \frac{k_0^2}{8k_1} \text{ for } |t' - t''| \leqq \delta, \qquad 0 \leqq t', \ t'' \leqq 2T,$$

where k_0 and k_1 denote the constants in (8.1) and (8.2). Hence, if $|t_1 - t_2| \leqq \delta$,

$$\frac{r^2}{\sigma^2} \geqq \frac{|W_m'(\tau_3)|^2 - |U_m'^2(\tau_3) - U_m'^2(\tau_1)| - |V_m'^2(\tau_3) - V_m'^2(\tau_2)|}{|W_m'(\tau_3)|^2}$$

$$\geqq 1 - 2\frac{2k_1}{k_0^2}\frac{k_0^2}{8k_1} = \frac{1}{2}.$$

If $|t_1 - t_2| \leqq \delta$ this proves (8.3). Let P_1 and P_2 be points for which $|t_1 - t_2| > \delta$. As $|W_m(t_1) - W_m(t_2)| - |W(t_1) - W(t_2)|$ and

$$\int_{t_1}^{t_2} [|W_m'(t)| - |W'(t)|]dt$$

approach zero uniformly for $0 \leqq t_1, t_2 \leqq 2T$ when $m \to \infty$ and

$$\left| \int_{t_1}^{t_2} |W_m'(t)| dt \right| \geqq k_0\delta,$$

(8.3) holds for all C_m for which m is sufficiently great, since (8.3) is valid if P_1 and P_2 are points on C. But (8.3) is obviously true for the finite set of the remaining C_m's.

9. Proof of Theorem V. Let

$$s = \sigma_m(t) = \int_0^t |W_m'(t)| dt, \qquad 0 \leqq t \leqq T,$$

denote the arc length along C_m. It follows from the Remark above that the functions $W_m^{(\nu)}(t)$ $(\nu = 1, 2, \cdots, n)$ are equicontinuous and uniformly bounded for all t and $m = 1, 2, \cdots$. Because of (8.2), the same is true for $1/\sigma_m'(t)$.

Suppose first $n \geqq 2$. We have

$$\overset{(1)}{\kappa_m}(s) = \overset{(1)}{\kappa_m}(\sigma_m(t)) = \frac{U_m'(t)V_m''(t) - V_m'(t)U_m''(t)}{(\sigma_m'(t))^3}.$$

By differentiating $\kappa^{(1)}(\sigma_m(t))$ ν times with respect to t we easily see that

$\kappa_m^{(\nu)}(s)$ $(\nu=2, 3, \cdots, n-1)$ is a polynomial in $U_m^{(\mu)}(t)$, $V_m^{(\mu)}(t)$ $(\mu=1, 2, \cdots, \nu+1)$ and in $1/\sigma_m'(t)$. From this fact we infer first, with the help of (8.2), that *the functions $\kappa_m^{(n-1)}(s)$, considered as functions of s, are equicontinuous and uniformly bounded for all s and all $m=1, 2, \cdots$.* Furthermore, we infer from that fact and hypothesis (3), with the help of §3, (1 a), (3), and (2) and the inequality

$$(9.1) \qquad 0 < k_0 \leqq \frac{\sigma_m(t) - \sigma_m(t')}{t - t'} \leqq k_1,$$

that the integrals

$$\int_0^a \left| \kappa_m^{(n-2)}(s+u) + \kappa_m^{(n-2)}(s-u) - 2\kappa_m^{(n-2)}(s) \right| \frac{du}{u^2}$$

exist and approach zero with a uniformly for all s and $m=1, 2, \cdots$. *Thus there exists a common convergence modulus for all these integrals.*

If $n=1$, let $w_m(s)$ be the parametric representation of C_m, the arc length s being the parameter. Let $\dot{w}_m(s)$ denote $dw_m(s)/ds$. Since

$$\dot{w}_m(s) = \frac{W_m'(t)}{\sigma_m'(t)}$$

we see that the functions $w_m(s)$ are equicontinuous for all s and all $m=1, 2, \cdots$. Furthermore, it follows from hypothesis (3) and (9.1) that

$$\int_0^a \left| \dot{w}_m(s \pm u) - \dot{w}_m(s) \right| \frac{du}{u}$$

exists and approaches zero uniformly with a for all s and $m=1, 2, \cdots$.

Since $W_m(t)$ converges uniformly toward $W(t)$ as $m\to\infty$, there is a number $D>0$, such that the diameter of every C_m is $\leqq D$, and a number $\rho>0$, such that the circle with center w_0 and radius ρ lies entirely in every $R_m (m=1, 2, \cdots)$ and in R. Furthermore, (8.3) is true for every C_m, with d independent of m. Therefore we may apply to every C_m Theorem III (c) if $n>1$ and an analogous theorem about the first derivative† if $n=1$. According to these theorems the functions $f_m^{(n)}(z)$ assume continuous boundary values over all of $|z|=1$, *which are equicontinuous in z and $m=1, 2, \cdots$ and uniformly bounded.*

Therefore we may apply to the functions $f_m^{(n)}(e^{i\vartheta})$, $0\leqq\nu\leqq 2\pi$, a theorem of Ascoli,‡ according to which every subsequence of the $f_m^{(n)}(e^{i\vartheta})$ contains a

† See WR, p. 440.

‡ See, for example, O. D. Kellogg, *Foundations of Potential Theory*, Berlin, Springer, 1929, p. 265.

subsequence $[f_{m_k}{}^{(n)}(e^{i\theta})]$ which converges uniformly in $0 \leqq \theta \leqq 2\pi$, as $k \rightarrow \infty$. Hence also $f_{m_k}{}^{(n)}(z)$ approaches uniformly a continuous function $F(z)$ in $|z| \leqq 1$. According to Weierstrass's convergence theorem, $F(z)$ is analytic in $|z| < 1$.

Now it follows from the fact that $W_m(t)$ approaches $W(t)$ uniformly, that, in every fixed circle $|z| \leqq \rho < 1$, $f_m(z)$ converges uniformly toward $f(z)$, and therefore also that $f_m{}^{(n)}(z)$ approaches $f^{(n)}(z)$ there. Hence, in $|z| < 1$, $F(z) \equiv f^{(n)}(z)$, so that $f^{(n)}(z)$ has continuous boundary values on $|z| = 1$.

Since *every* subsequence of the $f_m{}^{(n)}(z)$ contains a uniformly convergent subsequence approaching *always the same limit function* $f^{(n)}(z)$ in $|z| \leqq 1$, it follows that the sequence $f_m{}^{(n)}(z)$ itself approaches $f^{(n)}(z)$ uniformly in $|z| \leqq 1$. Hence the theorem is true.

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SIMPLY CONNECTED SETS†

BY

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The class of *simply connected sets*, which is the object of study of the present paper, is closely related to the class of unicoherent sets introduced by Vietoris‡ and Kuratowski.§ A connected set is unicoherent if, however it be expressed as the sum of two connected and relatively closed subsets, the common part of the latter is connected. For locally connected metric sets the two classes coincide. In order that a connected and locally arcwise connected subset M of the plane be simply connected, it is necessary and sufficient that the interior of every simple closed curve lying in M be a subset of M. The notion of *simple connectedness in the weak sense* is also defined. The properties of sets of these types have a variety of applications and furnish an interesting background for a number of well known theorems.

I wish to express my thanks to Professor R. L. Moore, who greatly encouraged me in the writing of this paper.

A set A is *closed* in a set B if A lies in B and contains every point of B which is a limit point of A. If A is closed in B it is called a *relatively closed subset* of B. If F and G are subsets of a reference space S, F is said to be a *closed subset* of G if F lies in G and is closed in S.

In a space S let C be a connected set and L any set whatever. L is said to *disconnect*, or *separate* C if $C-C\cdot L$ is not connected. Let H and K be two mutually exclusive subsets of C neither of which intersects L. Then L is said (1) to *separate* H from K in C if $C-C\cdot L$ can be expressed as the sum of two mutually separated sets which contain H and K respectively, (2) to *disconnect* H from K in C if every connected subset of C that intersects both H and K contains a point of L, and (3) to *weakly disconnect* H from K in C if every connected and relatively closed subset of C that intersects both H and K contains a point of L.

The three types of separation defined in (1), (2), and (3) are successively weaker. However, if C is a locally connected metric space, and $L\cdot C$ is closed in C, the three types are completely equivalent.

A connected set M is said to be *simply connected* if for each pair of points A and B of M, and any relatively closed subset L of M that separates A from

† Presented to the Society, October 28, 1933; received by the editors January 19, 1935.

‡ *Ueber stetige Abbildungen einer Kugelfläche*, Akademie van Wetenschappen, Amsterdam, Proceedings, vol. 29 (1926), p. 445. Vietoris uses the term "henkellos."

§ *Une caractérisation topologique de la surface de la sphère*, Fundamenta Mathematicae, vol. 13 (1929), p. 308.

B in M, there exists a connected subset of L which separates A from B in M. This definition becomes the criterion for a connected set to be ***simply connected in the weak sense*** if "separates" is replaced throughout by "weakly disconnects." The properties of being simply connected and simply connected in the weak sense are intrinsic and topologically invariant.

Every simply connected set is simply connected in the weak sense. If a metric and locally connected space is simply connected in the weak sense, it is simply connected.

A simply connected metric space need not be locally connected. For example, the plane set consisting of the origin and the points (x, y) for which $0<x\leqq 1$, $y=\sin(1/x)$, is simply connected but not locally connected. If in addition, however, the space is locally compact, its local connectivity follows.

In the definition of simply connected sets the separating set L was required to be closed. It can be shown that, for metric sets, this requirement may be omitted. A similar remark does not apply to the definition of sets that are simply connected in the weak sense.

THEOREM 1. *If A and B are two points of a simply connected metric space M, and $G=\{g_i\}$ is a countable collection of mutually exclusive closed sets no one of which separates A from B in M, and G^* is compact in M, then G^* does not separate A from B in M.*

For suppose G^* does separate A from B in M. Then there exists a closed subset L of G^* which separates A from B in M. Since M is simply connected, there exists a component λ of L which separates A from B in M. The continuum λ is not a subset of any element of G since no element of G separates A from B in M. Thus λ is the sum, $\sum\lambda\cdot g_i$, of a countable number (greater than one) of mutually exclusive closed sets. But λ is compact. This contradicts a theorem of Sierpiński.†

THEOREM 2. *If A and B are two points of a locally connected metric space M, and H and K are two closed sets neither of which separates A from B in M, and if the complementary domain D of $H\cdot K$ that contains $A+B$ is simply connected, then $H+K$ does not separate A from B in M.*

THEOREM 3. *Let A and B be two points of a connected and locally arcwise connected metric space S, and let $G=\{g_i\}$ be a countable collection of closed sets such that* (1) *the common part of every pair of elements of G is the closed set H (which may be vacuous),* (2) *if b_1 and b_2 are two arcs from A to B that lie in $S-H$, then b_1+b_2 lies in a compact set which is simply connected in the weak sense and whose closure contains no point of H, and* (3) *G^* is locally compact. If no element of G separates A from B in S, then G^* does not separate A from B in S.*

† *Un théorème sur les continus*, Tôhoku Mathematical Journal, vol. 13 (1918), pp. 300–303.

On the contrary supposition there exists a subset F of G^* which is closed, contains H, separates A from B in S, and is irreducible with respect to these three properties. Denote by G' the collection $\{g_i'\}$, where $g_i' = F \cdot g_i$. The set $F-H$, which is locally self-compact, is the sum of the countable number of relatively closed subsets $g_i' - H$. Hence, by a theorem of R. L. Moore,† there exists a set $g_k' - H$ that contains a point which is not a limit point of $(F-H)-(g_k'-H)$. It easily follows that there exists an arc b_1 from A to B which has no point in common with any element of G' except g_k'. Moreover, since by hypothesis g_k does not separate A from B in S, there exists an arc b_2 from A to B which has no point in common with g_k'. The continuum b_1+b_2 lies in a compact set W which is simply connected in the weak sense and such that $\overline{W} \cdot H = 0$. The set $W \cdot F$ separates A from B in W. Hence some component w of $W \cdot F$ weakly disconnects A from B in W, and thus intersects both b_1 and b_2. Consequently w contains points of at least two elements of G'. Thus the compact continuum $\overline{w}$ is the sum, $\sum \overline{w} \cdot g_i'$, of a countable number (more than one) of mutually exclusive closed sets, contrary to a theorem of Sierpiński.‡

Theorem 4. *If a metric space is simply connected in the weak sense, it is unicoherent.*

That the converse of Theorem 4 is not true can be illustrated by simple examples of compact plane continua.

Theorem 5. *A necessary and sufficient condition that a connected and locally connected metric space M be simply connected is that, if K is any subcontinuum of M, the boundary of every complementary domain of K be connected.*

The condition is necessary. For let M be simply connected and suppose there exists a continuum K such that the boundary B of a complementary domain D of K is not connected. Since M is connected and locally connected, the boundary of every complementary domain of K is a non-vacuous subset of K. Hence $M-D$ is a continuum. Thus M is the sum of two continua, $M-D$ and $\overline{D}$, whose intersection is the disconnected set B. It follows from Theorem 4 that M is not simply connected, contrary to hypothesis.

Assume next that the condition is satisfied. Let L be a closed set which separates the point P from the point Q in M. Denote by D the complementary domain of L that contains P, by B the boundary of D, and by Δ the complementary domain of the continuum $D+B$ that contains Q. The boundary β of Δ is a subset of B. But B is a subset of L. And by assumption β is con-

† *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, p. 11, Theorem 15.

‡ Loc. cit.

nected. Thus β is a connected subset of L that separates P from Q in M. Therefore M is simply connected.

It has been proved by C. Kuratowski† that for a connected and locally connected metric space M the following three propositions are equivalent:

A. *If K is any subcontinuum of M, the boundary of every complementary domain of K is connected.*

B. *However M be expressed as the sum of two continua, the intersection of the latter is connected.*

C. *If H and K are any two mutually exclusive closed subsets of M, and P and Q are points of H and K respectively, there exists a continuum which separates P from Q in M and contains no point of $H+K$.*

Thus, in view of Theorem 5, we have the following result.

THEOREM 6. *Each of the conditions* B *and* C *is necessary and sufficient that a connected and locally connected metric space be simply connected.*

We next prove a result which, for the compact case, is a consequence of a closely related theorem‡ of W. A. Wilson.

THEOREM 7. *Let M be a metric, connected, and locally arcwise connected space. If every simple closed curve lying in M is contained in a subset of M which is simply connected in the weak sense, then M is simply connected.*

For suppose M is not simply connected. By Theorem 5 there exists a continuum K having a complementary domain D whose boundary B is not connected. Let $B=B_1+B_2$, where B_1 and B_2 are mutually separated sets. Let P_1P_2 be an arc contained in D except for its end points, which lie in B_1 and B_2 respectively. Let F be a closed set which separates B_1 from B_2 in M. Denote by Q_1 and Q_2 the first points of F on P_1P_2 in the orders from P_1 to P_2 and from P_2 to P_1 respectively. About each point X of K consider a connected region R_X such that (1) $R_X\cdot F=0$ if $X\not\subset F$, (2) $R_X\cdot\overline{D}=0$ if $X\subset F$, (3) $R_X\cdot(P_1P_2)=0$ if $P_1\neq X\neq P_2$, and (4) $R_{P_1}\cdot(Q_1P_2)=R_{P_2}\cdot(Q_2P_1)=0$. The sum of these regions is a connected domain Δ whose intersection with F has no point in common with $\overline{D}$. Let h be an arc in Δ with end points P_1 and P_2. Denote by P_2' the first point of h in the order from P_1 to P_2 which lies on P_2Q_2, and by P_1' the first point of h in the order from P_2' to P_1 which lies on P_1Q_1. Denote by u the subarc $P_1'P_2'$ of P_1P_2, and by v the subarc $P_1'P_2'$ of h.

By hypothesis the simple closed curve $u+v$ lies in a subset N of M which is simply connected in the weak sense. The set F separates P_1' from P_2' in

† *Sur les continus de Jordan et le théorème de M. Brouwer*, Fundamenta Mathematicae, vol. 8 (1926), pp. 148–150.

‡ *On unicoherency about a simple closed curve*, American Journal of Mathematics, vol. 55 (1933), pp. 135–145, Theorem of §11.

M since it separates P_1 from P_2 in M. Hence the set $F \cdot N$, which is closed in N, separates P_1' from P_2' in N. Consequently some component F_0 of $F \cdot N$ weakly disconnects P_1' from P_2' in N. Thus F_0 contains a point U of u and a point V of v. But $U \subset D$ and $V \not\subset D$. It follows that F_0 contains a point of the boundary of D, contrary to the construction of F.

COROLLARY 1. *A metric, connected, and locally arcwise connected space M is a simple closed curve if and only if it fails to be simply connected but becomes simply connected upon the omission of any one of its points.*

COROLLARY 2. *Let M be a metric and locally arcwise connected space. If σ is a monotonic ascending sequence of simply connected subdomains of M, then the sum of the domains of σ is also simply connected.*

THEOREM 8. *If D is a connected subdomain of a compact, metric, continuous curve M, and the boundary B of D has at least n components (n an integer), then D contains a compact continuum K such that every subset of M which separates K from B in M has at least n components.*

Let B be expressed as the sum of n mutually separated sets $B_1, \cdots, B_n$. Let F_i $(i=1, \cdots, n)$ be a closed set which separates B_i from $B-B_i$ in M. There exists a continuum K which lies in D and contains the closed set $(F_1+ \cdots +F_n)\cdot D$. Let P be a point of K and consider an arc PQ_i $(i=1, \cdots, n)$ that is contained in D except for the point Q_i, which lies in B_i. Let P_i be the first point of Q_iP that lies in K. If L is a subset of M which separates K from B in M, then L must contain a point Z_i of the arc P_iQ_i $(i=1, \cdots, n)$. Denote by λ_i the component of L that contains Z_i. These components are subsets of D. Moreover no two of them coincide. For if $\lambda_i=\lambda_j$ $(i\neq j)$, then $\lambda_i+Q_iZ_i+Q_jZ_j$ is a connected set which intersects B_i and B_j and contains no point of F_i, contrary to the construction of F_i. It follows that L has at least n components.

THEOREM 9. *A compact, metric, continuous curve M is simply connected if and only if every two mutually exclusive subcontinua of M can be separated in M by a third subcontinuum of M.*

Theorem 9 can be proved with the aid of Theorems 5 and 8.

The following variation of Theorem 9 may be stated: A compact, metric, continuous curve M is simply connected if and only if, given any two mutually exclusive subcontinua H and K of M, there exists but one component of $M-(H+K)$ whose closure intersects both H and K.

THEOREM 10. *A compact, metric, one-dimensional continuum M is simply connected in the weak sense if and only if there exists but one irreducible subcontinuum of it between any two of its points.*

The condition is necessary. For let M be simply connected in the weak sense and suppose there exist in M two distinct continua, L_1 and L_2, each of which is irreducible between the point A and the point B. Clearly A and B lie in different components of $I=L_1\cdot L_2$. Let $I=I_A+I_B$, where I_A and I_B are mutually separated sets containing A and B respectively. Let F be a closed subset of M which separates I_A from I_B in M. About each point P of F consider a domain (relative to M) whose closure contains no point of I and such that its boundary is totally disconnected. There exists a finite number of these domains, $D_1, \cdots, D_n$, whose sum D covers F. If β_i denotes the boundary of D_i, the set $\beta_1+\cdots+\beta_n$ is totally disconnected. Hence the boundary β of D, being a subset of $\beta_1+\cdots+\beta_n$, is totally disconnected. Now F separates A from B in M, and β separates $A+B$ from F in M; hence β separates A from B in M. Consequently there exists a component Q of β which weakly disconnects A from B in M, and Q must be a point. Since Q weakly disconnects A from B in M, it must intersect both L_1 and L_2. Thus Q is a point of I, contrary to the construction of β.

The sufficiency of the condition is obvious.

THEOREM 11. *In a locally compact, locally connected, simply connected, metric space M let H and K be two closed sets of which H is compact and whose intersection T is totally disconnected. If A and B are points of $H-T$ and $K-T$ respectively, there exists a compact continuous curve, lying in M, which separates A from B in M and contains no point of $(H+K)-T$.*

There exists a domain D which contains H and is compact in M, and there exists a closed subset F of D which separates $H-T$ from $K-T$ in M. The set F is compact. Since M is simply connected, there exists a component ϕ of F which separates A from B in M. We shall construct a compact continuous curve which contains ϕ and has no point in common with $(H+K)-T$.

Let $\Delta_1, \Delta_2, \cdots$ be a sequence of compact domains closing down on $\phi\cdot T$ such that $\bar{\Delta}_{i+1}\subset\Delta_i$ $(i=1, 2, \cdots)$. Denote by L_0 the compact closed set $\phi\cdot(M-\Delta_1)$, and by $L_i (i=1, 2, \cdots)$ the compact closed set $\phi\cdot(\bar{\Delta}_i-\Delta_{i+1})$. For each point P of L_i $(i=0, 1, 2, \cdots)$ let r_P be a compact and connected domain containing P such that (1) $\bar{r}_P\cdot(H+K)=0$, and (2) $\bar{r}_P\subset(\Delta_{i-1}-\bar{\Delta}_{i+2})$ if $i>1$. By the Borel-Lebesgue theorem there exists a finite number of the closed domains $\bar{r}_P$ whose sum R_i covers L_i. Thus R_i is a compact closed set which has only a finite number of components and contains no point of $H+K$. Consequently† each component of R_i can be imbedded in a compact contin-

† Special cases of the theorem required here are due to H. M. Gehman, G. T. Whyburn, W. L. Ayres, and others. For references to their results and Wilder's generalization, see R. L. Wilder, *On the imbedding of subsets of a metric space in Jordan continua*, Fundamenta Mathematicae, vol. 19 (1932), pp. 45–64.

uous curve, lying in M, which has no point in common with $H+K$, and which, if $i>1$, is a subset of $(\Delta_{i-1}-\bar{\Delta}_{i+2})$. The sum S_i of these continuous curves is a closed and locally connected set which contains L_i. It readily follows that $S=\sum S_i+\phi\cdot T$ is a compact continuum, containing ϕ, which is locally connected at every point of $\sum S_i$ and hence, since $\phi\cdot T$ is totally disconnected, at every point of S. Therefore S is a compact continuous curve which separates A from B in M and has no point in common with $(H+K)-T$.

A sequence of sets lying in a space S is said to *close down* on a compact closed set K if K is common to all the sets of the sequence and if every domain containing K contains all but a finite number of the sets of the sequence. We now prove a proposition which generalizes a theorem of K. Borsuk† and is closely related to a theorem of Vietoris.‡

THEOREM 12. *In a compact metric space E let $\sigma=\{S_i\}$ be a sequence of sets closing down on a closed set S. If the sets of σ are simply connected in the weak sense, so also is S.*

Let A and B be any two points of S, and F any relatively closed subset of S which weakly disconnects A from B in S. Let $\{D_i\}$ be a sequence of domains closing down on F such that the closure of no one of them contains A or B. For each i the closed domain $\bar{D}_i$ weakly disconnects A from B in at least one of the sets of σ. For suppose $\bar{D}_i$ does not weakly disconnect A from B in any set of σ. Then for each j there exists a relative subcontinuum C_j of S_j which contains $A+B$ and has no point in common with $\bar{D}_i$. Some subsequence of $\{C_j\}$ has a sequential limiting set C. The set C is a subcontinuum of S that contains $A+B$ and has no point in common with D_i. Hence C has no point in common with F, contrary to the fact that F weakly disconnects A from B in S.

For each i let S_{n_i} be the first set of σ such that $\bar{D}_i$ weakly disconnects A from B in S_{n_i}. Since, by hypothesis, S_{n_i} is simply connected in the weak sense, there exists a component d_i of $S_{n_i}\cdot\bar{D}_i$ which weakly disconnects A from B in S_{n_i}. Some subsequence of $\{d_i\}$ has a sequential limiting set d. The continuum d is a subset of F since $\{D_i\}$ closes down on F. Furthermore d weakly disconnects A from B in S. For suppose L is a subcontinuum of S which contains $A+B$ but no point of d. Then there exists an integer r such that d_r and L are mutually exclusive. But this implies that d_r cannot weakly disconnect A from B in any set of σ, contrary to what was shown above.

† *Quelques théorèmes sur les ensembles unicohérents*, Fundamenta Mathematicae, vol. 17 (1931), p. 208.

‡ *Über den höheren Zusammenhang von Vereinigungsmengen und Durchschnitten*, Fundamenta Mathematicae, vol. 19 (1932), p. 266.

Thus d is a connected subset of F which weakly disconnects A from B in S. Therefore S is simply connected in the weak sense.

COROLLARY. *If, in a metric space, σ is a monotonic descending sequence of compact continua which are simply connected in the weak sense, the product of the sets of σ is also simply connected in the weak sense.*

We state next the fundamental lemma for simply connected subsets of the plane.

LEMMA K. *A plane continuum consisting of a simple closed curve and its interior is simply connected.*†

We note that a consequence of Lemma K and the Corollary of Theorem 12 is the following result due to Urysohn:

THEOREM K. *Every compact plane continuum which does not separate the plane is simply connected in the weak sense.*‡

We note further that Theorems 4 and K imply the following theorem of S. Janiszewski: If two compact plane continua intersect in a disconnected set, their sum separates the plane.§

The following generalization of Theorem K can be proved by considering an inversion of the plane.

THEOREM 13. *If A and B are two points of a compactly connected*‖ *plane continuum M which does not separate the plane, F is a closed subset of M that weakly disconnects A from B in M, and every component of F is compact, then some component of F weakly disconnects A from B in M.*

In accordance with Urysohn's theorem every compact plane continuum which does not separate the plane is simply connected in the weak sense. We next prove that certain continua which separate the plane have the same property.

THEOREM 14. *Every compact, indecomposable, plane continuum M which is the common boundary of all its complementary domains is simply connected in the weak sense.*

We shall show that if A and B are points of M there exists but one sub-

† For a proof of this lemma see R. L. Moore, *Foundations of Point Set Theory*, p. 194, Theorem 24′.

‡ *Ueber Räume mit verschwindender erster Brouwerscher Zahl*, Akademie van Wetenschappen, Amsterdam, Proceedings, vol. 31 (1928), pp. 808–810.

§ *Sur les coupures du plan faites par les continus*, Prace Matematyczno-Fizyczne, vol. 26 (1913), pp. 11–63.

‖ A connected set M is compactly connected if every two of its points lie together in a compact subcontinuum of M. See R. L. Moore, *Foundations of Point Set Theory*, p. 465.

continuum of M which is irreducible between A and B. Suppose L_1 and L_2 are two distinct subcontinua of M each of which is irreducible between A and B. The intersection of L_1 and L_2 cannot be connected; hence L_1+L_2 separates the plane. Since M is indecomposable there exists a point P of $M-(L_1+L_2)$. Denote by D a complementary domain of L_1+L_2 which does not contain P, and by Δ a complementary domain of M which lies in D. The boundary of D lies in L_1+L_2 and hence cannot contain P. Therefore P cannot lie on the boundary of Δ, contrary to hypothesis.

It follows that M is simply connected in the weak sense.

With the aid of this result it is easy to construct examples of compact plane continua which separate the plane, are not indecomposable, and are simply connected in the weak sense. It would be interesting to find necessary and sufficient conditions for a compact plane continuum to be simply connected in the weak sense.

THEOREM 15. *If A and B are two points of a compactly connected plane continuum M which does not separate the plane, and G is a countable collection of mutually exclusive closed subsets of M no one of which weakly disconnects A from B in M, and G^* is closed and compact, then G^* does not weakly disconnect A from B in M.*

The proof, based on Theorem 13, is similar to that of Theorem 1.

THEOREM 16. *If H and K are two plane continua one of which is compact, and G denotes the collection of those complementary domains of $H+K$ each of whose boundaries contains points of $H-H\cdot K$ and points of $K-H\cdot K$, then $G^*+H\cdot K$ is a connected set which is not disconnected by any element of G.*

If $H\cdot K=0$ then G contains only one element.

Suppose $H\cdot K\neq 0$. Consider the case where H and K are both compact and suppose there exists an element D of G such that $(G^*+H\cdot K)-D=N_1+N_2$, where N_1 and N_2 are mutually separated sets. The boundary of each element of G is connected and therefore contains a point of $H\cdot K$. Hence there exist points P_1 and P_2 of $H\cdot K$ which lie in N_1 and N_2 respectively. At least one of the sets N_1, N_2 is bounded. Hence there exists a compact closed subset F of $S-D$ (where S denotes the plane) which separates N_1 from N_2 in $S-D$. Now $S-D$ is a compactly connected continuum which does not separate S. Hence, by Theorem 13, there exists a component F_0 of F which weakly disconnects P_1 from P_2 in $S-D$. Hence F_0 intersects both H and K. The continuum F_0 contains a connected set f_0 which contains no point of $H+K$ but such that both H and K contain at least one limit point of f_0. Hence f_0 is a subset of an element of G. This is a contradiction. It follows also that $G^*+H\cdot K$ is connected.

The case where one of the sets H, K is not compact can be reduced to the one considered by an inversion of the plane about a circle lying in D.

It is interesting to observe that the Janiszewski theorem mentioned in connection with Theorem K is also a direct corollary of Theorem 16.

Another theorem due to Janiszewski† is the following: If H and K are two compact closed subsets of the plane neither of which separates the point A from the point B in the plane, and if $H\cdot K$ is connected, then $H+K$ does not separate A from B in the plane. With the aid of Theorem 13 this result can be generalized as follows.

THEOREM 17. *In a plane S let $G=\{g_i\}$ be a countable collection of closed sets such that G^* is compact. If A and B are two points of S, and M and N are two compact continua, each containing $A+B$, such that no element of G intersects both M and N, and such that the set of those points common to two or more elements of G is contained in a complementary domain D of $M+N$, then G^* does not separate A from B in S.*

For suppose the contrary. Then $(S-D)\cdot G^*$ separates A from B in $S-D$. Let F be a closed subset of $(S-D)\cdot G^*$ which separates A from B in $S-D$. Since $S-D$ is a compactly connected continuum which does not separate S, and F is compact, there exists, by Theorem 13, a component F_0 of F which weakly disconnects A from B in $S-D$. Therefore F_0 contains points of both M and N, and hence intersects at least two of the mutually exclusive closed sets $(S-D)\cdot g_i$. But this contradicts a theorem of Sierpiński.‡

THEOREM 18. *Let H and K be two plane continua whose common part is not connected. If N is a compact component of $H\cdot K$ such that $H\cdot K-N$ is closed, there exist two complementary domains Δ_1, Δ_2 of $H+K$ such that* (1) *the boundary of $\Delta_i (i=1, 2)$ intersects N, $H-H\cdot K$, and $K-H\cdot K$, and* (2) *$\Delta_1+N+\Delta_2$ contains a compact continuum L such that $L\cdot\Delta_1$ and $L\cdot\Delta_2$ are non-vacuous connected sets.*

Suppose first that H and K are compact.

Let E be a compact closed set which separates $H-H\cdot K$ from $K-H\cdot K$ in S, the plane. Let E' denote the sum of $H\cdot K$ and those points of E which lie in complementary domains of $H+K$ whose boundaries intersect both $H-H\cdot K$ and $K-H\cdot K$. The set E' is closed and separates $H-H\cdot K$ from $K-H\cdot K$ in S. Consider a component h' of $H-H\cdot K$ which has limit points in N and in $H\cdot K-N$. Denote by h the continuum which is the sum of h' and those components of $H\cdot K$ which contain limit points of h'. Let N be enclosed in a domain D such that $\bar{D}$ contains no point of $H\cdot K-N$ and such

† Loc. cit.
‡ Loc. cit.

that the boundary B of D consists of a finite number of mutually exclusive simple closed curves. There exist two finite collections T_h and T_K of subarcs of B such that (1) if b and c are elements of T_h and T_K, respectively, then $b\cdot h\neq 0$, $c\cdot K\neq 0$, and $(b+c)\cdot E'=b\cdot c=0$, and (2) $T_h^*\supset h\cdot B$ and $T_K^*\supset K\cdot B$. Denote by H' and K' the continua $h+T_h^*$ and $K+T_K^*$ respectively. We note that $H'\cdot K'=h\cdot K$. The set E' separates $H'-H'\cdot K'$ from $K'-H\cdot K$ in S. If F represents the sum of $H\cdot K$ and those points of E' which lie in complementary domains of $H'+K'$ whose boundaries intersect both $H'-H'\cdot K'$ and $K'-H\cdot K$, then F is a closed set which separates $H'-H'\cdot K'$ from $K'-H\cdot K$ in S. The collection R of those components of $\overline{D}\cdot(H'+K')-N$ having no limit points in N is finite. Let J be a simple closed curve which lies in D, intersects no element of R, and separates N from $H'\cdot K'-N$. There exists a subarc PQ of J whose end points lie in H' and K' respectively. The component u of $\overline{D}\cdot(H'+K')-N$ which contains P, and the component v of $\overline{D}\cdot(H'+K')-N$ which contains Q, have limit points in N. There exists a component F_1 of F which separates u from v in S. Thus F_1 contains a point of PQ and a point of N. Let L_1 be a subcontinuum of F_1 which is irreducible from N to $J+B$, and denote by d_1 the complementary domain of $H'+K'$ which contains the connected set $L_1-L_1\cdot N$.

The continuum $S-d_1$ is compactly connected and does not separate the plane. Hence, by Theorem 13, there exists a subarc $P'Q'$ of J which lies in $S-d_1$ and weakly disconnects N from a point X of $H'\cdot K'-N$ in $S-d_1$. There exists a subarc $p'q'$ of $P'Q'$ whose end points lie in H' and K' respectively. There exists, further, by Theorem 13, a component F_2 of $F\cdot(S-d_1)$ which weakly disconnects p' from q' in $S-d_1$. Hence F_2 contains a point of $p'q'$. If u' and v' denote the components of $\overline{D}\cdot(H'+K')-N$ which contain p' and q' respectively, then $u'+N+v'$ is a subcontinuum of $S-d_1$. Hence F_2 contains a point of N. Let L_2 be a subcontinuum of F_2 which is irreducible from N to $J+B$, and denote by d_2 the complementary domain of $H'+K'$ which contains $L_2-L_2\cdot N$.

The complementary domains δ_1 and δ_2 of $h+K$ which contain d_1 and d_2, respectively, are distinct. For suppose the contrary. Let Y_1 and Y_2 be points of d_1 and d_2 respectively. There exists an arc β from Y_1 to Y_2 which lies in δ_1 and hence contains no point of $h+K$. Since the boundaries of d_1 and d_2 intersect $K'-H\cdot K$, there exist two arcs Y_1Z_1 and Y_2Z_2 which lie in d_1 and d_2, respectively, except for the points Z_1 and Z_2, which are contained in $K'-H\cdot K$. The continuum $\Omega=Y_1Z_1+Y_2Z_2+K'$ contains Y_1+Y_2 and has no point in common with T_h^*. The common part of T_h^* and $h+K$ is a subset of h', and h' is a connected set having no point in common with $\beta+\Omega$. Hence, by Theorem 17, $H'+K=h+K+T_h^*$ does not separate Y_1 from Y_2 in S.

With this result it can be proved by a similar argument that $H'+K'$ does not separate Y_1 from Y_2 in S. But this is a contradiction.

It follows that the complementary domains Δ_1 and Δ_2 of $H+K$ which contain $L_1-L_1\cdot N$ and $L_2-L_2\cdot N$, respectively, are distinct. If we take $L=L_1+N+L_2$, the domains Δ_1, Δ_2 are seen to satisfy the conditions of the theorem.

The case where H and K are not assumed to be compact can be reduced to the one considered by performing an inversion of the plane about a circle whose center lies in $S-(H+K)$.

Corollary. *If H and K are two unbounded plane continua whose intersection is non-vacuous and compact, there exist two complementary domains Δ_1, Δ_2 of $H+K$ such that* (1) *Δ_i $(i=1, 2)$ contains an unbounded continuum, and* (2) *the boundary of Δ_i $(i=1, 2)$ intersects $H-H\cdot K$ and $K-H\cdot K$.*

Theorem 19. *A necessary and sufficient condition that a connected and locally arcwise connected subset M of the plane be simply connected is that the interior of every simple closed curve lying in M be a subset of M.*

That the condition is sufficient follows from Theorem 7 and Lemma K.

The condition is also necessary. For assume M to be simply connected and suppose M contains a simple closed curve J whose interior I contains a point Q which does not lie in M. Let l be a straight line which intersects I, and denote by P_1 and P_2 the two points of $l\cdot J$ such that the interval P_1P_2 of l contains $l\cdot J$. Join P_1 with P_2 by an arc b which contains Q and lies in I except for its end points. Let A and B denote interior points of the two arcs of J whose end points are P_1 and P_2. The open curve $h=(l-P_1P_2)+b$ separates A from B in the plane. Hence $h\cdot M$ separates A from B in M. But no component of $h\cdot M$ can separate A from B in M; for such a component would contain P_1+P_2 and hence Q. This contradicts the hypothesis that M is simply connected.

We get the following well known corollaries.

Corollary 1. *A bounded, connected subdomain of the plane is simply connected if and only if its complement is connected. This remains true if "complement" is replaced by "boundary."*

Corollary 2. *An unbounded, connected subdomain of the plane is simply connected if and only if every component of its complement is unbounded. This remains true if "complement" is replaced by "boundary."*

Corollary 3. *Every complementary domain of a plane closed set each component of which is unbounded is simply connected.*

COROLLARY 4. *If D is a complementary domain of a bounded plane continuum, and F is a bounded and relatively closed subset of D which separates a point A from a point B in D, then F contains a connected subset which separates A from B in D.*

An application of Corollary 3 and Theorem 2 is the following well known theorem: If H and K are two closed sets neither of which separates the point A from the point B in the plane, and if each component of $H\cdot K$ is unbounded, then $H+K$ does not separate A from B in the plane. This theorem can be generalized as follows:

THEOREM 20. *In a connected and locally arcwise connected subset M of a plane S let G be a countable collection of relatively closed sets such that* (1) *the common part of every pair of elements of G is the set H (which may be vacuous),* (2) *either $H+(S-M)$ is vacuous or every component of $H+(S-M)$ is unbounded, and* (3) *G^* is locally compact in M. If no element of G separates the point A from the point B in M, then G^* does not separate A from B in M.*

Let b_1 and b_2 be two arcs from A to B that lie in $M-H$. The set $H+(S-M)$, if not vacuous, is contained in the unbounded domain D which is complementary to b_1+b_2 in S. Hence the compact continuum $S-D$ is a subset of $M-H$. But $S-D$ does not separate S and therefore, by Theorem K, is simply connected in the weak sense. Hence, by Theorem 3, G^* does not separate A from B in M.

Another special case of this result is the following theorem of Anna Mullikin†: If G is a countable collection of mutually exclusive closed sets lying in the plane S, and no element of G separates the point A from the point B in S, then G^* does not separate A from B in S.

If the collection G is not restricted to be countable, we have a proposition related to a theorem‡ of Rutt and Roberts:

THEOREM 21. *In a connected and locally arcwise connected subset M of a plane S let G be any collection of connected sets which are closed in S such that* (1) *the common part of every pair of elements of G is the non-vacuous set H,* (2) *every component of $H+(S-M)$ is unbounded, and* (3) *G^* is closed in M. If no element of G separates the point A from the point B in M, then G^* does not separate A from B in M.*

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† *Certain theorems relating to plane connected point sets*, these Transactions, vol. 24 (1922), p. 148, Theorem 3.

‡ See N. E. Rutt, *On certain types of plane continua*, these Transactions, vol. 33 (1931), p. 815, Theorem IV and Corollary IV; and J. H. Roberts, *Concerning collections of continua not all bounded*, American Journal of Mathematics, vol. 52 (1930), p. 553, Theorem I.

In the outline of proof that follows let S be the space of reference.

Suppose the theorem false. Select an element g of G, choosing it to be unbounded if there are any unbounded elements of G. Let L denote the collection of all elements each of which is the sum of g and a component of a set obtained by subtracting H from an element of G. There exists a subset F of L^* which is closed in M, separates A from B in M, contains every element of L which has with it a point of L^*-g in common, and is irreducible with respect to these three properties.

With the aid of Theorem 20 there can be constructed four arcs AP_i $(i=1, \cdots, 4)$ lying in M such that (1) AP_i $(i=1, \cdots, 4)$ has in common with F the point P_i and this point only, and (2) $P_1, \cdots, P_4$ lie in $F-g$ and in distinct elements $l_1, \cdots, l_4$, respectively, of L. Let J be a simple closed curve which separates A from F in S such that, if D denotes the complementary domain of J that contains F, no two of the arcs AP_i have a point of $\overline{D}$ in common. Denote by $Q_i(i=1, \cdots, 4)$ the first point of P_iA which lies on J. Two of the points Q_i, say Q_1 and Q_3, separate the other two in J. The set $W=Q_1P_1+Q_3P_3+l_1+l_3$ contains Q_1+Q_3 and lies in $\overline{D}$. Moreover W is either a continuum or the sum of two or three unbounded continua. Therefore W separates Q_2 from Q_4, and hence P_2 from P_4, in $\overline{D}$. It follows that l_1+l_3 separates l_2-g from l_4-g in F. Thus $F-(l_1+l_3)=F_2+F_4$, where F_2 and F_4 are mutually separated sets containing l_2-g and l_4-g respectively. The sets $R_2=F_2+(l_1+l_3)$ and $R_4=F_4+(l_1+l_3)$ are closed in M and each contains every element of L which has with it a point of L^*-g in common. But R_2 and R_4 are proper subsets of F. Hence neither R_2 nor R_4 can separate A from B in M. Therefore, by Theorem 20, $R_2+R_4(=F)$ does not separate A from B in M, contrary to construction.

THEOREM 22. *If a bounded and locally arcwise connected subset M of the plane S separates the point A from the point B in S, then M contains a simple closed curve which separates A from B in S.*

By hypothesis $S-M=H+K$, where H and K are mutually separated sets containing A and B respectively. One of the sets H, K is bounded since the exterior of a circle enclosing M is a subset either of H or of K. Hence there exists a compact continuum F which separates A from B in S and contains no point of $H+K$. By a theorem of R. L. Wilder† there exists a compact continuous curve N which contains F and is a subset of M; and by a theorem of R. L. Moore‡ N contains a simple closed curve which separates A from B in S.

† Loc. cit.

‡ *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 260, Theorem 5.

This result enables us to generalize Corollary 1 of Theorem 19 as follows:

THEOREM 23. *A bounded, connected, and locally arcwise connected subset of the plane is simply connected if and only if it does not separate the plane.*

As further applications of simply connected sets we shall prove several theorems relating to the separation of a continuum by a closed set.

THEOREM 24. *Let H and K be two plane continua of which K is compact if H is compact. If H disconnects the boundary of some complementary domain of $H+K$ then H disconnects K.*

Suppose the contrary. Let B denote the boundary of a domain Δ complementary to $H+K$ such that $B-B\cdot H=N_1+N_2$, where N_1 and N_2 are mutually separated sets. The set N_1+N_2 is a subset of that complementary domain D of H which contains Δ. Let F be a relatively closed subset of D which separates N_1 from N_2 in D, and let F be compact if H is compact. By Corollaries 3 and 4 of Theorem 19 there exists a component F_0 of F that separates a point P_1 of N_1 from a point P_2 of N_2 in D. Hence F_0 contains a point of Δ, which implies that F_0 is a subset of Δ. But $K-K\cdot H$ is a connected subset of D which contains P_1 and P_2 but no point of Δ. This is a contradiction.

THEOREM 25. *In an n-dimensional euclidean space E_n the complement of every closed set K of dimension $n-3$ or less is simply connected.*

That the complement D of K is connected is well known. Suppose D is not simply connected. Then, by Theorem 6, $D=D_1+D_2$, where D_1 and D_2 are connected and relatively closed subsets of D whose intersection is not connected. Thus $D_1\cdot D_2=I_1+I_2$, where I_1 and I_2 are mutually separated sets. Let F be a closed set which separates I_1 from I_2 in E_n. Let P_1 and P_2 be points of I_1 and I_2 respectively. Then there exists a closed subset B of F which separates P_1 from P_2 in E_n and is the common boundary of two domains. Since the dimension of $B\cdot K$ cannot exceed $n-3$, it follows from a theorem of P. Alexandroff† that $B-B\cdot K$ is a connected set. But $B-B\cdot K$ contains a point of D_1 and a point of D_2 and is a subset of D_1+D_2. Hence $B-B\cdot K$ contains a point of $D_1\cdot D_2$, contrary to the fact that F, which contains B, has no point in common with $D_1\cdot D_2$.

The theorem of Alexandroff referred to is, as he has pointed out,‡ equivalent to the following theorem: If M and N are two closed subsets of E_n neither of which separates the point A from the point B in E_n, and if the

† *Sur les multiplicités cantoriennes et le théorème de Phragmèn-Brouwer généralisé*, Comptes Rendus, vol. 183 (1926), pp. 722–724. In this paper the implicit assumption seems to be made that the common boundary is compact. That this restriction is not necessary follows from an argument by inversion.

‡ Ibid.

dimension of $M \cdot N$ does not exceed $n-3$, then $M+N$ does not separate A from B in E_n. We note that Theorem 25 is another formulation of the same result.

THEOREM 26. *In n-dimensional euclidean space a closed set K of dimension $n-3$ or less disconnects a continuum M if and only if it disconnects the boundary of some complementary domain of M.*

Clearly K disconnects M if and only if $K \cdot M$ (whose dimension does not exceed $n-3$) disconnects M.

Suppose $M - K \cdot M = M_1 + M_2$, where M_1 and M_2 are mutually separated sets. Since $K \cdot M$ does not disconnect space there exists an arc P_1P_2 whose end points lie in M_1 and M_2, respectively, but which otherwise contains no point of M. The set $P_1P_2-(P_1+P_2)$ lies in a domain Δ complementary to M. If B denotes the boundary of Δ we have $B-K \cdot B = B \cdot M_1 + B \cdot M_2$, where $B \cdot M_1$ and $B \cdot M_2$ are mutually separated sets containing P_1 and P_2 respectively. Thus $K \cdot M$ disconnects the boundary of Δ.

The sufficiency of the condition can be proved with the aid of Theorem 25 and an argument similar to that of Theorem 24.

In particular a point of a continuum M in three or more dimensions is a cut point of M if and only if it is a cut point of the boundary of a complementary domain of M. It is interesting to note that this proposition, though true† for bounded continua in the plane, is not generally valid for unbounded plane continua.

As a second application of Theorem 3 we shall extend a result of R. L. Moore‡ to n-dimensional euclidean space ($n>2$).

THEOREM 27. *In a euclidean space E of three or more dimensions let G be a countable collection of closed sets of which the common part of each pair is the point O. If no element of G separates the point A from the point B in E, then G^* does not separate A from B in E.*

Let b_1 and b_2 be two arcs from A to B that lie in $E-O$. Consider two hyperspheres with centers at O and such that b_1+b_2 lies in the domain D included between them. D is homeomorphic to $E-O$ and hence, by Theorem 25, is simply connected. It follows from Theorem 3 that G^* does not separate A from B in E.

† See R. L. Moore, *Concerning the common boundary of two domains*, Fundamenta Mathematicae, vol. 6 (1924), p. 211, Theorem 8; and G. T. Whyburn, *Concerning continua in the plane*, these Transactions, vol. 29 (1927), p. 389, Theorem 19.

‡ *Foundations of Point Set Theory*, p. 298, Theorem 113.

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INTEGRATION OF FUNCTIONS WITH VALUES IN A BANACH SPACE†

BY

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1. Introduction. The central concern of this paper is the integration of functions with values in a complete normed vector space, or "Banach" space $\mathfrak{B}$. This question has already been studied by Graves and Bochner,§ but we shall approach it from an entirely independent angle, most easily understood as an extension of Fréchet's elegant interpretation|| of the Lebesgue integral.

Fréchet considers a function $f(p)$ from an abstract domain $\mathfrak{S}$ with a σ-field Σ of measurable sets, to the real number system R. To each partition Δ of $\mathfrak{S}$ into finite or enumerable sets σ_i (of measures $m(\sigma_i)$) of Σ he assigns a "relative upper integral"

$$J^*(f, \Delta) \equiv \sum_i m(\sigma_i) \cdot \sup_{p\epsilon\sigma_i} f(p)$$

and a dual "relative lower integral"

$$J_*(f, \Delta) \equiv \sum_i m(\sigma_i) \cdot \inf_{p\epsilon\sigma_i} f(p),$$

assuming that both series are unconditionally convergent.

It is evident that $J_*(f, \Delta) \leqq J^*(f, \Delta')$ for any Δ, Δ'. Therefore the intersection of the "relative integral ranges"

$$J_*(f, \Delta) \leqq x \leqq J^*(f, \Delta),$$

for fixed f and variable Δ, is not empty. If it consists of a single point $J(f)$ of R, then $f(p)$ is called "integrable," and $J(f)$ is called the "integral" of $f(p)$.

Our integral may be obtained from Fréchet's by making two alterations. R must be replaced by an arbitrary Banach space $\mathfrak{B}$, and the "relative integral range" must be redefined as the least closed convex set containing all sums $\sum_i m(\sigma_i) \cdot f(p_i)$ $[p_i \epsilon \sigma_i]$, assuming again the unconditional convergence of all such series.

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‡ Society of Fellows, Harvard University.

§ L. M. Graves, *Riemann integration and Taylor's theorem in general analysis*, these Transactions, vol. 29 (1927), pp. 163–77. S. Bochner, *Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind*, Fundamenta Mathematicae, vol. 20 (1933), pp. 262–76.

|| Fréchet, *Sur l'intégrale d'une fonctionnelle étendue à un ensemble abstrait*, Bulletin de la Société Mathématique de France, vol. 43 (1915), pp. 248–65.

The essential task is to prove that any two integral ranges of the same function overlap, leading immediately to the recognition of integrability as the property of having relative integral ranges of arbitrarily small diameter.

2. **Outline.** With this in mind, the outline of the paper is very easy to remember.

The essential technical facts are established in §§3–9 by a study of convexity and unconditional convergence.† Interesting incidental results are obtained, but the emphasis is on the large number of ways in which the sets of a given unconditionally convergent series of sets can be replaced, without destroying unconditional convergence or enlarging the closure of the "convex hull" of the vector sum.

The definition of the integral sketched in the introduction is then stated in full, together with some remarks on "completely additive set functions." These occupy §§10–14.

They are naturally followed by a discussion of the properties of the integral, a few of which may be stated here. The integral of any integrable function T is a completely additive set function depending linearly on T [§§15, 18]. Finite-valued functions are everywhere dense in the "space" of these set functions [§17]. If $\mathfrak{B}$ is separable, then two integrands define the same set function if and only if they are "equivalent" as functions [§20]. And any rectifiable curve in Hilbert space has a tangent at almost every point [§21].

The paper concludes in §§22–25 with counterexamples (such as of nowhere differentiable integrals), with a demonstration that our integral genuinely includes those of Graves and Bochner, and with the enumeration of some unsolved problems.

3. **Calculus of complexes.** The object of this section is to familiarize the reader with the formal properties of two natural operations on non-vacuous sets, or "complexes" of vectors.

Accordingly, let $\mathfrak{B}$ be any vector space,‡ whose elements we shall denote by Greek, and whose (real) coefficients by italic letters. Let further B_1, B_2, $B_3, \cdots$ denote complexes of elements of $\mathfrak{B}$, in the sense just defined.

We introduce the notation

$$b_1 \cdot B_1 + \cdots + b_r \cdot B_r = \sum_{i=1}^{r} b_i \cdot B_i$$

† First studied by M. W. Orlicz, *Beiträge zur Theorie der Orthogonalentwicklungen*, Studia Mathematica, vol. 1 (1929), pp. 1–39 and 249–55.

‡ As defined for instance in S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 26. We shall refer to this volume in later footnotes as Banach.

for the set of all sums $b_1\cdot\beta_1+\cdots+b_r\cdot\beta_r$ $[\beta_i\epsilon B_i]$. This is to be distinguished from $b_1\cdot B_1\dotplus\cdots\dotplus b_r\cdot B_r$, which will be used to denote the point-set sum of the $b_i\cdot B_i$, that is, the set of all elements of the form $b_i\cdot\beta_i$.

It is quite evident that if we denote the origin by Θ, then the following six properties of vector spaces hold for the vector sums of complexes:

$V1$: $B_1+B_2=B_2+B_1$,
$V2$: $B_1+(B_2+B_3)=(B_1+B_2)+B_3$,
$V3$: $b\cdot(B_1+B_2)=b\cdot B_1+b\cdot B_2$,
$V4$: $b_1\cdot(b_2\cdot B)=b_1b_2\cdot B$,
$V5$: $1\cdot B=B$,
$V6$: $B+\Theta=B$ and $0\cdot B=\Theta$.

Therefore if we define a "vectoroid" space to be any system satisfying conditions $V1$–$V6$, we can assert

THEOREM 1. *The non-vacuous subsets of $\mathfrak{B}$ are the elements of a vectoroid space.*

Calculations based on $V1$–$V6$ will be regarded as evident in the remainder of the paper, and performed without explanation.

4. **Convex hulls.** Let again B be any complex of $\mathfrak{B}$. By the "convex hull" of B [in symbols, $\mathrm{Co}(B)$] we mean the set of all elements of the form $b_1\beta_1+\cdots+b_r\beta_r$, where $b_i\geqq 0$, $\beta_i\epsilon B$, and $b_1+\cdots+b_r=1$. A convex complex is of course one which is its own convex hull, and any convex hull is convex, i.e., $\mathrm{Co}(\mathrm{Co}(B))=\mathrm{Co}(B)$. We observe in passing without proof

THEOREM 2. *$B\subset\mathfrak{B}$ is convex if and only if it satisfies $(m_1+m_2)B=m_1B+m_2B$ for all m_1, $m_2\geqq 0$.*

Now by $V4$, $b_1m\beta_1+\cdots+b_rm\beta_r=m(b_1\beta_1+\cdots+b_r\beta_r)$, which shows that $\mathrm{Co}(m\cdot B)=m\cdot\mathrm{Co}(B)$. Evidently also if $\alpha_i\epsilon A$, $\beta_i\epsilon B$, $c_i\geqq 0$, $c_i+\cdots+c_r=1$, and $\xi=c_1(\alpha_1+\beta_1)+\cdots+c_r(\alpha_r+\beta_r)$, then

$$\xi=(c_1\alpha_1+\cdots+c_r\alpha_r)+(c_1\beta_1+\cdots+c_r\beta_r)\epsilon\mathrm{Co}(A)+\mathrm{Co}(B)$$

proving $\mathrm{Co}(A+B)\subset\mathrm{Co}(A)+\mathrm{Co}(B)$.

But it is geometrically obvious† that given a_i, $b_j\geqq 0$ such that $a_1+\cdots+a_r=b_1+\cdots+b_s=1$, $c_{i,j}\geqq 0$ exists satisfying $\sum_j c_{i,j}=a_i$ and $\sum_i c_{i,j}=b_j$, whence $\sum_{i,j}c_{i,j}=1$. Therefore if $\eta=\sum_i a_i\alpha_i+\sum_j b_j\beta_j\epsilon\mathrm{Co}(A)+\mathrm{Co}(B)$, writing $\eta=\sum_{i,j}c_{i,j}(\alpha_i+\beta_j)$, we see that $\eta\epsilon\mathrm{Co}(A+B)$, proving $\mathrm{Co}(A)+\mathrm{Co}(B)\subset\mathrm{Co}(A+B)$.

This completes the proof of

† For both the a_i and the b_j can be regarded as dividing a unit line segment into disjoint intervals, the intersections of which are subintervals whose lengths $c_{i,j}$ have the desired properties.

THEOREM 3.† $\mathrm{Co}(m\cdot B)=m\cdot\mathrm{Co}(B)$ *and* $\mathrm{Co}(A+B)=\mathrm{Co}(A)+\mathrm{Co}(B)$.

That is, abstractly speaking, the correspondence $B\to\mathrm{Co}(B)$ is a homeomorphism carrying the vectoroid space of Theorem 1 into the vectoroid subspace of convex complexes.

5. The norm and diameter of convex hulls. We now add the permanent assumption that $\mathfrak{B}$ is "normed," that is, that there is associated with $\mathfrak{B}$ a rule assigning to every $\xi\in\mathfrak{B}$ a number $\|\xi\|$ called the "norm" of ξ, and satisfying

$N1$: $\|\Theta\|=0$ and $\|\xi\|>0$ for $\xi\neq\Theta$,

$N2$: $\|\xi+\eta\|\leqq\|\xi\|+\|\eta\|$,

$N3$: $\|c\cdot\xi\|=|c|\cdot\|\xi\|$.

Such a rule automatically associates with every bounded complex B of $\mathfrak{B}$ the "norm" $\|B\|\equiv\sup_{\beta\in B}\|\beta\|$. It also associates with B a "diameter" $\rho(B)=\|B-B\|\leqq 2\|B\|$. Moreover the norms of bounded complexes clearly satisfy $N1$–$N3$.

Because of the convexity of the norm function, we can prove

THEOREM 4. $\|\mathrm{Co}(B)\|=\|B\|$ *and* $\rho(\mathrm{Co}(B))=\rho(B)$.

For if $b_i\geqq 0$, $\sum_i b_i=1$, and $\beta_i\in B$, then

$$\left\|\sum_{i=1}^{r} b_i\cdot\beta_i\right\| \leqq \sum_{i=1}^{r}\|b_i\cdot\beta_i\| = \sum_{i=1}^{r} b_i\cdot\|\beta_i\| \leqq \sum_{i=1}^{r} b_i\cdot\|B\| = \|B\|$$

proving that $\|\mathrm{Co}(B)\|\leqq\|B\|$. But obviously $\|\mathrm{Co}(B)\|\geqq\|B\|$, and so $\|\mathrm{Co}(B)\|=\|B\|$. The second half of Theorem 4 follows since

$$\rho(\mathrm{Co}(B))\equiv\|\mathrm{Co}(B)-\mathrm{Co}(B)\| = \|\mathrm{Co}(B-B)\| = \|B-B\| = \rho(B).$$

6. Limits and closure. Hereafter we shall assume that $\mathfrak{B}$ is not only normed but "complete," that is, that every Cauchy sequence of elements of $\mathfrak{B}$ tends to a limit element of $\mathfrak{B}$. Further, we shall denote by $\overline{B}$ the closure of any complex B in $\mathfrak{B}$. The reader will find no difficulty in proving that $\overline{\mathrm{Co}(B)}$ is the least closed convex set containing B. The truth of the formula $\overline{A}+\overline{B}\subset\overline{A+B}$ is equally obvious.

THEOREM 5. *$\|\sum_{i=1}^{r}B_i\|$ is not altered if we replace the B_i by the closures of their convex hulls. Moreover if $0\leqq c_i\leqq 1$, then $\|\sum_{i=1}^{r}c_i\cdot B_i\|$ is bounded by the norm $\|\sum_{k=1}^{s}B_{i(k)}\|$ of the vector sum of some set of the B_i.*

† Theorem 3 has long been known. Cf. for instance T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Berlin, 1934, p. 29.

The first statement follows directly from Theorems 3–4 and the equality $\|\bar{A}\| = \|A\|$. To prove the second, it is sufficient to note that $c_i \cdot B_i \subset \mathrm{Co}(B_i + \Theta)$. Hence by Theorem 3,

$$\sum_{i=1}^{r} c_i \cdot B_i \subset \mathrm{Co}\left(\sum_{i=1}^{r} (B_i + \Theta)\right),$$

and

$$\left\| \sum_{i=1}^{r} c_i \cdot B_i \right\| \leqq \left\| \mathrm{Co}\left(\sum_{i=1}^{r} (B_i + \Theta)\right) \right\| = \left\| \sum_{i=1}^{r} (B_i + \Theta) \right\|$$
$$= \sup \left\| \sum_{k=1}^{s} B_{i(k)} \right\|$$

by what we have just proved, Theorem 4, and definition.

7. Unconditional summation of elements. The primary object of §§7–9 is to translate unconditional convergence into terms of the calculus of complexes and limitations on norm, in order to be able to handle the properties of relative integral ranges. For this purpose Theorem 6 is not strictly necessary.

An enumerable aggregate Ξ of elements $\xi_1, \xi_2, \xi_3, \cdots$ (which need not be distinct) of $\mathfrak{B}$ is called "unconditionally summable to ξ," if and only if every arrangement α of all the elements of Ξ gives a series $\Xi^{(\alpha)}$: $\xi_{\alpha(1)} + \xi_{\alpha(2)} + \xi_{\alpha(3)} + \cdots$ convergent to ξ. Under these conditions, the series $\Xi^{(\alpha)}$ are called "unconditionally convergent to ξ."

It is clear that Ξ ($\Xi^{(\alpha)}$) is unconditionally summable (convergent) to ξ if and only if to every $\epsilon > 0$ corresponds a number N so large that the sum η of any finite set of terms of Ξ ($\Xi^{(\alpha)}$) including $\xi_1, \cdots, \xi_N$ satisfies $\|\eta - \xi\| < \epsilon$.

Now let Ξ: $\xi_1 + \xi_2 + \xi_3 + \cdots$ and Ξ': $\xi_1' + \xi_2' + \xi_3' + \cdots$ be any two unconditionally convergent series. By $c \cdot \Xi$ we mean the series $c \cdot \xi_1 + c \cdot \xi_2 + c \cdot \xi_3 + \cdots$, and by $\Xi + \Xi'$ the series Ξ'': $\xi_1'' + \xi_2'' + \xi_3'' + \cdots$, where $\xi_i'' = \xi_i + \xi_i'$. The reader can easily see that these series are unconditionally convergent, and that the operations of addition and of multiplication by a scalar possess all of the usual vector properties. Therefore the unconditionally convergent series of $\mathfrak{B}$ are the elements of a vector space $\mathfrak{C}$.

Let $B(\Xi)$ denote the (bounded) set of the finite partial sums of the elements of Ξ. By the "norm" $\|\Xi\|$ of Ξ we mean $\|B(\Xi)\|$. Since $B(\Xi + \Xi') \subset B(\Xi) + B(\Xi')$ and $B(c \cdot \Xi) = c \cdot B(\Xi)$, we see that $\mathfrak{C}$ may be regarded as normed in the sense of §5.

We shall now prove that $\mathfrak{C}$ is complete relative to this norm, which amounts to asserting

THEOREM 6. *The unconditionally convergent series of $\mathfrak{B}$ are the elements of a second Banach space.*

Let $\Xi_1, \Xi_2, \Xi_3, \cdots$ be any sequence of unconditionally convergent series of elements of $\mathfrak{B}$, such that to any $\epsilon>0$ corresponds N so large that $m\geqq N$ and $n\geqq N$ imply $\|\Xi_m-\Xi_n\|<\epsilon$. Clearly the ith terms ξ_i^k of the Ξ_k are uniformly convergent Cauchy sequences, with limits ξ_i. Let Ξ denote the formal series $\xi_1+\xi_2+\xi_3+\cdots$. The proof is complete if Ξ is unconditionally convergent and

$$\lim_{n\to\infty}\|\Xi-\Xi_n\| = 0.$$

But to any $\epsilon>0$ corresponds N so large that if $m, n\geqq N$, then $\|\Xi_m-\Xi_n\|<\epsilon$. And we can find M so large that if $M<k(1)<\cdots<k(r)$, then $\|\sum_{i=1}^r\xi_{k(i)}^N\|<\epsilon$. It follows that under the same hypotheses,

$$\left\|\sum_{i=1}^r \xi_{k(i)}\right\| = \left\|\lim_{n\to\infty}\sum_{i=1}^r \xi_{k(i)}^n\right\|$$

$$\leqq \left\|\sum_{i=1}^r \xi_{k(i)}^N\right\| + \epsilon < 2\epsilon$$

so that Ξ must be unconditionally convergent. But now if we pick *any* $j(1)<\cdots<j(s)$, then for $n\geqq N$,

$$\left\|\sum_{i=1}^s (\xi_{j(i)} - \xi_{j(i)}^n)\right\| = \left\|\lim_{m\to\infty}\sum_{i=1}^s (\xi_{j(i)}^m - \xi_{j(i)}^n)\right\|$$

$$= \lim_{m\to\infty}\left\|\sum_{i=1}^s (\xi_{j(i)}^m - \xi_{j(i)}^n)\right\| \leqq \epsilon$$

by hypothesis. Hence $\|\Xi-\Xi_n\|=\epsilon$, completing the proof.

8. **Unconditional summation of complexes.** Suppose similarly Φ is an aggregate of enumerable complexes $B_1, B_2, B_3, \cdots$ of $\mathfrak{B}$. Φ will be called "unconditionally summable" to a given complex B if and only if every series $\beta_1+\beta_2+\beta_3+\cdots$ $[\beta_i\epsilon B_i]$ is unconditionally convergent, and B is the locus of the sums of such series. We shall abbreviate this by writing $\sum_i B_i=B$.

In order that Φ be unconditionally summable it is necessary as well as sufficient that to any $\epsilon>0$ correspond N so large that $N<k(1)<\cdots<k(r)$ implies $\|B_{k(1)}+\cdots+B_{k(r)}\|<\epsilon$. For otherwise we could form an infinite series of elements from a sequence of such sets of complexes which was not unconditionally convergent no matter how the gaps between the different terms were filled in by elements from the remaining complexes of Φ.

Keeping this in mind, we can prove without difficulty

THEOREM 7. *If* $\sum_i B_i = B$, *then* $\sum_i \overline{\mathrm{Co}(B_i)} \subset \overline{\mathrm{Co}(B)}$ *and*

$$\overline{\mathrm{Co}\,(B)} = \overline{\sum_i \mathrm{Co}\,(B_i)} = \overline{\sum_i \overline{\mathrm{Co}\,(B_i)}}.$$

In the first place (using the above notation), by Theorem 5

$$\left\| \sum_{i=1}^{r} \overline{\mathrm{Co}\,(B_{k(i)})} \right\| = \left\| \sum_{i=1}^{r} B_{k(i)} \right\| < \epsilon,$$

and consequently the aggregate of the $\overline{\mathrm{Co}(B_i)}$ is unconditionally summable. Secondly, by the theory of limits, for sufficiently large N it is true that every point of

$$\sum_{i=1}^{N} \overline{\mathrm{Co}\,(B_i)} \subset \overline{\mathrm{Co}\left(\sum_{i=1}^{N} B_i \right)}$$

is within ϵ of some point of $\overline{\mathrm{Co}(B)}$, and hence in the limit $\sum_i \overline{\mathrm{Co}(B_i)} \subset \overline{\mathrm{Co}(B)}$.

But conversely every point b of $\overline{\mathrm{Co}(B)}$ can be approximated by a sum $b^* = \sum_{i=1}^{s} m_i \beta_i^*$, where $m_i \geqq 0$, $\sum_{i=1}^{s} m_i = 1$, and $\beta_i^* \subset B$. And $\beta_i^* = \sum_{k=1}^{\infty} \beta_i^k$ $[\beta_i^k \epsilon B_k]$ whence $b^* = \sum_{i=1}^{s} m_i \sum_{k=1}^{\infty} \beta_i^k$. And since the sum of s unconditionally summable aggregates is itself unconditionally summable,

$$b^* = \sum_{k=1}^{\infty} \sum_{i=1}^{s} m_i \beta_i^k = \sum_{k=1}^{\infty} \bar{\beta}_k \qquad [\bar{\beta}_k \epsilon \mathrm{Co}(B_k)].$$

That is, $\overline{\mathrm{Co}(B)} \subset \overline{\sum_i \mathrm{Co}(B_i)}$.

Now consider the following triple inequality,

$$\overline{\sum_i \overline{\mathrm{Co}\,(B_i)}} \subset \overline{\mathrm{Co}\,(B)} \subset \overline{\sum_i \mathrm{Co}\,(B_i)} \subset \overline{\sum_i \overline{\mathrm{Co}\,(B_i)}}.$$

The first relation follows from $\sum_i \overline{\mathrm{Co}(B_i)} \subset \overline{\mathrm{Co}(B)}$, which was proved above. The second we have just proved, and the third is utterly obvious. This completes the demonstration of Theorem 7.

THEOREM 8. *If the aggregate* Φ *of complexes* B_i *is unconditionally summable, and the* $|m_i|$ *are uniformly bounded by a finite constant* K, *then the aggregate of complexes* $m_i \cdot B_i$ *is unconditionally summable, and*

$$\left\| \overline{\mathrm{Co}\left(\sum_i m_i \cdot B_i \right)} \right\| \leqq 2K \cdot \sup \left\| \sum_{k=1}^{r} B_{i(k)} \right\|.$$

Theorem 8 follows directly if we can prove unconditional summability from the relations

$$\left\|\overline{\operatorname{Co}\left(\sum_i m_i\cdot B_i\right)}\right\| = \left\|\sum_i m_i\cdot B_i\right\|$$
$$\leqq \left\|\sum_i |m_i|\cdot(B_i+\Theta) + \sum_i(-|m_i|)\cdot(B_i+\Theta)\right\|$$
$$\leqq 2K\cdot\left\|\sum_i \operatorname{Co}(B_i+\Theta)\right\| = 2K\cdot\left\|\sum_i (B_i+\Theta)\right\|$$
$$\equiv 2K\cdot\sup\left\|\sum_{k=1}^{r} B_{i(k)}\right\|$$

in which the first relation results from Theorem 4, the second from separating positive and negative coefficients, the third and fifth from inclusion of the left-hand complex in the right, and the fourth from Theorem 7.

But unconditional summability results from the three facts (1) $m_i\cdot B_i \subset K\cdot\operatorname{Co}(B_i\dotplus\Theta) - K\cdot\operatorname{Co}(B_i\dotplus\Theta)$, (2) Theorem 7, (3) the set of the finite partial sums of the $(B_i\dotplus\Theta)$ not involving the first N terms is the same as that of the B_i. For these show, taken in reverse order, that if $\sum_i B_i$ exists, then so do (1) $\sum_i(B_i\dotplus\Theta)$, (2) $\sum_i \operatorname{Co}(B_i\dotplus\Theta)$, and hence (by Theorem 6) $\sum_i[K\cdot\operatorname{Co}(B_i\dotplus\Theta) - K\cdot\operatorname{Co}(B_i\dotplus\Theta)]$, (3) $\sum_i m_i\cdot B_i$.

9. **Replacement of single series by double series.** We are now in a position to prove the essential

THEOREM 9. *If $m_i^j \geqq 0$, $\sum_j m_i^j = m_i$ for every i, $B_i^j \subset B_i$, every B_i is bounded, and $\sum_i m_i B_i = B$, then $\sum_{i,j} m_i^j\cdot B_i^j \subset \overline{\operatorname{Co}(B)}$, whence*

$$\overline{\operatorname{Co}\left(\sum_{i,j} m_i^j\cdot B_i^j\right)} \subset \overline{\operatorname{Co}(B)}.$$

For to any $\epsilon>0$ corresponds M so large that the norm of any finite sum of complexes B_k $[k>M]$ is less than $\frac{1}{2}\epsilon$. And since $B_1, \cdots, B_M$ are bounded, we can choose N satisfying

$$\text{(A)} \qquad \sum_{i=1}^{M}\sum_{j=N+1}^{\infty} m_i^j < \epsilon/2[\|B_1\| + \cdots + \|B_M\|].$$

If therefore we exclude the MN complexes $m_i^j\cdot B_i^j$ for which $i\leqq M, j\leqq N$, we see that any finite sum of the remaining $m_i^j\cdot B_i^j$ is composed of a set for which $i\leqq M, j>N$, plus a set for which $i>M$. But by the triangle inequality on norm and (A), the norm of the first sum $<\frac{1}{2}\epsilon$, while by Theorem 5 and construction the norm of the second sum $<\frac{1}{2}\epsilon$. Therefore the norm of the whole sum $<\epsilon$, and the $m_i^j\cdot B_i^j$ are unconditionally summable.

Again, every element of $\sum_{i=1}^{M}\sum_{j=1}^{N} m_j^i \cdot B_j^i$ lies within $\frac{1}{2}\epsilon$ of some element of $\overline{\mathrm{Co}(\sum_{i=1}^{M} m_i \cdot B_i)}$, and hence within ϵ of some element of $\overline{\mathrm{Co}(\sum_i m_i \cdot B_i)}$. Therefore in the limit, by Theorem 7, we obtain

$$\sum_{i,j} m_j^i \cdot B_j^i \subset \overline{\mathrm{Co}(B)}.$$

10. **Admissible domains.** Hitherto we have confined our attention to properties of the range of the functions which we shall try to integrate. We shall in the present section consider the domain.

We shall define as an "admissible domain" any space $\mathfrak{S}$ of points in which is defined a so-called "σ-ring" Σ of "measurable" point sets satisfying

*D*1: The complement $\mathfrak{S}-\sigma$ of any one set σ, and the product $\sigma_1 \cdot \sigma_2$ and the sum $\sigma_1 \dotplus \sigma_2$ of any two sets σ_1 and σ_2 of Σ, are in Σ.

*D*2: To every set σ of Σ corresponds a number $m(\sigma)$ called the "measure" of σ.

*D*3: $m(\sigma)$ is zero, finite and positive, or $+\infty$.

*D*4: If $\sigma = \sigma_1 \dotplus \sigma_2 \dotplus \sigma_3 \dotplus \cdots$ is the sum of finite or enumerable disjoint sets σ_i of Σ, then σ is in Σ, and $m(\sigma) = m(\sigma_1) + m(\sigma_2) + m(\sigma_3) + \cdots$.

By a "decomposition" of $\mathfrak{S}$ we mean any choice Δ_k of finite or enumerable disjoint non-vacuous measurable sets of finite measure, whose point-set sum is $\mathfrak{S}$. We shall adopt the fixed notation $\sigma_1^k, \sigma_2^k, \cdots$ for the sets of composition of Δ_k.

By the "product" $\Delta_1 \cdot \Delta_2$ of two decompositions Δ_1 and Δ_2 we mean the decomposition of $\mathfrak{S}$ into those sets $\sigma_i^1 \cdot \sigma_j^2$ which are non-vacuous.

11. **Completely additive set functions.** Since integration will be defined relative to the σ-ring Σ, it is only natural that we should define a (single-valued) "set function" as a function J assigning to each set σ of Σ a single "value" $J(\sigma)$ in $\mathfrak{B}$.

The "sum" $K = J_1 + J_2$ of two set functions J_1 and J_2, and the "product" $K^* = c \cdot J_1$ of J_1 by a real scalar c, are of course defined by the identities $K(\sigma) = J_1(\sigma) + J_2(\sigma)$ and $K^*(\sigma) = c \cdot J_1(\sigma)$. And J is called "completely additive" if and only if the hypothesis that σ is the sum of finite or enumerable disjoint sets σ_i of Σ implies the conclusion that the values $J(\sigma_i)$ are unconditionally summable to $J(\sigma)$.

LEMMA. *If J is completely additive, then the set of the $J(\sigma)$ $[\sigma \epsilon \Sigma]$ has a finite upper bound.*

Otherwise we could choose $\sigma_1, \sigma_2, \sigma_3, \cdots$ by induction so as to satisfy $\|J(\sigma_1)\| > 1$ and $\|J(\sigma_{i+1})\| > 3\|J(\sigma_i)\|$. And the series of the $(J(\sigma_i - \sigma_i \cdot (\sigma_1 \dotplus \cdots \dotplus \sigma_{i-1}))$ could not be unconditionally summable.

The (finite) least upper bound to the $\|J(\sigma_i)\|$ will be called the "norm" of J, denoted by $\|J\|$.

THEOREM 10. *The completely additive set functions of $\mathfrak{S}$ to $\mathfrak{B}$ are a Banach space $\mathfrak{F}(\mathfrak{S}; \mathfrak{B})$.*

Every property of Banach space is obvious except completeness. But since the $J_n(\sigma)$ are a uniformly convergent Cauchy sequence, it is obvious that they tend uniformly to a limit set function J.

It remains to prove that J is completely additive. But for each choice of $\sigma = \sigma_1 \dotplus \sigma_2 \dotplus \sigma_3 \dotplus \cdots$, this is a corollary of Theorem 6. This completes the proof of Theorem 10.

There are three superficial remarks, which, although apparently inconsequential, should perhaps be made. In the first place, the proof of Theorem 10 can be duplicated to show that $\mathfrak{F}(\mathfrak{S}; \mathfrak{B})$ is imbedded in the Banach "superspace" of bounded set functions. Secondly, every permutation $\sigma \rightarrow \pi(\sigma)$ of the sets of Σ induces an isometric linear transformation $\Pi: J \rightarrow J_\pi$ of this superspace into itself, defined by the equation $J_\pi(\sigma) = J(\pi^{-1}(\sigma))$. And thirdly, if this permutation preserves inclusion relations (i.e., is itself induced by a measure-preserving permutation of the points of $\mathfrak{S}$), then it carries $\mathfrak{F}(\mathfrak{S}; \mathfrak{B})$ into itself, and so defines an isometric linear transformation on it.

12. **Admissible point functions.** By a "function" (more precisely, point function) T of an admissible domain $\mathfrak{S}$ to a Banach space $\mathfrak{B}$ we shall mean from now on a rule assigning to each point p of $\mathfrak{S}$ one or more† "images" in $\mathfrak{B}$. More generally, if σ is any complex in $\mathfrak{S}$, we shall use $T(\sigma)$ to denote the complex of the images of the points of σ.

The "sum" $V = T + U$ of two such functions T and U, and the "product" $W = k \cdot T$ of a function T by a real scalar k are naturally defined by setting $V(p) = T(p) + U(p)$ and $W(p) = k \cdot T(p)$. This defines the admissible functions as elements of a "vectoroid" space, which becomes a vector space if we restrict ourselves to single-valued functions.

13. **Summability and integral ranges.** We now lay down

DEFINITION 1. *A function T is called "summable" under the decomposition Δ of $\mathfrak{S}$ if and only if each $T(\sigma_i)$ is bounded, and the aggregate of the $m(\sigma_i) \cdot T(\sigma_i)$ is unconditionally summable.*

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† The idea that since an integral range is a multiple-valued set function, we lose nothing by allowing T to be multiple-valued, is due to A. Kolmogoroff, *Untersuchungen über den Integralbegriff*, Mathematische Annalen, vol. 103 (1930), pp. 654–96.

DEFINITION 2. *If T is summable under Δ, then the set*

$$J_\Delta(T) \equiv \overline{\mathrm{Co}\left(\sum_i m(\sigma_i)\cdot T(\sigma_i)\right)}$$

is called the "integral range" of T relative to Δ.

THEOREM 11. *If the function T is summable under two decompositions Δ and Δ_1, then T is summable under the product decomposition $\Delta\cdot\Delta_1$, and*

$$J_{\Delta\cdot\Delta_1}(T) \subset J_\Delta(T)\cdot J_{\Delta_1}(T).$$

Therefore any two integral ranges of T overlap.

Suppose the sets of decomposition of $\mathfrak{S}$ under Δ and Δ_1 are σ_i and σ_i', respectively. Then if we denote $m(\sigma_i)$ by m_i, $T(\sigma_i)$ by B_i, $m(\sigma_i\cdot\sigma_j')$ by m_j^i, and $T(\sigma_i\cdot\sigma_j')$ by B_j^i, the hypotheses of Theorem 9 are clearly fulfilled. It follows that T is summable under the product $\Delta\cdot\Delta_1$ of Δ and Δ_1, and that $J_{\Delta\cdot\Delta_1}(T)\subset J_\Delta(T)$. The rest of the conclusion follows by symmetry.

14. The integrable functions and their integrals. We are now ready for

DEFINITION 3. *A function T will be called integrable if and only if the inferior limit of the diameters of its integral ranges is zero.*

THEOREM 12. *If T is integrable, then the intersection of the integral ranges of T is a single element $J(T)$ of $\mathfrak{B}$.*

We can choose a set of integral ranges $J_{\Delta_1}(T)$, $J_{\Delta_2}(T)$, $J_{\Delta_3}(T)$, $\cdots$ of diameters <1, $<\frac{1}{2}$, $<\frac{1}{4}$, $\cdots$. Since these are closed and overlap, their intersection is a point. But since every integral range of T is closed and overlaps every $J_{\Delta_k}(T)$, this point is contained in every integral range of T.

DEFINITION 4. *The $J(T)$ of Theorem 12 is called the integral of T over $\mathfrak{S}$.*

THEOREM 13. *T is integrable if and only if to every $\epsilon>0$ corresponds a decomposition Δ under which the aggregate $m(\sigma_i)\cdot T(\sigma_i)$ is unconditionally summable and has a diameter $<\epsilon$.*

For since the diameter is bounded, so is each $T(\sigma_i)$. And by Theorem 4, $\rho(J_\Delta(T))<\epsilon$. And these are the only facts about integrability not assumed.

15. Integrals are completely additive set functions. In this section it will be shown that the integral of any integrable function T is a completely additive set function depending linearly on T. To this end we prove

THEOREM 14. *If the function T is integrable over $\mathfrak{S}$, then it is integrable over every set σ of Σ to an element of $\mathfrak{B}$ which will be denoted by $J(T, \sigma)$, and the set function $J(T, \sigma)$ is completely additive.*

Let Δ be any decomposition of $\mathfrak{S}$ into sets σ_i under which T is summable. Then writing $\sigma_i^1=\sigma\cdot\sigma_i$ and $\sigma_i^2=(\mathfrak{S}-\sigma)\cdot\sigma_i$, we see $T(\sigma_i^k)\subset T(\sigma_i)$ $[k=1, 2]$ and $\sum_k m(\sigma_i^k)=m(\sigma_i)$. Therefore by Theorem 9

$$\sum_{i,k} m(\sigma_i^k)\cdot T(\sigma_i^k) \subset \sum_i m(\sigma_i)\cdot T(\sigma_i),$$

whence obviously

$$\sum_i m(\sigma_i^1)\cdot T(\sigma_i^1) + \sum_i m(\sigma_i^2)\cdot T(\sigma_i^2) \subset J_\Delta(T).$$

It is a corollary† that $\sum_i m(\sigma_i^1)\cdot T(\sigma_i^1)$ is of diameter at most $\rho(J_\Delta(T))$, and hence by Theorem 13 T is integrable over σ.

Similarly, if Δ and Δ_1 are any two decompositions of $\mathfrak{S}$ into subsets σ_i and σ_i^1 respectively, then by Theorem 9

$$\sum_i J(T, \sigma_i) \subset \sum_i J_{\Delta\cdot\Delta_1}(T, \sigma_i) \subset J_{\Delta\cdot\Delta_1}(T) \subset J_\Delta(T),$$

whence, in the limit, $\sum_i J(T, \sigma_i)=J(T)$. Now replacing $\mathfrak{S}$ by an arbitrary set σ of Σ, we complete the proof.

Definition 5. *By the "norm" of an integrable function T, is meant the real number* $\|T\|\equiv\sup_{\sigma\in\Sigma}\|J(T, \sigma)\|$.

Theorem 15. *If Δ is any decomposition of $\mathfrak{S}$, if the function T is integrable over every set σ_i of composition of Δ, and the aggregate of the $J(T, \sigma_i)$ is unconditionally summable, then T is integrable over $\mathfrak{S}$ and* $J(T)=\sum_i J(T, \sigma_i)$.

Decompose each σ_i by a decomposition Δ_i under which $\|J_{\Delta_i}(T, \sigma_i) - J(T,\sigma_i)\| < \epsilon/2^i$. Then the corresponding decomposition of $\mathfrak{S}$ will be summable, and its integrated range will be within a sphere of radius ϵ of $\sum_i J(T, \sigma_i)$.

Theorem 16. *If T and U are integrable functions, and m is a real number, then $m\cdot T$ and $T+U$ are integrable, $J(m\cdot T)=m\cdot J(T)$, and $J(T+U)=J(T) +J(U)$.*

The conclusions about $m\cdot T$ are evident, since if $\rho(J_\Delta(T))<\epsilon$, then $J_\Delta(m\cdot T)=m\cdot J_\Delta(T)$ is of diameter $<m\epsilon$. Those about $T+U=V$ follow since if $\rho(J_\Delta(T))<\epsilon$ and $\rho(J_{\Delta_1}(U))<\epsilon$, then

$$J_{\Delta\cdot\Delta_1}(V) \subset J_\Delta(T) + J_{\Delta_1}(U)$$

which is of diameter less than 2ϵ.

Corollary. $\|m\cdot T\|=|m|\cdot\|T\|$ *and* $\|T+U\|\leqq\|T\|+\|U\|$.

16. Multiplication by a scalar function. We shall now prove a very powerful result,

† Since $\rho(A)\leqq\rho(A+B)\leqq\rho(A)+\rho(B)$, for any A and B.

THEOREM 17. *If T is integrable over $\mathfrak{S}$, and $f(p)$ is any real-valued bounded Lebesgue integrable function over $\mathfrak{S}$, then the function $U(p)=f(p)\cdot T(p)$ is integrable, and*

$$\|U\| \leqq 2F\cdot\|T\| \quad \textit{where} \quad F \equiv \sup_{p\epsilon\mathfrak{S}} | f(p) | .$$

Let $\epsilon>0$ be given. By taking the product of a finite number of suitably chosen decompositions, we arrive at a decomposition Δ into subsets σ_i which satisfies the following conditions.

(1) On any σ_i, $-F\leqq c_i\leqq f(p)\leqq c_i+\epsilon\leqq F+\epsilon$.

(2) The aggregate $m(\sigma_i)\cdot T(\sigma_i)$ is unconditionally summable.

(3) $\|\sum_i m(\sigma_i)\cdot T(\sigma_i)-J(T)\|<\epsilon$.

Now by Theorem 8, the aggregates $m(\sigma_i)\cdot U(\sigma_i)$, $c_i\cdot J(T, \sigma_i)$, and $\epsilon_i\cdot J(T, \sigma_i)$ $[0\leqq \epsilon_i\leqq\epsilon]$ are unconditionally summable. Moreover for any choice of $p_i\epsilon\sigma_i$,

$$\left\| \sum_i m(\sigma_i)\cdot U(p_i) - \sum_i c_i\cdot J(T, \sigma_i)\right\|$$

$$\text{(B)} \qquad \leqq \left\| \sum_i c_i\cdot \left[m(\sigma_i)T(p_i) - J(T, \sigma_i)\right]\right\| + \left\| \sum_i \epsilon_i m(\sigma_i)\cdot T(p_i)\right\|$$

$$\leqq 2F\cdot\epsilon + \epsilon[\|T\| + \epsilon] = \epsilon[2F + \|T\| + \epsilon].$$

(The first inequality comes from writing $U(p_i)=c_iT(p_i)+\epsilon_iT(p_i)$; the first half of the second inequality from Theorem 8, and the second half from assumption (3) and Theorem 8.)

But $\epsilon[2F+\|T\|+\epsilon]$ tends to zero with ϵ; consequently, by Theorem 13, $U(p)$ is integrable. The relation $\|J(U)\|\leqq 2F\cdot\|T\|$ now follows by Theorem 8, since $\|T\|$ bounds the norms of the partial sums of the $J(T, \sigma_i)$ and $|c_i|\leqq F$. Replacing $\mathfrak{S}$ by σ, we have similarly $\|J(U, \sigma)\|\leqq 2F\cdot \sup_{\sigma^*\subset\sigma}\|J(T, \sigma^*)\|\leqq\|T\|$ (by definition of $\|T\|$). This completes the proof of the theorem.

17. **Denseness of finite-valued functions.** We shall next show that in a certain sense every integrable function can be approximated arbitrarily closely by finite-valued functions.

THEOREM 18. *Functions assuming only a finite number of distinct values are everywhere dense in the vector space of integrable functions normed by Definition 5.*

To any integrable function T and number $\epsilon>0$ corresponds a decomposition Δ of $\mathfrak{S}$ into subsets σ_i satisfying $\|J_\Delta(T)-J(T)\|<\frac{1}{2}\epsilon$. Moreover by Theorem 8 we can choose n so large that

$$\left\| \sum_{n+1}^{\infty} m_i^* \cdot T(\sigma_i) \right\| < \tfrac{1}{2}\epsilon$$

for all m_i^* satisfying $0 \leqq m_i^* \leqq m(\sigma_i)$.

Define the function U by the equations

$$U(p) = \begin{cases} J(T, \sigma_i)/m(\sigma_i) & \text{if } p\epsilon\sigma_i, \quad m(\sigma_i) > 0, \quad \text{and} \quad k \leqq n, \\ \Theta & \text{otherwise.} \end{cases}$$

U is clearly finite-valued and integrable.

But if σ is any set of Σ, and $\sigma \cdot \sigma_i$ and $\sigma_i - \sigma \cdot \sigma_i$ are denoted by σ_i^1 and σ_i^2 respectively, then $0 \leqq m(\sigma_i^1) \leqq m(\sigma_i)$, and so

$$\left\| \sum_{n+1}^{\infty} J(T - U, \sigma_i^1) \right\| = \left\| \overline{\mathrm{Co}\left(\sum_{n+1}^{\infty} m(\sigma_i^1) \cdot T(\sigma_i^1) \right)} \right\| < \tfrac{1}{2}\epsilon.$$

Again, $\Theta\epsilon[J_\Delta(T, \sigma_i^k) - J(T, \sigma_i^k)]$ for all i, k. Hence

$$\sum_{i=1}^{n} [J_\Delta(T, \sigma_i^1) - J(T, \sigma_i^1)] \subset \sum_{i,k} [J_\Delta(T, \sigma_i^k) - J(T, \sigma_i^k)],$$

and so

$$\left\| \sum_{i=1}^{n} J(T - U, \sigma_i^1) \right\| \leqq \|J_\Delta(T) - J(T)\| < \tfrac{1}{2}\epsilon.$$

By the complete additivity of $J(T, \sigma)$, we now get $\|J(T-U)\| < \epsilon$.

Theorem 18 can be extended even further to a result whose proof, although not difficult, is so long that we shall omit it. Accordingly, we state without proof

THEOREM 19. *The integrable functions of euclidean space to any separable Banach space $\mathfrak{S}$ are a separable space under the norm of Definition* 5.

The only construction involved is that of replacing each of the first n σ_i by an approximating point-set sum of a finite number of intervals with rational coordinates, and $J(T, \sigma_i)/m(\sigma_i)$ by a nearby element of $\mathfrak{B}$. The everywhere dense functions are then defined as Θ except on a finite number of intervals with rational coordinates, and constant on each such interval.

18. **Effect of linear transformation of the range.** It is almost self-evident that all of the arguments so far are preserved under linear transformations, since these preserve both sums and limits. Accordingly, we prove

THEOREM 20. *If T is any integrable function of $\mathfrak{S}$ to $\mathfrak{B}$, and α: $\beta \rightarrow \alpha(\beta)$ is any linear transformation of $\mathfrak{B}$ into the Banach space $\mathfrak{A}$, then* (i) *the function*

$U(p) \equiv \alpha(T(p))$ of $\mathfrak{S}$ to $\mathfrak{A}$ is integrable, (ii) $J(U) = \alpha(J(T))$, (iii) *if we denote the modulus*† *of α by a, then $\|U\| \leqq a \cdot \|T\|$.*

If T is summable under a decomposition Δ of $\mathfrak{S}$, then so is U, and $J_\Delta(U) = \alpha(J_\Delta(T))$. This is true by definition of U for single terms $m(\sigma_i) \cdot U(\sigma_i)$; it remains true for finite sums since α is additive, and under passage to the limit since $\|\alpha(\beta) - \alpha(\beta')\| / \|\beta - \beta'\|$ is bounded. The proof is completed by letting $\rho(J_\Delta(T))$ tend to zero.

Theorem 20, and the fact that for real-valued functions our integral reduces to Fréchet's interpretation of the Lebesgue integral, leads us to formulate

THEOREM 21. *Let T be any integrable function of $\mathfrak{S}$ to the Banach space $\mathfrak{B}$, and $f(\xi)$ a variable linear functional with domain $\mathfrak{B}$. For each f, $f(J(T))$ is the Lebesgue integral $\int_{\mathfrak{S}} f(T(p))dm(\mathfrak{S})$ of the real function $f(T(p))$ with domain $\mathfrak{S}$, and so $J(T)$ is the intersection (as f varies) of the hyperplanes of elements η of $\mathfrak{B}$ satisfying $f(\eta) = \int_{\mathfrak{S}} f(T(p))dm(\mathfrak{S})$.*

The first statement is a corollary of Theorem 19, and the second follows from the fact‡ that to every $\xi \neq J(T)$ there corresponds a linear functional f such that $f(\xi) \neq f(J(T))$.

We can deduce from Theorem 21 and known§ results

COROLLARY 1. *If $\sum_{k=1}^{\infty} \|T_k\| < +\infty$, $T(p) = \sum_{k=1}^{\infty} T_k(p)$ is defined almost everywhere and integrable, then $J(T) = \sum_{k=1}^{\infty} J(T_k)$.*

COROLLARY 2. *A function of a square to a Banach space $\mathfrak{B}$ can only fail to satisfy the Theorem of Fubini because of non-integrability.*

COROLLARY 3. *If $T_0(t), T_1(t), \cdots, T_n(t)$ are functions of the line interval $[0, x]$ to $\mathfrak{B}$ such that*

$$T_k(t) - T_k(0) = J(T_{k+1}, [0, t])$$

for $k = 0, 1, \cdots, n-1$ and $0 \leqq t \leqq x$, and if B_x denotes the range of $T_n(t)$ on $[0, x]$, then

$$T_0(x) \subset \sum_{k=0}^{n-1} (x^k/k!) \cdot T_k(0) + (x^n/n!) \cdot \overline{\mathrm{Co}\,(B_x)}.$$

(This is Taylor's formula with the remainder.)

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† That is, $\sup_{\beta \neq \Theta} \|\alpha(\beta)\| / \|\beta\|$. Banach calls this the norm of α; we have not conformed to his usage for fear of confusion with the other norms which we have defined.

‡ Banach, p. 55.

§ C. Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig, 1927. We refer to Theorem 13, p. 441, and Theorem 1, p. 627. I am stating Corollary 3 without reference.

19. **Uniform convergence and the method of iteration.** We can prove by a very superficial argument

THEOREM 22. *If* $T_1, T_2, T_3, \cdots$ *are integrable functions of* $\mathfrak{S}$ *to* $\mathfrak{B}$, $m(\mathfrak{S})$ *is finite, and the* T_n *converge uniformly to* T, *then* T *is integrable and* $J(T) = \lim_{n\to\infty} J(T_n)$.

For n exists so large that $\|T(p)-T_n(p)\| < \epsilon/(2m(\mathfrak{S}))$ for all p, and Δ such that every element of $J_\Delta(T_n)$ lies within $\frac{1}{2}\epsilon$ of $J(T_n)$. Under these circumstances every element of $J_\Delta(T)$ lies within $\frac{1}{2}\epsilon$ of some element of $J_\Delta(T_n)$, and hence within ϵ of $J(T_n)$. The theorem is now obvious.

THEOREM 23. *If* $K(x, y)$ *is any function of the square* $0 \leqq x, y \leqq 1$ *to* $\mathfrak{B}$ *which is uniformly continuous in* x, *Lebesgue integrable in* y, *and satisfies* $|K(x, y)| \leqq M < +\infty$, *then the integral equation*

$$S(x) = T(x) + c\int_0^1 K(x, y)S(y)dy \tag{1}$$

has a unique solution $S(x)$ *provided* $2cM < 1$.

First we construct a solution by iteration. We set $S_k(x) = \Theta$, and $S_{k+1}(x) = T(x) + c\int_0^1 K(x, y)\cdot S_k(y)dy$. By Theorem 17 these integrals exist, and satisfy

$$\|S_{k+1}(x) - S_k(x)\| = [2cM]^k\cdot\|T\|$$

by induction and Theorem 17. Hence the series of differences converges uniformly to a limit satisfying (1).

Theorem 17 may also be used to show that as the homogeneous equation corresponding has no solution, the solution obtained in this way by iteration is unique.

As a matter of fact, inspection shows that the entire Fredholm theory of integral equations of the form (1) carries over to the case where $S(x)$ and $T(x)$ are permitted to be vector functions. The only change is that when the determinant of $K(x, y)$ vanishes, the solutions to the equation

$$S(x) = c\int_0^1 K(x, y)S(y)dy \tag{2}$$

are of the form $f_1(x)\xi_1 + \cdots + f_r(x)\xi_r$, where the $f_i(x)$ are linearly independent *real* solutions of (2), and the ξ_i are arbitrary vector constants.

20. **Consequences of separability.** Since the sum of enumerable sets of measure zero is again of measure zero, it is possible to extend certain theorems concerning the Lebesgue integral to our integral in the case that $\mathfrak{B}$ is *separable*, that is, contains an enumerable everywhere dense set of elements.

In the first place, if we define two functions T and U of $\mathfrak{S}$ to $\mathfrak{B}$ as "equivalent" if and only if $T(p)=U(p)$ except on a set of measure zero, then

THEOREM 24. *If $\mathfrak{B}$ is separable, then two point functions T and U of $\mathfrak{S}$ to $\mathfrak{B}$ give rise to the same set function $J(T,\sigma)=J(U,\sigma)$ if and only if T and U are equivalent.*

The sufficiency of equivalence is obvious. But conversely, if $J(T,\sigma)=J(U,\sigma)$, then by Theorem 21 and the theory of the Lebesgue integral, $f(T(p))=f(U(p))$ except on a set of measure zero, for any functional $f(\xi)$. But we can define functionals $f_i(\xi)$ of modulus unity† such that $f_i(\xi_i)=\|\xi_i\|$ for each of an enumerable everywhere dense set of elements ξ_i of $\mathfrak{B}$.

But if $T(p)-U(p)=\xi\neq\Theta$, then choosing ξ_i satisfying $\|\xi_i-\xi\|<\frac{1}{2}\|\xi\|$, clearly $f_i(\xi)>0$. Hence except on the set of measure zero on which some $f_i(T(p)-U(p))\neq 0$, $T(p)=U(p)$. This proves Theorem 24.

COROLLARY. *If $\mathfrak{B}$ is separable, then every integrable function is single-valued except on a set of measure zero.*

Again, suppose $\mathfrak{S}$ is the line interval $[a, b]$. A function $T(x)$ of $\mathfrak{S}$ to $\mathfrak{B}$ is said to have the "strong" derivative $T'(x)$ at the point x if and only if $(1/h)\cdot[T(x+h)-T(x)]$ tends to $T'(x)$ as $|h|$ tends to zero; it is said to have the "weak" derivative $T'(x)$ if and only if for every linear functional $f(\xi)$,

$$\lim_{|h|\to 0}[f(T(x+h))-f(T(x))]/h=f(T'(x)).$$

THEOREM 25. *If the space of the functionals of $\mathfrak{B}$ is separable, then the integral $J(T,[a,x])$ of any integrable function T of a line interval to $\mathfrak{B}$ is weakly differentiable except on a set of measure zero.*

Theorem 25 is a corollary of Theorem 21 and the truth of the theorem for the real functions $f(J(T,[a,x]))$.

21. Rectifiable curves in Hilbert space. It is well known that any real function of bounded variation is differentiable almost everywhere. We shall extend this proposition.

THEOREM 26. *Every function $h(x)$ of bounded variation‡ of an interval $[a, b]$ to Hilbert space $\mathfrak{H}$ is (strongly) differentiable except on a set of measure zero. Moreover its derivative is integrable to $h(b)-h(a)$.*

In the first place, if $\mathfrak{F}$ and $\mathfrak{G}$ are orthogonal complements§ in $\mathfrak{H}$, then the

† Banach, p. 55. Of course the f_i are not in general everywhere dense in the space of the functionals of $\mathfrak{B}$.

‡ I.e., such that $\sup\sum_1^n\|h(x_i)-h(x_{i-1})\|<+\infty$ $(x_0<\cdots<x_n)$. This upper limit we shall call the "total variation" of $h(x)$.

§ We have borrowed without explanation standard terms such as orthogonal complement and orthonormal vector from M. H. Stone's *Linear Transformations in Hilbert Space*, New York, 1932.

projections $f(x)$ of $h(x)$ onto $\mathfrak{F}$ and $g(x)$ of $h(x)$ onto $\mathfrak{G}$ are both of bounded variation. While if we denote by $|f|$, $|g|$, and $|h|$ the total variations of $f(x)$, $g(x)$, and $h(x)$ respectively, then

$$(1) \qquad (|f|^2 + |g|^2)^{1/2} \leqq |h| \leqq |f| + |g|.$$

Now divide $[a, b]$ so finely by points $a = x_0 < \cdots < x_n = b$ that

$$(2) \qquad |h| - \epsilon = \sum_1^n \|h(x_i) - h(x_{i-1})\| \leqq |h|.$$

The $h(x_i)$ will all lie on a linear manifold $\mathfrak{F}$ of dimensions at most $n+1$. And if $\mathfrak{G}$ is the orthogonal complement of $\mathfrak{F}$, then by elementary algebra and conditions (1)–(2),

$$(3) \qquad |g| \leqq (|h|^2 - |f|^2)^{1/2} \leqq (2\epsilon \cdot |h|)^{1/2}.$$

Further, construct the non-decreasing real variation function

$$v_g(x) = \int_a^x \|dg(t)\| \leqq |g| \leqq (2\epsilon \cdot |h|)^{1/2}.$$

By the theory of real functions, $v_g'(x)$ exists and does not exceed $(2\epsilon \cdot |h|)^{1/4}$ except on a set S of measure at most $(2\epsilon \cdot |h|)^{1/4}$. But clearly

$$(4) \qquad \left\| \frac{g(x + \Delta x) - g(x)}{\Delta x} \right\| \leqq \frac{1}{\Delta x} \int_x^{x+\Delta x} \|dg(t)\| = \frac{1}{\Delta x} \int_x^{x+\Delta x} dv_g.$$

Therefore if x is not on S, then, for small enough Δx,

$$(5) \qquad \begin{aligned} \left\| \frac{h(x + \Delta x) - h(x)}{\Delta x} - \frac{f(x + \Delta x) - f(x)}{\Delta x} \right\| &= \left\| \frac{g(x + \Delta x) - g(x)}{\Delta x} \right\| \\ &\leqq (2\epsilon \cdot |h|)^{1/4}. \end{aligned}$$

Consequently except on a set S of measure $= (2\epsilon \cdot |h|)^{1/4}$ of $[a, b]$, for each x and sufficiently small Δx, the difference quotients $\|\{g(x+\Delta x) - g(x)\}/\Delta x\|$ are bounded by $(2\epsilon \cdot |h|)^{1/4}$.

But since $\mathfrak{F}$ has only a finite number of dimensions, all the components of $f(x)$, and hence $f(x)$ itself, have finite derivatives except on a set S' of measure zero. Therefore if $|\Delta x| < M_x$, a sufficiently small positive function of x, then the diameter of the set of the $\{h(x+\Delta x) - h(x)\}/\Delta x$ is at most $2 \cdot (2\epsilon \cdot |h|)^{1/4}$ for any x not on $S \dotplus S'$ [of measure $\leqq (2\epsilon \cdot |h|)^{1/4}$]. The proof of the first assertion is completed by letting ϵ tend to zero.

To prove the second assertion, decompose $[a, b]$ so that the integral ranges of $f'(x)$ and $v_g'(x)$ are both of diameters $< \frac{1}{2}\epsilon$. It will follow that the

integral range of $h(x)$ lies within $\epsilon+|g|$ of $f(b)-f(a)$. The remainder of the proof is obvious.

If we use the variation-function $s(x)=\int_a^x\|dh(t)\|$ as the parameter for the domain, then we get the geometrical

COROLLARY 1. *Any rectifiable curve in Hilbert space has a tangent at almost every point.*

We are more interested in the analytical

COROLLARY 2. *If $h(t)$ is a bounded function of a line interval to Hilbert space, and $g(x)=J(h, [a, x])$, then $g(x)$ is (strongly) differentiable almost everywhere to $h(x)$.*

Corollary 2 depends on Theorem 24.

22. **Examples of non-differentiable integrals.** In this and the following section we shall give seven examples illustrating various theoretical points. In all seven we shall understand the symbol $\mathfrak{H}$ to denote Hilbert space, and $\mathfrak{B}$ to denote the space of real bounded functions $y=y(x)$ on the interval $[0, 1]$, with the norm $\|y\|=\sup_x y(x)$. We shall also use $\xi_{i,j}$ to denote a doubly infinite set of orthonormal vectors in $\mathfrak{H}$.

We shall first show by two examples that the hypotheses of Corollary 2 of Theorem 26 are indispensable.

EXAMPLE 1. *There exists a totally discontinuous bounded integrable function of the interval $[0, 1]$ to $\mathfrak{B}$, whose integral, although of bounded variation, is nowhere even weakly differentiable.*

Let y_r in $\mathfrak{B}$ correspond to the function $y_r(x)=0$ on $[0, r)$, $y_r(x)=1$ on $[r, 1]$. Graves observed† that the function $T(r)=y_r$ of $[0, 1]$ to $\mathfrak{B}$ was bounded, totally discontinuous, and integrable. Since $y(a)$ is a linear functional $f_a(y)$ of the elements y: $y(x)$ of $\mathfrak{B}$, and $U(s)=J(T, [0, s])$ is not differentiable at a with respect to f_a, $U(s)$ is nowhere even weakly differentiable, although there is a bound on the norm of its differential quotients $\Delta U/\Delta s$.

EXAMPLE 2. *There exists an integrable function of the interval $[0, 1)$ to $\mathfrak{H}$, whose integral is nowhere strongly differentiable.*

Set $T_i(t)=2^i\cdot\xi_{i,j}$ on $[j/2^i, 2^{-2i}+j/2^i)$ $[j=0, \cdots, 2^i-1]$. The function $V(t)=\sum_1^\infty T_i(t)$ exists almost everywhere, and is integrable. The proofs of these facts, and of the nowhere strong differentiability of $W(x)=J(V, [0, x])$, are left to the reader.

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† Loc. cit., p. 164. Cf. also S. Bochner, *Absolut-additive Mengenfunktionen*, Fundamenta Mathematicae, vol. 21 (1933), pp. 211–13, in which a function essentially like $U(s)$ is shown to be nowhere strongly differentiable.

Example 1 is not integrable in the sense of Bochner, and Example 2 is not integrable in the sense of Graves. This shows that our integral is included in neither [cf. §24].

23. **Other counterexamples.** We shall list below examples showing that the hypotheses of various other theorems proved above cannot be eliminated.

EXAMPLE 3. *There exists a function T of the interval* $[0, 1]$ *to* $\mathfrak{B}$ *such that* $\|T(x)\| \equiv 1$ *and yet* $\|T\| = 0$.

Let T assign to each point x of $[0, 1]$ the function f_x: $f_x(x) = 1$, $f_x(t) = 0$ for $t \neq 0$. The proofs that $T(x)$ is integrable and that $J(T, \sigma) = 0$ for any σ of Σ are left to the reader. This example shows that we cannot drop the hypothesis of separability in Theorem 24.

EXAMPLE 4. *To any* $\epsilon > 0$ *corresponds a function T of the interval* $[0, 1)$ *to* $\mathfrak{H}$ *such that* $\|T(x)\| \equiv 1$ *and yet* $\|T\| < \epsilon$.

Let us choose n so large that $n\epsilon > 1$. Let $T(x) = \xi_{k,1}$ on $[(k-1)/n^2, k/n^2)$ $[k = 1, \cdots, n^2]$. The proof that $\|T\| < \epsilon$ is left to the reader.

EXAMPLE 5. *There exists a Cauchy sequence of integrable functions of the interval* $[0, 1)$ *to* $\mathfrak{H}$ *converging* (*relative to the norm of Definition* 5) *to no limit function.*

Let $T_i(x) = \xi_{i,j}$ on $[(j-1)/2^i, j/2^i)$ $[j = 1, \cdots, 2^i]$. Evidently $\|T_i\| = 2^{-i/2}$; hence if $U_n = T_1 + \cdots + T_n$, then the sequence $\{U_n\}$ is a Cauchy sequence (under Definition 5).

Suppose $U(x)$ existed, satisfying $\lim_{n\to\infty}\|U_n - U\| = 0$. By Theorem 21 the scalar product (which is a linear functional) of $U(x)$ with each $\xi_{i,j}$ would have to be that of $T_i(x)$ with $\xi_{i,j}$, except on a set of measure zero. But there can be no $U(x)$ in $\mathfrak{H}$ having the necessary $\xi_{i,j}$-components; the sum of the squares tends to infinity.

Example 5 shows that although integrable functions constitute a normed vector space under Definition 5 (provided we consider two integrands as the "same" if they give rise to the same set function), this space is not in general complete.

EXAMPLE 6. *There is a function of the unit square to* $\mathfrak{H}$ *not satisfying the Theorem of Fubini.*

We shall use the functions $T_i(x)$ of Example 5, and set $U(x, y) = 2^i \cdot T_i(x)$ on $2^{-i} \leq y < 2^{-i+1}$, and Θ where not otherwise defined. The function $U(x, y)$ cannot be integrated with respect to y on a single line $x = \text{constant}$, and yet it is integrable over the square to

$$\sum_{i=1}^{\infty} \sum_{j=1}^{2^i} (2^{-i}) \cdot \xi_{i,j}.$$

Example 7. *There exists an integrable function of the interval* $[0, 1)$ *to* $\mathfrak{H}$, *such that* $\|T(x)\|$ *is not integrable.*

Set $T(x) = (2^i/i) \cdot \xi_{i,1}$ on $[2^{-i}, 2^{-i-1})$. The proofs are left to the reader.

24. **Relation to integrals of Graves and Bochner.** In §22 it was noted that the integral used in this paper was included in neither Graves' nor Bochner's. It will now be shown that on the contrary it includes both.

Without repeating Graves' definition, it may be said that if $T(x)$ is integrable in his sense, then the integral of T is arbitrarily near every Riemann sum $\sum_i T(x_i)\Delta x_i$ under a suitable decomposition of the domain of T into a finite number of intervals. But if this is so, then by Theorems 12–13 proved above, the function is integrable to the same integral in the sense of Definitions 1–4.

Again, let $T(x)$ be any function integrable in the sense of Bochner over an abstract domain $\mathfrak{S}$ of finite measure M. That is, suppose that

(1) the real function $\|T(p)\|$ is summable;

(2) $T(p) = \lim_{n\to\infty} T_n(p)$ for almost every p in $\mathfrak{S}$;

(3) each $T_n(p)$ is a "finite-valued" function.

Let $\epsilon > 0$ be given. On a set $\mathfrak{S}_\epsilon$ of measure $> M - \epsilon$, the $T_n(p)$ converge *uniformly* to $T(p)$.

Again, let $\delta > 0$ be given. Choose n so large that $\|T(p) - T_n(p)\| < \delta/M$ on $\mathfrak{S}_\epsilon$. Then decompose $\mathfrak{S}_\epsilon$ into sets σ_i $[i = 1, \cdots, s]$ of measures m_i, on which $T_n(p)$ is constant and has the values ξ_i.

The integral range $J_\Delta(T, \mathfrak{S}_\epsilon)$ under this decomposition clearly lies within $M \cdot (\delta/M) = \delta$ of $\sum_{i=1}^{s} m_i \cdot \xi_i$. Hence T is integrable in my sense over $\mathfrak{S}_\epsilon$, and the integral $J(T)$ of T is $\lim_{n\to\infty} \sum_{i=1}^{n} m_i \cdot \xi_i$, i.e., equal to Bochner's integral.

Since both Bochner's integral and mine are completely additive set functions, this can be extended to $\mathfrak{S}$, and thence to domains which are the sums of enumerable sets of finite measure.

25. **Open questions.** There are several open questions of a technical nature concerning the subject we have just treated.

One is as to the replaceability of the closure of the convex hull by the convex hull alone, without affecting the program. For instance, is $\sum_i \mathrm{Co}(B_i) \subset \mathrm{Co}(\sum_i B_i)$? The accomplishment of this might simplify the definitions and argument.

Another is that of generalizing the integral to the case that the range is merely a *topological* vector space, in which $\alpha_n \to \alpha$ and $\beta_n \to \beta$ imply $\alpha_n + \beta_n \to \alpha + \beta$, and $b_n \to b$ and $\beta_n \to \beta$ imply $b_n\beta_n \to b \cdot \beta$. Both these problems are trivial if we restrict ourselves to Graves' Riemann integral, but with unconditional convergence, they become very deep.

Two questions which I have been unable to answer are, do there exist (non-integrable) functions, the intersection of whose integral ranges is vacuous? or such that when we apply Theorem 21, the intersection of the hyperplanes

$$f(\eta) = \int_{\mathfrak{S}} f(T(p))dm(\mathfrak{S})$$

turns out to be vacuous? [Added in proof: the answer to the second question is yes.]

Is it true that if the "conjugate" space of the linear functionals of $\mathfrak{B}$ is separable, then the integral of any bounded integrable function is strongly differentiable almost everywhere? Or that every rectifiable curve in $\mathfrak{B}$ has a tangent at almost every point?

And is it true that if unconditional convergence implies absolute convergence in $\mathfrak{B}$, then the vector space of integrable functions is complete under the norm of Definition 5?

It may be proved that any vectoroid space satisfying

$V7$: $(b_1+b_2)\cdot B = b_1\cdot B + b_2\cdot B$

is a vector space. Is it true that if we merely require

$V7'$: $B+(-1)\cdot B = \Theta$

we can still deduce all the properties of vector space? How, if at all, can $V1$–$V6$ be replaced by properties of the calculus of vector complexes in such a way that the addition of $V7'$ does give a complete set of axioms for vector spaces?

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FUNCTIONAL DEPENDENCE*

BY

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1. Introduction. The condition for dependence of n functions of $n+p$ variables is roughly that every determinant of order n formed from the matrix of the first partial derivatives vanish identically. The theorem easiest to prove assumes condition (A): *One of the determinants of highest order which do not vanish identically is different from zero at a given point.* The first theorem free from condition (A) is due to Bliss,† who established an analytic relation in the case of two analytic functions of not more than two variables. Osgood‡ proved that, for the case of three or more analytic functions of as many variables, the identical vanishing of the Jacobian does not necessarily imply that the functions satisfy an analytic relation. No result of a positive nature was given in this case.

More recently, Knopp and Schmidt§ have established a relation for the case of n real functions, of class C',‖ of not more than n variables. The result is obtained in the large, and free of condition (A).¶

In the present paper we treat first the real case, extending the results

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† G. A. Bliss, *Fundamental Existence Theorems*, Colloquium Publications of the American Mathematical Society, vol. 3, part 1, 1913. See also Osgood, *Lehrbuch der Funktionentheorie*, vol. 2, part 1, chapter 2, §24, where a treatment involving parameters is given. We refer to the latter book as Osgood II.

‡ W. F. Osgood, *On functions of several complex variables*, these Transactions, vol. 17 (1916), pp. 1–8.

§ K. Knopp and R. Schmidt, *Funktionaldeterminanten und Abhängigkeit von Funktionen*, Mathematische Zeitschrift, vol. 25 (1926), pp. 373–381. We refer to this paper, and to the authors, as K and S.

‖ A function of class $C^{(k)}$ is one having all partial derivatives, continuous, of order k. A function of class $C^{(\infty)}$ is one having all partial derivatives, continuous, of every finite order.

¶ For a proof of the same results under the weaker hypotheses that the functions need not be of class C' but are merely differentiable in the sense of Stolz, see A. Ostrowski, *Funktionaldeterminanten und Abhängigkeit von Funktionen*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 36 (1927), pp. 129–134.

We mention also a paper by G. Doetsch, *Die Funktionaldeterminante als Deformationsmass einer Abbildung und als Kriterium der Abhängigkeit von Funktionen*, Mathematische Annalen, vol. 99 (1928), pp. 590–601. He defines a point as regular if the matrix of first partial derivatives has the same rank there as at all sufficiently nearby points, otherwise singular. He establishes a functional relation without restriction on the number of variables, under each of the following hypotheses: (i) there are no singular points in the set under consideration; (ii) the singular points are mapped on a set of measure zero; (iii) the singular points lie on an at most denumerably infinite set of n-dimensional manifolds.

of K and S to the case that the number of variables is arbitrary (but finite). The proof of K and S does not generalize in any obvious manner so as to give this result. No conditions like those of Doetsch are imposed. Conditions of differentiability are imposed. It would be interesting to know to what extent, if any, these conditions are necessary. [Added in proof: See footnote to Theorem 4.II.]

In the second part we treat the case of n analytic functions of any finite number of variables. We establish here an important property which would obtain if an analytic relation did exist, namely that the point set determined by the given functions in the representing space for the values of the functions has the properties that it is nowhere dense* and that it cannot disconnect any region† of the space.

Finally we treat the case of two analytic functions of n complex variables, extending the results of Bliss to the case that n is no longer restricted not to exceed 2. Our relation, like that of Bliss, is satisfied only by the values of the given function. In this case we construct a new proof, different from that of Bliss.

In all cases parameters are included. In the case of reals, a relation is obtained which is satisfied only by the values of the functions, which is not the case with K and S.‡

2. **Preliminary lemmas.** We now introduce some lemmas which are helpful in the subsequent proofs.

LEMMA 2.I. *Let a transformation*

$$u_j = v_j(x_1, \cdots, x_n) = v_j(x) \qquad (j = 1, \cdots, m),\ m > 1,\ n > 1, \tag{2.1}$$

be given, where $v_j(x)$ is real and continuous over a closed bounded set K of real (x)-space, with B a closed subset of K. Then if each point of B has a neighborhood on B whose image under (2.1) *is nowhere dense in real (u)-space, and if each point of $K-B$ has a neighborhood on $K-B$ whose image is likewise nowhere dense, the image of K in (u)-space is nowhere dense.*§

Since the sum of a finite number of nowhere dense sets is nowhere dense, it follows easily upon use of the Heine-Borel theorem that the image of B is nowhere dense. If Q is any point of (u)-space, and $\mathcal{N}$ any neighborhood of

* This first property also follows easily from the results in the case of reals.

† Region denotes connected open set, hence connected by curves (Hausdorff, *Mengenlehre*, 2d edition, p. 154, Theorem VIII). Domain denotes (non-vacuous) open set.

‡ To obtain this result we modify the treatment in one of the two parts of K and S. It was found necessary to replace the other part of K and S by a different treatment.

The present paper is complete in itself.

§ A set S is dense at a point if the point has a neighborhood which consists of limit points of S.

Q, a set $\mathcal{D}(B)$ can be found open on K and containing B, with image having no point on a sub-neighborhood $\mathcal{N}_1$ of $\mathcal{N}$. By applying the Heine-Borel theorem to $K-\mathcal{D}(B)$, we then find that the image of K has no point on a sub-neighborhood $\mathcal{N}_2$ of $\mathcal{N}_1$. As Q is arbitrary, we infer that the lemma is true.

LEMMA 2.II. *Under the hypotheses of the preceding lemma, suppose each point of B has a neighborhood on B whose image in (u)-space is not only nowhere dense but also has the property that it cannot disconnect any region of (u)-space, and each point of $K-B$ has a neighborhood on $K-B$ whose image satisfies the same conditions. Then the image of K in (u)-space is nowhere dense and cannot disconnect any region of (u)-space.*

As the proof is similar to that of Lemma 2.I, we omit it.

LEMMA 2.III. *Given the equations* (2.1) *with $v_j(x)$ of class C', and with the matrix Δ of first partial derivatives of rank $\leqq r$ neighboring a point $(x^0)=(x_1^0, \cdots, x_n^0)$, with $0<r$, suppose $\partial v_1/\partial x_1 \neq 0$ at (x^0). If we substitute the solution*

$$x_1 = \xi_1(u_1, x_2, \cdots, x_n) \tag{2.2}$$

of the first equation in (2.1), *determined at (x^0), in the remaining equations* (2.1), *obtaining*

$$u_j = \zeta_j(u_1, x_2, \cdots, x_n) \qquad (j = 2, \cdots, m), \tag{2.3}$$

then the matrix of the first partial derivatives of the ζ's with respect to $x_2, \cdots, x_n$ has rank $\leqq r-1$ for $(u_1, x_2, \cdots, x_n)$ near $[v_1(x^0), x_2^0, \cdots, x_n^0]$.

If either $n-1$ or $m-1$ is less than r, the result is obvious. In the contrary case, take any r-rowed determinant of the partial derivatives, as

$$\|\alpha\| = \left\| \frac{\partial(\zeta_2, \cdots, \zeta_{r+1})}{\partial(x_2, \cdots, x_{r+1})} \right\|.$$

With the help of the identity

$$\zeta_j(u_1, x_2, \cdots, x_n) \equiv v_j[\xi_1(u_1, x_2, \cdots, x_n), x_2, \cdots, x_n]$$

and the fact that (2.2) is the solution of the first equation (2.1), the $(j-1)$st row of $\|\alpha\|$ can then be written $(j=2, 3, \cdots, r+1)$:

$$\left\| \left[\frac{\partial v_j}{\partial x_1}\left(-\frac{\partial v_1}{\partial x_2}\Big/\frac{\partial v_1}{\partial x_1}\right) + \frac{\partial v_j}{\partial x_2}\right] \cdots \left[\frac{\partial v_j}{\partial x_1}\left(-\frac{\partial v_1}{\partial x_{r+1}}\Big/\frac{\partial v_1}{\partial x_1}\right) + \frac{\partial v_j}{\partial x_{r+1}}\right] \right\|.$$

Since each bracket is a sum of two terms, α equals a sum of 2^r determinants. But any of these which contain at least two columns of first terms are zero, since those columns are proportional. There remain only $r+1$ of the 2^r determinants, and their sum is seen to be the expansion of

$$\beta = \frac{\partial(v_1, v_2, \cdots, v_{r+1})}{\partial(x_1, x_2, \cdots, x_{r+1})}$$

by minors of the first row, except for a factor $\partial v_1/\partial x_1$ of each term. Since Δ is of rank $\leqq r$, $\beta = 0$. Hence $\alpha = 0$ and the lemma is true.

3. A nowhere dense map. Before stating the next lemma, we introduce the following notation. If m and n are integers, $m \neq 0$, let k_m^n and $t(m, n)$ be the integers defined as follows:

$$n/m \leqq k_m^n < (n/m) + 1; \tag{3.1}$$

$$t(m, n) = k_m^n \text{ if } n \leqq 2m, \tag{3.2}$$

$$t(m, n) = t(m, n - 1) + k_m^n - 2 \text{ if } n > 2m. \tag{3.3}$$

LEMMA 3.I. *Given the functions $v_j(x_1, \cdots, x_n)$, $j = 1, \cdots, m$, of class $C^{(t)}$, $t \geqq 1$, in a domain $\mathcal{D}$ of real (x)-space, let L be a closed bounded subset of $\mathcal{D}$, at each point of which all the first partial derivatives of all the v's are zero. Suppose $t \geqq t(m, n)$. Then under the transformation*

$$u_j = v_j(x_1, \cdots, x_n) \equiv v_j(x) \qquad (j = 1, \cdots, m), \tag{3.4}$$

the image of L in (u)-space is nowhere dense.

From (3.2) and (3.3) we see that $k_m^n \leqq t(m, n)$, and since $t \geqq t(m, n)$ it follows that $t \geqq k_m^n$.

Let $k = k_m^n$ and L_1 be the locus of points of L at which all partial derivatives of orders 1 to k inclusive, of all the v's, are zero. Let $b > 0$ be a constant such that the distance from L to the boundary, if any, of $\mathcal{D}$ is $> n^{1/2}b$. Given $\eta > 0$, we choose δ, with $0 < \delta < b$, so that at all points not farther than $n^{1/2}\delta$ from L_1, any kth-order partial derivative of v_j, $j = 1, \cdots, m$, is in absolute value less than η. Next we subdivide (x)-space into n-cubes by planes $x_j = p/2^h$, p any integer, choosing h as a positive integer such that $\epsilon = 1/2^h < \delta$. We consider those closed cubes q of this subdivision each of which contains a point of L_1. If we let P' be the image in (u)-space, under (3.4), of a point P on L_1 in a cube q, it follows from Taylor's theorem with the remainder that the other points of q are transformed into points whose coordinates differ from those of P' by less, in any case, than

$$\frac{n^k \epsilon^k}{k!} \cdot \eta = \frac{\zeta}{2} \epsilon^k \qquad \left(\zeta = \frac{2n^k}{k!} \cdot \eta\right).$$

Therefore the transforms of the points of q lie in a cube of edge $\leqq \zeta\epsilon^k$, hence of volume $\leqq \zeta^m \epsilon^{km}$. Since the volume of the original cube q is ϵ^n, the ratio of the volumes is $\zeta^m \epsilon^{km-n}$. Now $km - n \geqq 0$, by (3.1), since $k = k_m^n$; and ζ and ϵ

can be made as small as we like. Since the total volume of the cubes q does not increase as $\epsilon = 1/2^h$ becomes smaller, it follows that L_1 can be enclosed in a finite set of closed cubes whose image has Jordan measure less than a fixed preassigned constant. Hence the image of L_1 has Jordan measure zero, and since the image is a closed set, it must be nowhere dense.*

If $n \leqq m$ then $k=1$ and $L_1 = L$, and the proof of Lemma 3.I is completed. If $n > m$ then $k > 1$ and we continue.

From Lemma 2.I we see that we can now confine our attention to a neighborhood of an arbitrary point of $L - L_1$. First we take the case of a point where all partial derivatives of orders 1 up to $k-1$, of all the v's, are zero. As we are going to treat separately the case (which can arise only if $k > 2$) of a point at which at least one derivative of order $k-1$ is different from zero, it follows, again from Lemma 2.I, that for the present we need merely consider a closed set, say L_2, in a neighborhood of a point Q of $L - L_1$, with all partial derivatives of orders up to $k-1$ vanishing at each point of L_2.

Since Q is not on L_1, at least one derivative of order k is not zero at Q, say $\partial^k v_1/\partial x_1{}^k \neq 0$. Since $n > m$, $n > 1$. Now we apply the implicit function theorem to the locus $\partial^{k-1} v_1/\partial x_1{}^{k-1} = 0$, which contains L_2, obtaining that locus in the form

$$(3.5) \qquad x_1 = \xi_1(x_2, \cdots, x_n), \quad \text{of class } C^{(t-k+1)}.\dagger$$

We take L_2 small enough so that (3.5) contains L_2. Let l_2 be the locus of points of $(x_2, \cdots, x_n)$-space which, under (3.5), give points of L_2. Now we substitute in (3.4) obtaining the equations

$$(3.6) \qquad u_j = v_j[\xi_1(x_2, \cdots, x_n), x_2, \cdots, x_n] \qquad (j = 1, \cdots, m).$$

Let $b_1 > 0$ be a constant such that for points of $(x_2, \cdots, x_n)$-space within distance $(n-1)^{1/2} b_1$ of l_2, (3.5) still holds and gives points within distance $n^{1/2} b$ of L. Let M be the larger of 1 and $(n-1) \cdot$(maximum of $|\partial \xi_1/\partial x_2|, \cdots, |\partial \xi_1/\partial x_n|$ for points within distance $(n-1)^{1/2} b_1$ of l_2). Let $\epsilon_1 > 0$ be a constant, with $0 < \epsilon_1 < b_1$. Then it follows from the law of the mean that if the coordinates of a point $(x_2, \cdots, x_n)$ differ from those of a point of l_2 by not more than ϵ_1, the corresponding values of x_1 differ by at most $M\epsilon_1$. Let H be an upper bound for the absolute values of the partial derivatives of order k, of all the v's, for points of $\mathcal{D}$ within distance $n^{1/2} b$ of L. Now we apply Taylor's theorem with the remainder to the functions in (3.4), for two points of the locus (3.5),

* In using this property to show that the closed set is nowhere dense, namely that it has Jordan measure zero, we follow K and S.

† That the class of ξ_1 is at least that of $\partial^{k-1} v_1/\partial x_1{}^{k-1}$ is seen easily from the formulas for the derivatives of ξ_1.

one on L_2, whose respective coordinates $x_2, \cdots, x_n$ differ by at most ϵ_1. Hence their coordinates x_1 differ by at most $M\epsilon_1 \geqq \epsilon_1$. In applying Taylor's theorem, all derivatives of orders less than k are taken at the first point, and the remainder term is a sum of derivatives of order k. Therefore, by (3.6), the corresponding two points in (u)-space have coordinates respectively differing by less than

$$\frac{n^k(M\epsilon_1)^k}{k!}\cdot H = \frac{H_1}{2}(\epsilon_1)^k.$$

Therefore if we cover a portion of $(x_2, \cdots, x_n)$-space containing l_2 by a network of $(n-1)$-cubes, of edge ϵ_1, each of which has at least one point in l_2, each of these cubes is mapped by (3.5) and (3.4) on a subset of a cube in (u)-space of edge $H_1(\epsilon_1)^k$. The ratio of the volumes of two such cubes is $(H_1)^m(\epsilon_1)^{km-n+1}$. Since $k=k_m^n$, $km-n+1>0$. As ϵ_1 can be made as small as we like, we conclude that the image of L_2 has Jordan measure zero, hence is nowhere dense.

If $k=2$, Lemma 3.I is now proved. If $n\leqq 2m$ then k must equal 1 or 2, and we see that we have now proved the following:

LEMMA 3.II. *If $n\leqq 2m$, Lemma* 3.I *is true.*

If $k>2$, we continue with the proof of Lemma 3.I. Again by use of Lemma 2.I we find that we can next turn our attention to a closed subset L_3 of L, neighboring a point of L, with all partial derivatives of $v_1, \cdots, v_m$, of orders 1 to $k-2$ inclusive, zero at each point of L_3, but with some particular $(k-1)$st-order partial derivative, say $\partial^{k-1}v_1/\partial x_1^{k-1}$, not zero in L_3. Applying the implicit function theorem to the locus $\partial^{k-2}v_1/\partial x_1^{k-2}=0$, we obtain $x_1=\xi_2(x_2, \cdots, x_n)$, ξ_2 of class $C^{(t-k+2)}$. We substitute this function into the equations (3.4), obtaining

$$u_j = \zeta_j(x_2, \cdots, x_n) \qquad (j = 1, \cdots, m), \tag{3.7}$$

with ζ_j of class $C^{(t-k+2)}$. We can now apply mathematical induction, since if we reduce the situation to that of m functions of a number of variables not greater than $m/2$, the corresponding k will be not greater than 2, and we can then apply Lemma 3.II. Since in (3.7) we have m functions of $n-1$ variables, of class $C^{(t-k+2)}$, it follows that we can apply the inductive process provided $t-k+2\geqq t(m, n-1)$. Since $t\geqq t(m, n)$ and $k=k_m^n$, (3.3) is seen to ensure the fulfillment of this condition when $n>2m$.

Continuing in this way we finally find that Lemma 3.I is valid provided $t-k+\nu\geqq t(m, n-1)$, for a finite (possibly vacuous) set of values of $\nu>2$. As this inequality is a consequence of the one above, we conclude that Lemma 3.I is true.

Let (cf. (3.1))

$$s(m, n, r) = k_{m-r}^{n-r} + k_{m-r}^{n-2m+r-1} + k_{m-r}^{n-2m+r-2} + \cdots + k_{m-r}^{1} \quad \text{if } n - 2m + r > 1, \tag{3.8}$$

$$s(m, n, r) = k_{m-r}^{n-r} \quad \text{if } n - 2m + r \leqq 1. \tag{3.9}$$

THEOREM 3.III. *Given the functions $v_j(x_1, \cdots, x_n)$, of class $C^{(s)}$, $s \geqq 1$ $(j=1, \cdots, m)$, in a domain $\mathfrak{D}$ of real (x)-space, let J denote the matrix of first partial derivatives. Let B be a closed bounded subset of $\mathfrak{D}$, at each point of which J has rank $\leqq r < m$, with r a non-negative integer. Suppose $s \geqq s(m, n, r)$. Then under the transformation*

$$u_j = v_j(x_1, \cdots, x_n) \qquad (j = 1, \cdots, m), \tag{3.10}$$

*the image of B in (u)-space is nowhere dense.**

According to Lemma 3.I, if L is the subset of B at each point of which all the first partial derivatives of all the u's are zero, the image of L is nowhere dense provided

$$t(m, n) \leqq s. \tag{3.11}$$

For the moment we defer the verification of (3.11), and proceed with the rest of the argument.

If L is all of B, nothing remains but to verify (3.11). If L is not all of B, then r must be positive and we proceed as follows. According to Lemma 2.I we need now consider only a closed subset Λ of B consisting of points in a neighborhood of some point Q of $B-L$. Since Q is not on L, some first partial derivative of a v_j is different from zero at Q, say $\partial v_1/\partial x_1 \neq 0$. Applying the implicit function theorem, we solve the first equation (3.10) in the form

$$\begin{aligned} x_1 &= \xi_s(u_1, x_2, \cdots, x_n), \quad &&\text{if } n > 1, \\ x_1 &= \xi_s(u_1), &&\text{if } n = 1, \end{aligned} \tag{3.12}$$

with ξ_s of class $C^{(s)}$, and substituting the result in the remaining equations (3.10) obtain

$$\begin{aligned} u_j &= \phi_j(u_1, x_2, \cdots, x_n) \quad &&(j = 2, \cdots, m), \text{ if } n > 1, \\ u_j &= \phi_j(u_1) &&(j = 2, \cdots, m), \text{ if } n = 1, \end{aligned} \tag{3.13}$$

with ϕ_j of class $C^{(s)}$. (Note that $m>1$ since $r>0$ and $m>r$.) We treat the case $n>1$, as it is then obvious how to treat the case $n=1$. We take Λ so small that it is part of the locus for which (3.12) is valid. We now let λ be

* If $n \leqq m$, the only condition is that the u's be of class C', which is the result of K and S.

the projection of Λ on $(x_2, \cdots, x_n)$-space, and see, by Lemma 2.I, that it will be sufficient to show that the set in $(u_1, \cdots, u_n)$-space obtained from (3.13) by taking $(x_2, \cdots, x_n)$ anywhere in λ and u_1 anywhere in a closed neighborhood of the value determined by (3.10) at Q, is nowhere dense. It is sufficient to prove that the set in $(u_2, \cdots, u_n)$-space obtained from (3.13) for each fixed value of u_1 is nowhere dense, for if a closed set is dense at a point in (u)-space it must contain a neighborhood of the point. According to Lemma 2.III, the rank of the matrix of first partial derivatives of the ϕ's in (3.13) with respect to $x_2, \cdots, x_n$ is $\leqq r-1$. As the theorem is, by Lemma 3.I, proved for $r=0$ (except for verifying (3.11)), we can use induction with respect to the rank, and since in (3.13) with $u_1=$constant we have $m-1$ functions of $n-1$ variables, it follows that to complete the proof of Theorem 3.III we need merely show, in addition to (3.11), that

$$(3.14) \qquad s(m, n, r) \leqq s(m-1, n-1, r-1) \quad \text{if } r > 0.$$

But if m, n, r are each reduced by unity, $n-r$, $m-r$ and $n-2m+r$ are all unchanged. Hence, by (3.8) and (3.9), (3.14) holds with the equality sign when $r>0$. Therefore it remains only to verify (3.11).

First we note that if $n=2m+1$, then from (3.2) and (3.3) we have $t(m, n)=2+k_m^n-2=k_m^n=3$. Hence

$$(3.15) \qquad t(m, n) = k_m^n \quad \text{if } n \leqq 2m+1.$$

Next, with the help of (3.1), we rewrite (3.3) in the following form, with n replaced by ν:

$$(3.16) \qquad t(m, \nu) = k_m^{\nu-2m} + t(m, \nu-1), \quad \text{if } \nu > 2m.$$

Writing (3.16) for $\nu=n$, then substituting from (3.16) in the result with $\nu=n-1$, etc., we obtain the following, when $n>2m+1$:

$$(3.17) \qquad t(m, n) = k_m^{n-2m} + k_m^{n-2m-1} + \cdots + k_m^1 + t(m, 2m).$$

From (3.2), $t(m, 2m)=2$, and combining the 2 with the first term of the right hand member of (3.17) we obtain

$$(3.18) \quad t(m, n) = k_m^n + k_m^{n-2m-1} + k_m^{n-2m-2} + \cdots + k_m^1, \quad \text{if } n > 2m+1.$$

Case I. $n \geqq 2m+2$. Then $n-2m+r>1$ and (3.8) holds. Since in this case $n>m$, $k_{m-r}^{n-r} \geqq k_m^n$, a relation between the first terms of the sums in (3.8) and (3.18) respectively. The second term in (3.8) is obviously (from (3.1)) at least as great as the second term in (3.18), etc. As there are at least as many

terms in (3.8) as in (3.18), it follows that in Case I (3.11) is satisfied, since $s \geqq s(m, n, r)$.

Case II. $m < n \leqq 2m+1$. As in Case I, $k_{m-r}^{n-r} \geqq k_m^n$. It then follows from (3.15) that (3.11) is satisfied, whether (3.8) or (3.9) is used.

Case III. $n \leqq m$. According to (3.15), $t(m, n) = 1$, hence again (3.11) is satisfied. This completes the proof of Theorem 3.III.

We see from (3.8), (3.9), (3.15), and (3.18) that $s(m, n, 0) = t(m, n)$. Since, as we have observed, (3.14) is satisfied with the equality sign, and the values of $t(m, n)$ were derived from (3.2) and (3.3) which were used in the proof of Lemma 3.I, it follows that our treatment does not admit any smaller value for s than $s(m, n, r)$ as given by (3.8) and (3.9).

4. Dependence of real functions. After a preliminary theorem, which is an extension of a theorem of K and S, we obtain our theorem on dependence of real functions, following the procedure of K and S.

THEOREM 4.I. *If M is a closed set of the real number space of $w_1, \cdots, w_n$, there exists a function $F(w_1, \cdots, w_n) \equiv F(w)$ of class $C^{(\infty)}$ in all of finite (w)-space, which vanishes only at the points of M in (w)-space.*

First divide (w)-space by the hyperplanes $w_k = 0, \pm 1, \pm 2, \cdots$ $(k = 1, \cdots, n)$, and let $[q_1]$ denote the set of the resulting closed hypercubes which contain no points of M. For $s = 2, 3, \cdots$, we consider the hyperplanes $w_k = 0, \pm 1/2^s, \pm 2/2^s, \pm 3/2^s, \cdots$ $(k = 1, \cdots, n)$, and, for each s, let $[q_s]$ denote the set of those of the closed hypercubes into which these planes divide (w)-space, which contain no point of M and no inner point of any cube q_j with $j < s$. Next, for $s = 1, 2, 3, \cdots$, we define

$$(4.1) \qquad f_s(w) = s^{-s} \cdot \exp \lfloor - (\sin (2^s \pi w_1) \sin (2^s \pi w_2) \cdots \sin (2^s \pi w_n))^{-2} \rfloor$$

in each cube of $[q_s]$, and $f_s(w) = 0$ elsewhere. Then we define $f(w) = f_1(w) + f_2(w) + f_3(w) + \cdots$. It is easily verified that $f(w)$ is of class $C^{(\infty)}$. Evidently $f(w) = 0$ at each point of M, but $f(w) \not\equiv 0$ in any region.*

Now let $a_1 = 0$ and a_t denote the positive square root of the $(t-1)$st positive integer which is not a perfect square, $t = 2, 3, \cdots, n+1$. Thus $a_2^2 = 2$, $a_3^2 = 3$, $a_4^2 = 5$, etc. For $t = 1, 2, \cdots, n+1$, let M_t denote the set obtained from M by subjecting it to the transformation $w_j' = w_j + a_j$ $(j = 1, \cdots, n)$, and $\phi_t(w)$ the function defined exactly as $f(w)$ was defined, but with M replaced by M_t in determining $\phi_t(w)$. We denote $\phi_t(w + a_t) = \phi_t(w_1 + a_t, w_2 + a_t, \cdots, w_n + a_t)$. Let $F(w) = \phi_1(w + a_1) + \phi_2(w + a_2) + \cdots + \phi_{n+1}(w + a_{n+1})$. Then $F(w)$ is of class $C^{(\infty)}$ in (w)-space, since $f(w)$ is, and obviously vanishes

* This paragraph is taken from K and S, where further details are given. We may replace the function in (4.1) by a simpler one, as they did, if $f(w)$ is not required to be of class $C^{(\infty)}$.

on M. Furthermore, $F(w)$ vanishes only on M. For, since $f(w) \geqq 0$, if $F(b)=0$, then $\phi_1(b+a_1)=\phi_2(b+a_2)=\cdots=\phi_{n+1}(b+a_{n+1})=0$. From the definition of $\phi_t(w)$ it follows that for $t=1, 2, \cdots, n+1$ at least one of the sums b_1+a_t, $b_2+a_t, \cdots, b_n+a_t$ would be rational if (b) were not on M. Let $b_{m_t}+a_t$ be one such, $t=1, \cdots, n+1$. Since there are only n b's, we see that one of them must occur twice in such a sum. Thus b_m+a_p and b_m+a_q must both be rational, $p<q$, for some m, p, q. Thus a_p would equal a_q+r, r rational and $\neq 0$ since $a_p \neq a_q$; and by equating the squares of these two expressions, we would have a_q rational. But if a positive integer a_q is not a perfect square its square root is irrational, as follows from the theorem of unique factorization of positive integers. Hence we would have a contradiction, and it follows that Theorem 4.I is true.*

Next we state the definition of functional dependence, including the case that parameters are involved.

DEFINITION 1. *Functions* $v_j(x_1, \cdots, x_n, y_1, \cdots, y_p)$, $j=1, \cdots, m$, *defined on a closed bounded set B of* (x, y)-*space, are said to be dependent in* $x_1, \cdots, x_n$ *on B if there is a function* $F(u_1, \cdots, u_m, y_1, \cdots, y_p)$ *with the following properties.*

(i) $F(u, y)$ *is defined in all of real* (u, y)-*space and has continuous partial derivatives of the first order there.*

(ii) *For each* (y^0), $F(u, y^0) \not\equiv 0$ *in each region of* (u)-*space.*

(iii) $F[v_1(x, y), \cdots, v_m(x, y), y_1, \cdots, y_p] \equiv \Phi(x, y)=0$ *at each point of B.*

DEFINITION 2. *Functions* $v_j(x_1, \cdots, x_n, y_1, \cdots, y_p)$, $j=1, \cdots, m$, *defined in a domain* $\mathcal{D}$ *of* (x, y)-*space, are said to be dependent in* $x_1, \cdots, x_n$ *on* $\mathcal{D}$ *if they are dependent in* $x_1, \cdots, x_n$ *on every closed subset of* $\mathcal{D}$.

THEOREM 4.II. *Let* $v_j(x_1, \cdots, x_n, y_1, \cdots, y_p)$ *be given of class* $C^{(s)}$, $s \geqq 1$, *in a domain* $\mathcal{D}$ *of real* (x, y)-*space,* $j=1, \cdots, m$, *and where the y's may be lacking. Let K be a closed subset of* $\mathcal{D}$ *at each point of which the matrix of the first partial derivatives of the v's with respect to the x's is of rank* $\leqq r$, *where* $0 \leqq r < m$, *r a fixed integer. Suppose* $s \geqq s(m, n, r)$ [*see* (3.1), (3.8), (3.9)]. *Then the functions* $v_j(x, y)$ *are dependent in* $x_1, \cdots, x_n$ *on K.*†

Let M denote the locus of all points of (u, y)-space satisfying the condition that there be at least one set $\xi_1, \cdots, \xi_n$ such that (ξ, y) is on K and $u_j=v_j(\xi, y)$, $j=1, \cdots, m$. Since K is closed and bounded, M is closed (and bounded). Hence, by Theorem 4.I, it is seen that Theorem 4.II is true if M

* A more direct proof without use of the irrational quantities is easily given, but the exposition would be cumbersome.

† The theorem of K and S requires that $n \leqq m$. Added in proof: E. Kamke has proved this theorem for the special case $n=m+1$; see Mathematische Zeitschrift, vol. 39 (1935), pp. 672–676.

is nowhere dense. Since M is closed, it is sufficient to prove that for each fixed set $(y_1, \cdots, y_p)$, the corresponding subset of M is nowhere dense. But this follows from Theorem 3.III. Hence Theorem 4.II is true.

THEOREM 4.III. *Theorem* 4.II *remains true if* $K=\mathfrak{D}$ (*hence not closed*).

This follows from Definition 2.

THEOREM 4.IV. *Theorems* 4.II *and* 4.III *remain true if, in Definition* 1, *F is required to be of class* $C^{(\infty)}$, *and* $F(u, y)=0$ *only at points for which* $u_i=v_i(x, y)$ *for some* (x) *with* (x, y) *on* B.*

Theorem 4.IV was actually proved in establishing Theorems 4.II and 4.III.

5. **Several functions of several complex variables.** When for $m>2$ analytic functions of several complex variables the rank of the matrix of first partial derivatives is less than m, it follows from Osgood's examples that we cannot establish the existence of an analytic relation, even in the small. However, we prove a geometric result, which applies in the large.

LEMMA 5.I. *Let the functions* $f_j(x_1, \cdots, x_n, y_1, \cdots, y_p)\equiv f_j(x, y)$, $j=1, \cdots, m$, *be analytic in a domain* $\mathfrak{D}$ *of the real* $(2n+2p)$*-space of the* $n+p$ *complex variables* $x_1, \cdots, y_p$. *Let B be a closed subset of* $\mathfrak{D}$ *at each point of which* $\partial f_j/\partial x_r=0$, $j=1, \cdots, m$, $r=1, \cdots, n$. *Then the points* $(u_1, \cdots, u_m, y_1, \cdots, y_p)$ *of the set* B' *obtained by use of the equations*

$$u_j = f_j(x, y) \qquad (j = 1, \cdots, m), \tag{5.1}$$

at all points of B, *form a set nowhere dense in the* $(2m+2p)$*-dimensional* (u, y)*-space, and having the property that it cannot disconnect any region of that space.*

First we prove that B' is nowhere dense. Let B_0 be the part of B in any subspace defined by $y_k=y_k^0$, $k=1, \cdots, p$, and B_0' the corresponding part of B'. If there are no y's, we take $B_0=B$, and $B_0'=B'$. Then we have the functions $u_j=f_j(x, y^0)$ of the x's only, satisfying $\partial f_j/\partial x_r=0$. Now neighboring any point of B_0 the simultaneous solution of the latter mn equations is given, according to the Second Weierstrass Preparation Theorem,† by a finite number of configurations, each with a certain positive number of independent variables, provided not all the left hand members of the equations are identically zero. From the conditions $\partial f_j/\partial x_r=0$ holding on each of these configurations it follows that on each of them $f_j=u_j=$constant, $j=1, \cdots, m$. In the case that all the left members of the equations are identically zero,

* The last part of this theorem is not proved by K and S in their case $n\leqq m$.

† Osgood II, chapter 2, §17.

each f_i is obviously constant. Since B_0 is closed we can apply the Heine-Borel theorem to it, and it follows that

(B_0'): *B_0' contains only a finite number of points.*

Since B' is closed it must then be nowhere dense, for a closed dense set must contain a region.

To prove the second part, we must show that if $\mathfrak{R}$ is any region (connected open set) of (u, y)-space, and $\mathfrak{R}_1=\mathfrak{R}-\mathfrak{R}B'$, then $\mathfrak{R}_1$ is a region. It is obviously open. Now if there are no y's, the result follows from (B_0'). Hence we may suppose that there are some y's. Let C and D be any two points of $\mathfrak{R}_1$, and let them be joined by a path l consisting of straight line segments CS_1, $S_1S_2, \cdots, S_{t-1}S_t, S_tD$, all in $\mathfrak{R}$. We take the segments (by inserting additional points if necessary) in length $<d$, where the distance from l to any boundary point of $\mathfrak{R}$ is greater than $3d$. From property (B_0') it follows that the S's can be changed slightly, in each case keeping the y's fixed, so that none of them is on B'. Let this be done, with everything mentioned above still holding.

We now show how to replace the line segments by paths in $\mathfrak{R}_1$, if they are not already entirely in $\mathfrak{R}_1$. We take CS_1, say. Let C_y, S_{1y} and L denote the projections on (y)-space of C, S_1 and the segment CS_1, respectively. For each point of L we now determine a subsegment of L as follows. To a point A of L, not C_y or S_{1y}, we first let correspond a point $P(A)$ of $\mathfrak{R}_1$ which projects onto (y)-space in the point A. The existence of such a point $P(A)$ follows from property (B_0'). The subsegment of L determined by A is now chosen as one with mid-point at A and so short that the parallel segment through $P(A)$ in (u, y)-space, of the same length and with mid-point at $P(A)$, is within $\mathfrak{R}_1$ and also within a sphere Σ with radius $2d$ and center at the mid-point of CS_1. If A is at C_y or S_{1y}, the corresponding subsegment of L is similarly defined, but has A as one end point. We now apply the Heine-Borel theorem and choose a finite set of these subsegments of L which cover L, and shorten some of them if necessary so that they just cover L but no two of them have more than one point in common. If $C_y, E_1, E_2, \cdots, E_s, S_{1y}$ are the points A determining these subsegments, the corresponding parallel line segments through the points $P(C_y)=C$, $P(E_1), \cdots, P(E_s)$, $P(S_{1y})=S_1$ will constitute part of the path in $\mathfrak{R}_1$ joining C to S_1. The remainder of that path consists of a finite number of curves constructed as follows.

Let F_1 and F_2 denote end points of two of these line segments which project onto a single point of L (common end point of two of the subsegments of L). It is then sufficient to join F_1 and F_2 by a curve in $\mathfrak{R}_1$. But F_1 and F_2 lie in a space $\mathfrak{S}$ defined by $y_k=\text{constant}$, $k=1, \cdots, p$, which, according to

(B_0'), contains only a finite number of points of B'. Hence any curve interior to Σ and on $\mathcal{S}$, joining F_1 to F_2 and avoiding this finite set of points, will be satisfactory. It follows that Lemma 5.I is true.

THEOREM 5.II. *Let the functions* $f_j(x_1, \cdots, x_n, y_1, \cdots, y_p) \equiv f_j(x, y)$, $j=1, \cdots, m$, *be analytic in a domain* $\mathcal{D}$ *of the real* $(2n+2p)$*-space of the* $n+p$ *complex variables* $x_1, \cdots, y_p$. *Let* K *be a closed subset of* $\mathcal{D}$ *at each point of which the matrix of the first partial derivatives of the* f*'s with respect to the* x*'s is of rank* $\leqq r < n$. *Then the points* $(u_1, \cdots, u_m, y_1, \cdots, y_p)$ *of the set* K' *obtained by use of* (5.1) *at all points of* K, *form a set nowhere dense in the* $(2m+2p)$*-dimensional* (u, y)*-space, and having the property that it cannot disconnect any region of that space.*

The theorem is also true if there are no y's.

Let B denote the (closed) subset of K at each point of which every derivative $\partial f_i/\partial x_j = 0$. We now apply Lemma 2.II,as follows. The x's of Lemmas 2.I and 2.II are the real and imaginary parts of the x's and y's of Theorem 5.II. For equations (2.1) we now have the equations resulting from (5.1) involving those parts and the real and imaginary parts of the u's, together with the $2p$ equations resulting from

$$u_j = y_{j-m} \qquad (j = m+1, \cdots, p). \tag{5.2}$$

Thus the m of Lemma 2.II is $2m+2p$ of Theorem 5.II. Now according to Lemma 5.I, the image B' of B under (5.1) and (5.2) satisfies the conclusion of Theorem 5.II, hence satisfies the first hypothesis of Lemma 2.II. To complete the proof of the theorem we now see that it will be sufficient to show that the second hypothesis of Lemma 2.II is also satisfied.

If $r=0$, $K-B$ is vacuous and no further proof is necessary.

If $K-B$ is not vacuous, at any point of $K-B$ at least one first partial derivative is not zero, say $\partial f_1/\partial x_1 \neq 0$. By the implicit function theorem the first equation of (5.1) then has a solution which can be substituted in the remaining equations (5.1), yielding

$$u_j = g_j(x_2, \cdots, x_n, y_1, \cdots, y_p, u_1) \qquad (j = 2, \cdots, m), \tag{5.3}$$

g_j analytic, with u_1 now one of the parameters, say $u_1 = y_{p+1}$. According to Lemma 2.III the matrix of the first partial derivatives of the g's with respect to $x_2, \cdots, x_n$ is of rank $\leqq r-1$. Since Theorem 5.II is true when $r=0$ it is true when $m=1$. Hence if $m>1$ we can apply mathematical induction and assume that it is true for $m-1$ functions. Since $r-1$ is less than $m-1$, it then follows that Theorem 5.II is true for the $m-1$ functions of $n-1$ variables and $p+1$ parameters given by (5.3), here considered only neighboring a

point. Hence the second hypothesis of Lemma 2.II is satisfied, and Theorem 5.II is true.

6. **The case of two analytic functions.** We prove the following theorem.*

THEOREM 6.I. *Let $f(x, y) \equiv f(x_1, \cdots, x_n, y_1, \cdots, y_p)$ and $g(x, y)$ be given, analytic, with the matrix of the first partial derivatives of f and g with respect to $x_1, \cdots, x_n$ of rank <2, neighboring a point P: (x^0, y^0) of the space of the complex variables $x_1, \cdots, y_p$, and with $f(x, y^0)$ not identically constant. Then f and g satisfy an analytic relation.*

More precisely, let $u_0 = f(x^0, y^0)$ and $v_0 = g(x^0, y^0)$. Then there exists a function $G(y_1, \cdots, y_p, u, v) \equiv G(y, u, v)$, a polynomial in v with coefficients analytic in (y, u) near (y^0, u_0), such that $G[y, f(x, y), g(x, y)] \equiv 0$; and if $G(y^1, u_1, v_1) = 0$ for (y^1, u_1, v_1) in a neighborhood of (y^0, u_0, v_0), then $u_1 = f(x, y^1)$, $v_1 = g(x, y^1)$, for a $(2n-2)$-dimensional set of points (x) near (x^0).†

The theorem is also true if there are no y's.

By a change of variables if necessary we may assume that $0 = u_0 = v_0 = x_1^0 = \cdots = y_p^0$. We also make a change in the x's only so that $f(x_1, 0, \cdots, 0; 0, \cdots, 0) \not\equiv 0$. That this is possible follows from one of the hypotheses. We now establish

$$\phi(x, y, u) \equiv f(x, y) - u \equiv H(x, y, u) \cdot \Omega(x, y, u), \tag{6.1}$$

where $\Omega(x, y, u)$ is analytic and not zero near $(0, 0, 0)$, and

$$\begin{aligned} H(x, y, u) \equiv x_1^m + A_1(x_2, \cdots, x_n, y, u) \cdot x_1^{m-1} + \cdots \\ + A_m(x_2, \cdots, x_n, y, u) \end{aligned} \tag{6.2}$$

where the A's are analytic near $(0, 0, 0)$, and $A_j(0, 0, 0) = 0$, $j = 1, \cdots, m$. In (6.1) the first identity simply defines ϕ. Next we note that since $f(x_1, 0, \cdots, 0) \not\equiv 0$, $\phi(x_1, 0, \cdots, 0, 0) \not\equiv 0$, so that we can apply the Weierstress Preparation Theorem‡ to ϕ, giving us the second identity in (6.1), together with (6.2).

Since $\phi_u = -1 \neq 0$, ϕ is irreducible§ at $(0, 0, 0)$, hence we see from (6.1) that H is irreducible at $(0, 0, 0)$. Therefore the discriminant‖ $\Delta(x_2, \cdots, x_n,$

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* The theorem of Bliss requires that $n \leq 2$. Cf. Bliss or Osgood, loc. cit.

† A 0-dimensional set is non-vacuous.

‡ Osgood II, chapter 2, §2.

§ An analytic function is reducible at a point P if, neighboring P, it is expressible as the product of two analytic functions, each vanishing at P.

‖ Any definition of discriminant may be used, since if one of them vanishes identically, so do the others.

y, u) of H is not identically zero, and where $\Delta \neq 0$ the roots of $H=0$ are distinct analytic functions.* Let the roots be

$$x_1 = \xi_j(x_2, \cdots, x_n, y, u) \qquad (j = 1, \cdots, m) \tag{6.3}$$

and define

$$\begin{aligned} &F(x_2, \cdots, x_n, y, u, v) \\ &\quad \equiv \prod_{j=1}^{m} \{ v - g[\xi_j(x_2, \cdots, x_n, y, u), x_2, \cdots, x_n, y] \}. \end{aligned} \tag{6.4}$$

Then F is easily seen to be single-valued and analytic where $\Delta \neq 0$ near $(0, 0, 0, 0)$, and bounded in modulus. Hence, by a theorem of Kistler,† F is analytic in a neighborhood of $(0, 0, 0, 0)$. We shall now show that F is independent of $x_2, \cdots, x_n$.

At any point $(x_2, \cdots, x_n, y, u)$ where $\Delta \neq 0$, for $s > 1$,

$$\begin{aligned} F_{x_s} = &- \sum_{q=1}^{m} \Bigg[\left(g_{x_1} \cdot \frac{\partial \xi_q}{\partial x_s} + g_{x_s} \right) \\ &\cdot \prod_{j \neq q} \{ v - g[\xi_j(x_2, \cdots, x_n, y, u), x_2, \cdots, x_n, y] \} \Bigg], \end{aligned} \tag{6.5}$$

with ξ_q as the first argument of g_{x_1} and of g_{x_s}. Now $f_{x_1} \neq 0$ at any point $[\xi_q(x_2, \cdots, x_n, y, u), x_2, \cdots, x_n, y]$, for since (6.3), solution of $H=0$, is satisfied, we see from (6.1) that if $f_{x_1}=0$ then $H_{x_1}=0$, and as both H and H_{x_1} would thus be zero, Δ would $=0$, contrary to hypothesis. Since $f_{x_1} \neq 0$, and (6.3), solution of $H=0$, is by (6.1) also a solution of $f(x, y) - u = 0$, it follows that

$$\frac{\partial \xi_q}{\partial x_s} = - f_{x_s} / f_{x_1},$$

with $x_1 = \xi_q$ in evaluating the right hand member. Substituting this value in (6.5), $q = 1, \cdots, m$, and using the hypothesis about the rank of the matrix of partial derivatives, we see from (6.5) that $F_{x_s} = 0$ where $\Delta \neq 0$. By continuity, $F_{x_s} \equiv 0$. Hence F is independent of the x's, and we can write

$$F(x_2, \cdots, x_n, y, u, v) \equiv G(y, u, v). \tag{6.6}$$

Now if $u = f(x, y)$ and $v = g(x, y)$ for (x, y) near $(0, 0)$ so that (u, v) is near $(0, 0)$, then from (5.1) we see that $H(x, y, u) = 0$, so that x_1 must be one of the roots ξ_j of that equation given in (6.3). Hence, in one of the m factors

* Cf. Osgood II, chapter 2, §9; chapter 1, §6.

† Osgood II, chapter 3, §5, Theorem 1.

on the right hand side of (6.4), the first argument of g is x_1. Since $v=g(x, y)$, that factor is zero, so that $F=0$, and from (6.6) we then see that $G=0$. Thus $G[y, f(x, y), g(x, y)]\equiv 0$, as was to be proved.

Conversely, if $G(y, u, v)=0$ at a point near $(0, 0, 0)$, if $n>1$ and we take any $(x_2, \cdots, x_n)$ near $(0, \cdots, 0)$, then we see from (6.6) that at least one factor on the right hand side of (6.4) is zero. Hence if we let x_1 be a proper one of the roots (6.3) of $H=0$, then v will equal $g(x_1, x_2, \cdots, x_n, y)$, and, since $H(x, y, u)=0$, from (6.1) we see that $u=f(x_1, \cdots, x_n, y)$. Hence the second conclusion of the theorem is true when $n>1$. If $n=1$ there are no variables $x_2, \cdots, x_n$, and the argument simplifies, giving the stated result. Hence Theorem 6.I is true.

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NOTE ON IRREDUCIBLE QUARTIC CONGRUENCES*

BY
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Introduction. In the study of the metabelian subgroups in the holomorph of the abelian group of order p^n and type $1, 1, \cdots$ it becomes necessary to classify irreducible quartic congruences belonging to the modular field defined by the prime p under the group of linear fractional transformations with coefficients belonging to the same field.† This classification is offered here because it does not depend on the group problem from which it arose, and because it is believed that the results should have application in many other connections.

The case $p=2$ is excluded from consideration; it usually requires special treatment. The case $p=3$ must also be excluded from any argument that depends on (3) or the form of the general quartic which follows (3). The theorem at the end of §1, which is established by an argument which is principally geometric, is valid for all odd primes. It is easy to show that when $p=3$ any irreducible quartic is conjugate to x^4+x+2 or to x^4+x^2+2.

1. **Classification of quartics.** We consider the irreducible quartic

$$(1) \qquad x^4 - \gamma x^2 + \alpha x - \beta \equiv 0 \pmod{p},$$

where α, β, γ are residues of the integers, mod p, and the group of linear fractional transformations

$$(2) \qquad x = (ax' + b)/(cx' + d),$$

where a, b, c, d are in the same modular field. The congruence (1) defines a Galois field $GF(p^4)$ in which every quartic and every quadratic belonging to the $GF(p)$ is reducible. If λ is a root of (1), then its four roots are $\lambda, \lambda^p, \lambda^{p^2}, \lambda^{p^3}$. The marks of $GF(p^4)$ may be considered as points of a line, and (2) is a projective transformation of that line into itself. Any quartic into which (1) may be transformed by (2) has roots $\mu, \mu^p, \mu^{p^2}, \mu^{p^3}$ whose cross ratio is the same as that of the roots of (1). All six cross ratios σ of the roots of (1), corresponding to the ways in which $\lambda, \lambda^p, \lambda^{p^2}, \lambda^{p^3}$ may be arranged, satisfy the relation

$$(3) \qquad I^3/J^2 = 108(1 - \sigma + \sigma^2)^3/[(\sigma + 1)^2(\sigma - 2)^2(2\sigma - 1)^2],$$

* Presented to the Society, April 6, 1934; received by the editors July 7, 1934.

† The connection between the two problems is indicated in the paper *On cubic congruences* which appears in the Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 962–969.

where I and J are semi-invariants of (1).* For convenience we give the definitions of I and J for the quartic

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 \equiv 0.$$

They are

$$(4)\qquad \begin{aligned} I &= a_0a_4 - 4a_1a_3 + 3a_2^2, \\ J &= a_0a_2a_4 - a_0a_3^2 - a_1^2a_4 + 2a_1a_2a_3 - a_2^3. \end{aligned}$$

Now since I and J are polynomials in the coefficients of the quartic, the absolute invariant $i = I^3/J^2$ of (1) is in the $GF(p)$, or, if $I \neq 0$ and $J = 0$, is infinity. If both I and J are zero, (1) has equal roots and in such quartics we are not interested here. Two quartics which are conjugate under the group (2) have the same absolute invariant i, and i may conceivably take on any one of the values $0, 1, 2, \cdots, p-1, \infty$.

We shall show first that two irreducible quartics belonging to the $GF(p)$ and having the same σ are conjugate under the group (2). Let λ and μ be respective roots of the two quartics. Then

$$(5)\qquad \mu = k_0 + k_1\lambda + k_2\lambda^2 + k_3\lambda^3,$$

where the k's are integers. The condition that there exist a transformation (2) which puts λ into μ is

$$k_0 + k_1\lambda + k_2\lambda^2 + k_3\lambda^3 \equiv (a\lambda + b)/(c\lambda + d).$$

Clearing of fractions, making use of the fact that λ is a root of (1) and of no congruence of lower degree with integer coefficients, we obtain a system of linear homogeneous congruences in a, b, c, d. The condition that the system have a solution different from 0, 0, 0, 0 is

$$(6)\qquad k_1k_3 + \gamma k_3^2 - k_2^2 \equiv 0.$$

Conversely, if (6) is satisfied there exists a solution with a, b, c, d all integers. The determinant of the transformation in the general case is

$$d^2(k_2^4 - \gamma k_2^2k_3^2 + \alpha k_2k_3^3 - \beta k_3^4)/(k_2^2k_3),$$

which cannot be zero unless k_2 and k_3 are both zero, or d is zero. If both k_2 and k_3 are zero, we have the solution $a = k_1$, $b = k_0$, $c = 0$, $d = 1$. If $d = 0$, a, b, and c cannot all be zero and hence $k_2 = 0$. In this case the determinant is $-bc = -\beta k_3c^2$ which is zero only if $k_3 = 0$. Consequently, if μ is not an integer and (6) is satisfied, there exists a non-singular transformation (2) with integer coefficients which puts λ into μ.

* Cf. Clebsch, *Theorie der Binären Algebraischen Formen*, Leipzig, 1872, p. 170. The difference in coefficients comes from the definitions of I and J given below.

Now since the coefficients of (2) are integers, if (2) transforms λ into μ it transforms λ^p into μ^p, etc., and consequently transforms (1) into the second quartic under consideration. Let us define the cross ratio of the ordered set $\lambda, \lambda^p, \lambda^{p^2}, \lambda^{p^3}$ as

$$(7) \qquad \sigma(\lambda) = (\lambda - \lambda^p)(\lambda^{p^2} - \lambda^{p^3})/[(\lambda - \lambda^{p^3})(\lambda^{p^2} - \lambda^p)].$$

If we set $\sigma(\lambda) = \sigma(\mu)$, substitute for μ from (5), and make use of the fact that λ is a root of (1), we obtain (6). Consequently, if μ is such that $\sigma(\lambda) = \sigma(\mu)$, then (6) is satisfied and there exist integers a, b, c, d such that (2)transforms λ into μ.

From the definition of $\sigma(\lambda)$ in (7) it is obvious that $\sigma(\lambda^{p^2}) = \sigma(\lambda)$, and that $\sigma(\lambda^p) = \sigma(\lambda)$ if and only if $\sigma(\lambda)$ is -1. Consequently every irreducible quartic is transformed into itself by an operator of order two of the group (2), and every irreducible quartic for which $\sigma(\lambda) = -1$ is transformed into itself by an operator of order four of the group (2). No operator of (2) other than those just described and powers of them can transform an irreducible quartic into itself, for if $T(x)$ is an operator of (2) and $T(\lambda) = \mu$, then $T(\lambda^p) = \mu^p$.

The order of (2) is $p(p^2-1)$. Every irreducible quartic such that $\sigma(\lambda) \neq -1$ is one of a set of $p(p^2-1)/2$ conjugates under (2). Let the number of such sets be k. Each other irreducible quartic belongs to a set of $p(p^2-1)/4$ conjugates, and there is not more than one such set. The number of irreducible quartics is $p^2(p^2-1)/4$.* From the relation

$$kp(p^2 - 1)/2 + mp(p^2 - 1)/4 = p^2(p^2 - 1)/4$$

where $m = 0$ or 1, it follows that $m = 1$ and $k = (p-1)/2$. We state the principal result in the following theorem:

The irreducible quartics belonging to a $GF(p)$ constitute $(p+1)/2$ sets of conjugates under the linear fractional group with coefficients in the $GF(p)$.

2. **Characterization in terms of the absolute invariant.** We have characterized the $(p+1)/2$ types of irreducible quartic in terms of the cross ratio of the roots in a given cyclic order. It follows from (3) that the absolute invariant i of an irreducible quartic is restricted to a set of not more than $(p+1)/2$ of the numbers $0, 1, 2, \cdots, p-1, \infty$. We shall show that two irreducible quartics of different types have different values for i, and therefore that the $(p+1)/2$ types are characterized by exactly $(p+1)/2$ values of the absolute invariant.

Since $[\sigma(\lambda)]^{p^2} = \sigma(\lambda^{p^2}) = \sigma(\lambda)$, it follows that $\sigma(\lambda)$ is always in the $GF(p^2)$ contained in the $GF(p^4)$ defined by the irreducible quartic. Also, since

* Dickson, *Linear Groups*, Leipzig, 1901, p. 18.

$[\sigma(\lambda)]^p = \sigma(\lambda^p)$ and is equal to $\sigma(\lambda)$ only if $\sigma(\lambda) = -1$, it follows that $\sigma(\lambda)$ is an integer only if $\sigma(\lambda) = -1$. When $\sigma(\lambda) = -1$, then $i = \infty$; the other possible values of $\sigma(\lambda)$ for $i = \infty$ are 2 and 1/2, neither of which can be the $\sigma(\lambda)$ of an irreducible quartic, being integers. There is therefore just one type with $i = \infty$, and we may confine our attention to the other $(p-1)/2$ types and assume that $\sigma(\lambda)$ is not an integer.

If our conjecture that the type is determined by the value of i is correct then of the six values of σ obtained by using a suitable value of i in (3) it should be possible to isolate two, either of which could be the $\sigma(\lambda)$ of an irreducible quartic. Let us suppose (3) to be written as a sextic polynomial equal to zero. Since the irreducible quartic has no multiple root and since harmonic quartics were disposed of in the preceding paragraph, it follows that the sextic we are dealing with now is one which corresponds to the "general" quartic or else to the equianharmonic quartic. The sextic has six distinct roots or two triple roots. Since σ is not an integer and is in the $GF(p^2)$, the sextic polynomial is the product of three quadratic factors belonging to and irreducible in the $GF(p)$. Let λ_1 be a root of the quartic and denote $\sigma(\lambda_1)$ by σ_1. Then $\sigma(\lambda_1^p) = 1/\sigma_1$. Since the sum of σ_1 and $1/\sigma_1$ is equal to its pth power it is in the $GF(p)$. Hence σ_1 and $1/\sigma_1$ satisfy a quadratic relation with integer coefficients. Since the sum of $(1-\sigma_1)$ and $1/(1-\sigma_1)$ has for a pth power the sum of $\sigma_1/(\sigma_1-1)$ and $(\sigma_1-1)/\sigma_1$, it follows that neither pair satisfies a quadratic relation with integer coefficients unless we are dealing with the equianharmonic case. Hence, if the sextic is written with 1 for the coefficient of σ^6, just one of its irreducible quadratic factors has the constant term equal to 1 or else the sextic is the cube of a quadratic. In either case the zeros of the quadratic factor with the constant term 1 are the two possible values of the cross ratio of $\lambda, \lambda^p, \lambda^{p^2}, \lambda^{p^3}$, where λ is a root of an irreducible quartic. Therefore,

Two irreducible quartics with the same value of the absolute invariant are conjugate under (2).

It follows from this theorem that there are $(p+1)/2$ values of i, including ∞, such that there exist irreducible quartics having those values of i. Any quartic with integer coefficients having for i a number not among those $(p+1)/2$ values is necessarily reducible. This agrees with the statement of the conditions for irreducibility of a quartic given by Dickson.* It is of some interest to note that there exists a quartic of the form (1) having for its ab-

* *Criteria for the irreducibility of functions in a finite field*, Bulletin of the American Mathematical Society, vol. 13 (1906), p. 7.

solute invariant any given integer or infinity, and to see why for certain values of i such a quartic is reducible.

Let us suppose that I and J are any two integers, and let us write (1) in the form

$$x^4 + 6a_2x^2 + 4a_3x + a_4 \equiv 0.$$

If we use (4) to express a_3 and a_4 in terms of a_2, I, and J, we have

$$a_4 = I - 3a_2^2, \text{ and } a_3^2 = Ia_2 - 4a_2^3 - J.$$

Writing the second congruence as

$$a_2^3 - Ia_2/4 + (J + a_3^2)/4 \equiv 0,$$

we note that as a_3 is allowed to run through the numbers of the $GF(p)$ we have $(p+1)/2$ cubic congruences of which no more than $(p+1)/3$ can be irreducible.* Hence, a_3 may be selected and then a_2 and a_4 determined so that the resulting quartic has the semi-invariants I and J. If I is fixed and J runs through the values $0, 1, 2, \cdots, p-1$ then i runs through the non-zero squares if I is a square or the not-squares if I is a not-square. If I, J, is $0, 1$, or $1, 0$ then i is 0 or ∞. An easy computation shows that there are exactly $(p-1)/2$ integers such that if i takes any one of those values the sextic is the product of two irreducible cubics or the product of six linear functions with integer coefficients.

The values of i which are suitable for irreducible quartics are readily determined from the fact that the discriminant of the quartic must be a not-square,† viz., I^3-27J^2 is a not-square. Since $I^3=iJ^2$, it follows that $i-27$ must be a not-square. As i runs through the numbers of the $GF(p)$, $(p-1)/2$ such i's are obtained, and the remaining one is $i=\infty$.

3. **Determination of a quartic of a given type.** Having given an i for which there exists an irreducible quartic it does not follow that every quartic having that value of i is irreducible. I and J may be selected in many ways. J may always be selected so that $-J$ is a square in the $GF(p)$. Then consider the quartic

$$x^4 + 4(-J)^{1/2}x + I \equiv 0. \tag{8}$$

It has semi-invariants I and J. The condition that it be reducible is found from Ferrari's method of solution of the quartic. The resolvent cubic is

$$t^3 - 4It + 16J \equiv 0. \tag{9}$$

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* Cf. *On cubic congruences*, loc. cit., p. 968.

† Dickson, the second reference preceding.

This congruence is reducible and has one integral root t_1. The condition that (8) be reducible is that t_1 be a square if $i \neq \infty$, or that $-I$ be a square if $i = \infty$. In the latter case a proper choice of I makes (8) irreducible.

Suppose $i \neq \infty$ and (8) reducible. Then consider

$$x^4 + 6a_2x^2 + 4a_3x + a_4 \equiv 0, \tag{10}$$

where $a_4 = I - 3a_2^2$, and $a_3^2 = Ia_2 - 4a_2^3 - J$, the I and J being the same as in the last paragraph. The resolvent cubic of (10) may be readily shown to have the root $t_1 + 2a_2$, and (10) is reducible if $t_1 - 4a_2$ is a square. Since t_1 depends only on I and J there are $(p-1)/2$ values of a_2 such that $t_1 - 4a_2$ is not a square. Any of these values which makes the quantity $Ia_2 - 4a_2^3 - J$ a square in the $GF(p)$ gives an irreducible congruence (10) with integer coefficients. Since irreducible congruences with the given i have been shown to exist and since every quartic can be transformed into the form (10) by means of an operator of (2) it follows that a number a_2 exists satisfying the given conditions. We have thus a straightforward method of writing a member of each of the $(p+1)/2$ conjugate sets of irreducible quartics.

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CONTINUITY AND SUMMABILITY FOR DOUBLE FOURIER SERIES*

BY

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1.1. Introduction. The object in this paper is, first, to consider two extensions to double series of Riesz's theorem‡ on the equivalence of the Riesz and Cesàro methods of summation for simple series,§ and, secondly, to consider three extensions to double Fourier series of Hardy and Littlewood's theorem,|| as refined by Paley,¶ Bosanquet,** and Wiener,†† on the equivalence of continuity in the mean of a function and the summability of its Fourier series. We consider the question of summability in Part I and that of continuity and summability in Part II. The results in Part II are based on those in Part I.

Part I

2.1. Extensions of Riesz's Theorem. We consider here a double series

$$\sum_{m,n=0}^{\infty} a_{m,n}. \tag{2.11}$$

The definitions for Cesàro and Rieszian summability of this series are analogous to those for simple series.‡‡ Let $-1<\alpha$, $-1<\beta$. Let m, n be integers, positive or 0. Let

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† The first results of this paper were obtained while Dr. Gergen was Peirce instructor at Harvard University, and Dr. Littauer was National Research Fellow, also at Harvard University.

‡ For a statement of this theorem, its proof, and references, see Hobson, **8**, pp. 90–98.

Numbers in bold face type refer to the bibliography at the end of this paper.

§ One extension of Riesz's theorem has been given by Merriman, **11**, p. 526. Merriman's theorem is that, if $0\leqq\alpha$, $0\leqq\beta$, if each column, $\sum_{m=0}^{\infty}a_{m,n}$, of the series (2.11) is summable by Cesàro or by Rieszian means of order α, and if each row, $\sum_{n=0}^{\infty}a_{m,n}$, is summable by Cesàro or by Rieszian means of order β, then the series is summable $(C;\alpha,\beta)$ to sum s if, and only if, it is summable $(R;\alpha,\beta)$ to sum s. This theorem, which is plainly contained in Theorem II, is not very satisfactory in treating double Fourier series. Merriman's proof like ours is based on Hobson's proof of Riesz's theorem, but it takes a different form from ours.

|| Hardy and Littlewood, **5**, p. 70.

¶ Paley, **14**, p. 180 and p. 190.

** Bosanquet, **3**, p. 147 and p. 153.

†† Wiener, **18**, and **19**, p. 78.

‡‡ Cesàro means for double series have been considered by many authors. Among the earlier of these might be mentioned Moore, **13**, and Young, **20**. In addition to Merriman's paper, **11**, might be mentioned Mears' paper, **10**, in connection with Rieszian summability.

$$C_{\alpha,\beta}(m, n) = S_{\alpha,\beta}(m, n) \Big/ \left\{ \binom{\alpha + m}{m} \binom{\beta + n}{n} \right\},$$

where

$$S_{\alpha,\beta}(m, n) = \sum_{p=0}^{m} \binom{\alpha + m - p}{m - p} \sum_{q=0}^{n} \binom{\beta + n - q}{n - q} a_{p,q},$$

$$\binom{a}{b} = \frac{\Gamma(a + 1)}{\Gamma(b + 1)\Gamma(a - b + 1)}.$$

Then the series (2.11) is summable by Cesàro means of order (α, β), or, more shortly, is summable $(C; \alpha, \beta)$ to sum s if $C_{\alpha,\beta}(m, n) \to s$ as $(m, n) \to (\infty, \infty)$.

On the other hand, let

$$R_{\alpha,\beta}(x, y) = x^{-\alpha} y^{-\beta} \sigma_{\alpha,\beta}(x, y) = \sum_{p<x} (x - p)^{\alpha} \sum_{q<y} (y - q)^{\beta} a_{p,q}.$$

Then, the series (2.11) is summable by Rieszian means of order (α, β), or summable $(R; \alpha, \beta)$, to sum s if $R_{\alpha,\beta}(x, y) \to s$ as $(x, y) \to (\infty, \infty)$.

Now the natural extension of Riesz's theorem is that, if $0 \leqq \alpha$, $0 \leqq \beta$, then the series (2.11) is summable $(R; \alpha, \beta)$ to sum s if, and only if, it is summable $(C; \alpha, \beta)$ to sum s. This result is however in question. In our extensions we find it essential to introduce additional conditions. In the first we use the idea of ultimate boundedness, and in the second, that of ordinary boundedness. We say that the series is bounded [ultimately bounded] $(R; \alpha, \beta)$ if $R_{\alpha,\beta}(x, y)$ is bounded independently of x, y for $0 < x$, $0 < y$ [sufficiently large x, y]. Similar definitions hold for Cesàro summability, the condition $0 < x$, $0 < y$ being replaced by $0 \leqq m$, $0 \leqq n$. When first presented for publication this paper contained no reference to ultimate boundedness, and accordingly, it contained neither Theorem I nor VI. The truth of Theorem I and one of the type of Theorem VI was conjectured by the referee, Professor Szász, who kindly communicated his ideas to the authors. It was his suggestion as to the possible use of Agnew's fundamental lemma,* Lemma 4 below, that directed our efforts in the proofs of these theorems.

Our extensions of Riesz's theorem are as follows:

THEOREM I. *Let* $0 \leqq \alpha$, $0 \leqq \beta$. *Then* (a) *the series* (2.11) *is summable* $(C; \alpha, \beta)$ *to sum s if it is ultimately bounded* $(C; \alpha, \beta)$ *and if it is summable* $(R; \alpha, \beta)$ *to sum s. Moreover,* (b) *the series is summable* $(R; \alpha, \beta)$ *to sum s if it is ultimately bounded* $(R; \alpha, \beta)$ *and is summable* $(C; \alpha, \beta)$ *to sum s.*

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* For this lemma see Agnew, 1, p. 649. For theorems and references to theorems of the same general type as Theorem I, see Agnew, 1 and 2.

THEOREM II. *Let* $0 \leqq \alpha$, $0 \leqq \beta$. *Then* (a) *the series* (2.11) *is bounded* $(C; \alpha, \beta)$ *if, and only if, it is bounded* $(R; \alpha, \beta)$. *In addition,* (b) *if the series is bounded either* $(C; \alpha, \beta)$ *or* $(R; \alpha, \beta)$, *it is summable* $(C; \alpha, \beta)$ *to sum s if, and only if, it is summable* $(R; \alpha, \beta)$ *to sum s.*

The second part of Theorem II is of course a corollary of the first part and Theorem I. The proofs of Theorem I and part (a) of Theorem II are based on the lemmas of §§3.2 to 3.5. The last of these is Agnews' lemma; the other three are on simple series and are modeled, to some extent, after some given by Hobson in his proof of Riesz's theorem. Hobson's lemmas in general are not sufficiently precise for our purposes. Incidentally, we might point out two results which follow from our lemmas but do not seem to be in the literature. The first is the analogue of part (a) of Theorem II, and the second is to the effect that a series of functions $\sum_{m=0}^{\infty} a_m(x)$ is uniformly summable on the interval $a \leqq x \leqq b$ by Cesàro means of order α, $0 \leqq \alpha$, if, and only if, it is uniformly summable there by Rieszian means of order α.

3.1. **Lemmas for Theorems I and II.** In these lemmas and throughout the rest of the paper we suppose that x, y are positive numbers, that m, n, p, q are integers, positive or 0, and that M denotes a number independent of those of the variables x, y, m, n, p, q with which we are concerned at the moment. The range for these variables is understood to be

$$0 < x, \qquad 0 < y, \qquad 0 \leqq m, \qquad 0 \leqq n, \qquad 0 \leqq p, \qquad 0 \leqq q,$$

or that part of this range indicated.

In Lemmas 1 to 3 we consider a series

$$\sum_{m=0}^{\infty} A_m. \tag{3.11}$$

Here M is understood to be independent of the values of the A's. We denote by k a fixed positive number, by K, the largest integer less than k, and by μ, the largest integer less than x. We write

$$S_a(m) = \sum_{p=0}^{m} \binom{a+m-p}{m-p} A_p, \quad \sigma_a(x) = \sum_{p<x} (x-p)^a A_p.$$

We define $\binom{m}{p}$ as 0 for $p = m+1, m+2, \cdots$, and set

$$e(x) = \sum_{p<x} (-1)^p \binom{K+1}{p} (x-p)^k, \qquad \lambda = k - K,$$

$$E(x) = e(x) - \Gamma(k+1) \binom{\mu + \lambda - 1}{\mu}, \qquad T(x) = \sigma_k(x) - \Gamma(k+1) S_k(\mu).$$

3.2. We consider first $E(x)$. We have

LEMMA 1. *If k is an integer then $E(x)$ is bounded for all x and vanishes for $k<x$. If k is not an integer then*

$$|E(x)| \leqq M(x+1)^{\lambda-2}.$$

For $K+1<x$ we have*

$$\begin{aligned} &|e(x) - k(k-1)\cdots(k-K)x^{\lambda-1}| \\ &\quad = M\int_{x-1}^{x} du_1 \int_{u_1-1}^{u_1} du_2 \cdots \int_{u_K-1}^{u_K} (u_{K+1}^{\lambda-1} - x^{\lambda-1})du_{K+1} \\ &\quad \leqq M|(x-K-1)^{\lambda-1} - x^{\lambda-1}|. \end{aligned}$$

If k is an integer this is 0, and the lemma follows in this case. If k is not an integer it is $O(x^{\lambda-2})$ as $x\to\infty$. Accordingly, since†

$$k(k-1)\cdots(k-K)\mu^{\lambda-1} - \Gamma(k+1)\binom{\mu+\lambda-1}{\mu} = O(x^{\lambda-2}),$$

$$x^{\lambda-1} - \mu^{\lambda-1} = O(x^{\lambda-2})$$

as $x\to\infty$, the lemma follows in this case also.

3.3. We turn now to

LEMMA 2. *We have*

$$|\sigma_k(x)| \leqq M \max_{m<x} |S_k(m)|.$$

In addition, corresponding to each positive integer m_0, we can write

$$\sigma_k(x) = \sum_{p<m_0} B_p(x)A_p + H(x)$$

for $m_0<x$, where the B's are independent of the A's,

$$(3.31) \qquad |B_p(x)| \leqq M \quad \textit{for} \quad p < m_0 < x,$$

and

$$|H(x)| \leqq M_0 \max_{m_0\leqq m<x} |S_k(m)|$$

for $m_0<x$, M_0 being independent of m_0 as well as x and the A's.

We have‡

* Compare Hobson, 8, p. 90.
† For the former see Hobson, 8, p. 91.
‡ See Hobson, 8, (4), p. 71.

$$A_p = \sum_{q=0}^{p} (-1)^{p+q} \binom{K+1}{p-q} S_K(q).$$

Hence

$$\sigma_k(x) = \sum_{p<x} (x-p)^k \sum_{q=0}^{p} (-1)^{p+q} \binom{K+1}{p-q} S_K(q) = \sum_{q<x} e(x-q) S_K(q).$$

On the other hand,*

$$S_k(\mu) = \sum_{q<x} \binom{\mu - q + \lambda - 1}{\mu - q} S_K(q).$$

Thus,

$$(3.32) \qquad T(x) = \sum_{q<x} E(x-q) S_K(q).$$

Now,†

$$S_K(q) = \sum_{m=0}^{q} (-1)^{q+m} \binom{\lambda}{q-m} S_k(m).$$

Hence,

$$(3.33) \qquad \begin{aligned} T(x) &= \sum_{q<x} E(x-q) \sum_{m=0}^{q} (-1)^{q+m} \binom{\lambda}{q-m} S_k(m) \\ &= \sum_{m<x} D(x-m) S_k(m), \end{aligned}$$

where

$$D(x) = \sum_{n<x} (-1)^n E(x-n) \binom{\lambda}{n}.$$

Consider $D(x)$. If k is not an integer, then, since‡

$$\left| \binom{\lambda}{n} \right| \sim \frac{1}{|\Gamma(-\lambda)|} \frac{1}{n^{\lambda+1}}$$

as $n \to \infty$, we have, by Lemma 1,

$$\begin{aligned} |D(x)| &\leqq M \left\{ \sum_{n<x/2} + \sum_{x/2 \leqq n < x} \right\} (x+1-n)^{\lambda-2} (n+1)^{-\lambda-1} \\ &\leqq M \{ (x+1)^{\lambda-2} + (x+1)^{-\lambda-1} \}. \end{aligned}$$

Hence,

$$(3.34) \qquad \begin{aligned} \sum_{m<x} |D(x-m)| &\leqq M \sum_{m<x} \{ (x-m+1)^{\lambda-2} + (x-m+1)^{-\lambda-1} \} \\ &\leqq M. \end{aligned}$$

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* See Hobson, 8, (5), p. 72.
† See Hobson, 8, (6), p. 72.
‡ See Hobson, 8, pp. 71–72.

On the other hand, if k is an integer then D is bounded for all x and vanishes for $k+1<x$. It follows that (3.34) is valid in this case also. We choose M_0 so that, for all x,

$$\Gamma(k+1)+\sum_{m<x}|D(x-m)| \leqq M_0.$$

Allowing for the moment m_0 to have one of the values $0, 1, \cdots$, we set, for $m_0<x$,

$$H(x) = \Gamma(k+1)S_k(\mu) + \sum_{m=m_0}^{\mu} D(x-m)S_k(m).$$

Then, for $0 \leqq m_0 < x$,

$$|H(x)| \leqq M_0 \max_{m_0\leqq m<x} |S_k(m)|.$$

Taking $m_0=0$, H reduces to σ_k, and the first part of the lemma follows. In addition, for $0<m_0<x$, we have

$$\sigma_k(x) - H(x) = \sum_{m<m_0} D(x-m)\sum_{p=0}^{m}\binom{k+m-p}{m-p}A_p = \sum_{p<m_0} B_p(x)A_p,$$

where

$$B_p(x) = \sum_{m=p}^{m_0-1} D(x-m)\binom{k+m-p}{m-p}.$$

Since these B's are independent of the A's and satisfy (3.31) the second part likewise follows.

3.4. We proceed to the proof of

LEMMA 3. *We have*

$$|S_k(m)| \leqq M \max_{x\leqq m+1} |\sigma_k(x)|.$$

In addition, corresponding to each positive integer m_0, we can write

$$S_k(m) = \sum_{p<m_0} C_p(m)A_p + I(m)$$

for $m_0 \leqq m$, where the C's are independent of the A's,

$$(3.41)\qquad |C_p(m)| \leqq M \qquad \textit{for } p < m_0 \leqq m,$$

$$(3.42)\qquad |I(m)| \leqq M_0 \max_{m_0\leqq x\leqq m+1} |\sigma_k(x)| \qquad \textit{for } m_0 \leqq m,$$

M_0 being independent of m_0 as well as m and the A's.

In this lemma we set $\sigma_0(x)=\sum_{p\leq x}a_p$. We note then that*

$$S_K(q) = \sum_{r=0}^{K} M\sigma_r(q).$$

Hence, by (3.32),

$$\begin{aligned} S_k(m) &= M\sigma_k(m+1) + \sum_{q=0}^{m} E(m+1-q)\sum_{r=0}^{K} M\sigma_r(q) \\ &= M\sigma_k(m+1) + \sum_{r=0}^{K} M\sum_{q=0}^{m} E(m+1-q)\sigma_r(q). \end{aligned}$$

Accordingly, it is enough to prove the lemma with $S_k(m)$ replaced by

$$T_r(m) = \sum_{q=0}^{m} E(m+1-q)\sigma_r(q),$$

where $r=0, 1, \cdots$ or K.

We set

$$W(x) = \int_{\mu}^{x} (x-t)^{\lambda-1}\sigma_K(t)dt,$$

integrate by parts K times, and take $x=q+1/(K+1)$, $q+2/(K+1)$, $\cdots$, $q+1$, successively. We get, for $p=1, 2, \cdots, K+1$,

$$\{(K+1)/p\}^{\lambda}W\{q+p/(K+1)\} = \sum_{n=0}^{K} \alpha_n p^n \sigma_{K-n}(q),$$

$$\alpha_n(K+1)^n\lambda \cdots (\lambda+n)\cdot(K-n)! = K!.$$

Noting that for $1\leq K$ the determinant of the σ's in these equations is a nonzero multiple of the Vandermonde formed with the numbers $1, 2, \cdots, K+1$, we see that $\sigma_r(q)$ can be expressed in the form

$$\sigma_r(q) = \sum_{i=1}^{K+1} MW\{q+i/(K+1)\}.$$

We thus have

$$T_r(m) = \sum_{i=1}^{K+1} M\sum_{n=0}^{m} E(m+1-q)W\{q+i/(K+1)\}.$$

Hence, it is enough to prove the lemma with $S_k(m)$ replaced by

$$U(m) = \sum_{n=0}^{m} E(m+1-q)W(q+\xi),$$

* Compare Hobson, 8, p. 91.

where ξ is a fixed number satisfying $0<\xi\leqq 1$.

We have*

$$W(x) = M\sigma_k(x) + M\int_0^{\mu} \sigma_k'(v)\psi(v, x),$$

where ψ remains positive and never increases as v increases from 0 to μ, and

$$\psi(0, x) \leqq M.$$

Allowing m_0 to have one of the values $0, 1, \cdots$, and denoting by M_0 a number independent of m, m_0 and the A's, we can then write

$$\begin{aligned} U_1 &= \sum_{q=m_0}^{m} E(m+1-q)W(q+\xi) = M_0\sum_{q=m_0}^{m} E(m+1-q)\int_0^{m_0}\sigma_k'(v)\psi(v, q+\xi)dv \\ &\quad + \sum_{q=m_0}^{m} E(m+1-q)\left\{M_0\sigma_k(q+\xi) + M_0\int_m^{q}\sigma_k'(v)\psi(v, q+\xi)dv\right\} \\ &= U_2 + U_3, \end{aligned}$$

say. Now, applying the second mean-value theorem, we have

$$\begin{aligned} |U_3| &\leqq M_0 \max_{m_0+\xi\leqq x\leqq m+1} |\sigma_k(x)|\cdot\sum_{q=m_0}^{m}|E(m+1-q)| \\ &\leqq M_0 \max_{m_0+\xi\leqq x\leqq m+1} |\sigma_k(x)|. \end{aligned}$$

From this inequality the first part of the lemma follows on taking $m_0=0$. In addition we can obtain the second part.

We have, for $0<m_0\leqq m$,

$$U - U_3 = \sum_{q<m_0} E(m+1-q)W(q+\xi) + U_2 = U_4 + U_2,$$

say. Now

$$\begin{aligned} U_4 &= \sum_{q<m_0} E(m+1-q)\int_q^{q+\xi}(q+\xi-t)^{\lambda-1}\sigma_K(t)dt \\ &= \sum_{p<m_0} A_p \sum_{p\leqq q<m_0} E(m+1-q)\int_q^{q+\xi}(q+\xi-t)^{\lambda-1}(t-p)^{K}dt, \end{aligned}$$

and it is plain that the coefficient of A_p here satisfies (3.41). On the other hand,

$$U_2 = M\sum_{q=m_0}^{m} E(m+1-q)\int_0^{m_0}\psi(u, q+\xi)\sum_{p<u}(u-p)^{k-1}A_p du$$

* See Hobson, 8, pp. 94–95.

$$= M \sum_{p<m_0} A_p \sum_{q=m_0}^{m} E(m+1-q) \int_p^{m_0} (u-p)^{k-1}\psi(u, q+\xi)du,$$

and we have, for $p=0, 1, \cdots, m_0-1$,

$$\left| \sum_{q=m_0}^{m} E(m+1-q) \int_p^{m_0} (u-p)^{k-1}\psi(u, q+\xi)du \right|$$

$$\leqq M \sum_{q=m_0}^{m} \left| E(m+1-q)\psi(m_0, q+\xi) \right| \leqq M.$$

The lemma follows.

3.5. We consider finally

Lemma 4. *Let m_0 be a positive integer. Let $g_p(x)$, $G_p(y)$, $p=0, 1, \cdots, m_0-1$, be defined for sufficiently large values of their arguments. For each p let*

$$g_p(x) = o(1)$$

as $x \to \infty$; and for sufficiently large x, y let

$$\left| \sum_{p<m_0} g_p(x)G_p(y) \right| \leqq F(x),$$

where F is independent of y. Then

$$\sum_{p<m_0} g_p(x)G_p(y) = o(1) \quad as \quad (x, y) \to (\infty, \infty). \tag{3.51}$$

This lemma with x, y replaced by integral variables is the lemma of Agnew previously cited. It is plain that the lemma is likewise valid when x or y is replaced by an integral variable. We shall have occasion to use it in various forms.

The lemma in the general case is an immediate corollary of Agnew's result. For suppose that (3.51) is false. Then there exist a positive ϵ and two sequences of numbers $\{x_m\}$, $\{y_n\}$, $m, n=1, 2, \cdots$, such that $x_m \to \infty$ as $m \to \infty$, $y_n \to \infty$ as $n \to \infty$, and

$$\left| \sum_{p<m_0} g_p(x_m)G_p(y_m) \right| \geqq \epsilon.$$

But, by Agnew's result,

$$\sum_{p<m_0} g_p(x_m)G_p(y_n) = o(1)$$

as $(m, n) \to (\infty, \infty)$. This gives us the contradiction.

4.1. **Proof of Theorem I.** As the proofs of (a) and (b) are similar in character we shall confine ourselves to the proof of (a). In addition, we shall as-

sume that $0<\alpha$, $0<\beta$. When $\alpha=\beta=0$ there is nothing to prove, and when one and not the other is 0 the proof requires similar steps to the following, but fewer of them.

To begin with we note that we can suppose that $s=0$. In fact, for either type of summability, the series (2.11) is summable to sum s [ultimately bounded] if, and only if, the series

$$\sum_{m,n=0}^{\infty} b_{m,n}, \qquad b_{0,0} = a_{0,0} - s, \quad b_{m,n} = a_{m,n} \quad \text{for} \quad 0 < m^2 + n^2,$$

is summable to sum 0 [ultimately bounded] by means of the same order. The theorem for $s\neq 0$ is then a consequence of the theorem for $s=0$.

We now set

$$P(m, y) = \sum_{p=0}^{m} \binom{\alpha + m - p}{m - p} \sum_{n<y} (y-n)^{\beta} a_{p,n},$$

and proceed to show that

$$(4.11) \qquad P(m, y) = o(m^{\alpha} y^{\beta}) \quad \text{as} \quad (m, y) \to (\infty, \infty).$$

Let $0<\epsilon$ be arbitrarily small. Then, denoting by M_0 the constant of (3.42), we select an M and a positive integer m_0 so that

$$|S_{\alpha,\beta}(m, n)| \leqq M m^{\alpha} n^{\beta}, \qquad M_0 |\sigma_{\alpha,\beta}(x, y)| \leqq \epsilon x^{\alpha} y^{\beta},$$

the former for $m_0 \leqq m$, $m_0 \leqq n$, and the latter for $m_0 \leqq x$, $m_0 \leqq y$.

Next we apply Lemma 3. We can write, for $m_0 \leqq m$,

$$\begin{aligned} P(m, y) &= \sum_{p<m_0} C_p(m) \sum_{n<y} (y-n)^{\beta} a_{p,n} + I(m, y) \\ &= I_1(m, y) + I, \end{aligned}$$

say, where $C_p(m)$ is bounded for $p<m_0\leqq m$, and

$$(4.12) \qquad |I| \leqq M_0 \max_{m_0 \leqq x \leqq m+1} |\sigma_{\alpha,\beta}(x, y)| \leqq \epsilon (m+1)^{\alpha} y^{\beta}$$

for $m_0 \leqq m$, $m_0 \leqq y$.

Consider I_1. By Lemma 2, we have, for $m_0 < y$,

$$P(m, y) = \sum_{q<m_0} B_q(y) \sum_{p=0}^{m} \binom{\alpha + m - p}{m - p} a_{p,q} + H(m, y),$$

where $B_q(y)$ is bounded for $q<m_0<y$, and

$$|H| \leqq M \max_{m_0 \leqq n < y} |S_{\alpha,\beta}(m, n)| \leqq M m^{\alpha} y^{\beta}$$

for $m_0 \leqq m$, $m_0 < y$. We conclude that, for $m_0 \leqq m$, $m_0 < y$,

$$|I_1| \leqq |P| + |I| \leqq |H| + M \sum_{p=0}^{m} \left| \binom{\alpha + m - p}{m - p} \right| \sum_{q<m_0} |a_{p,q}|$$

$$\leqq M(m+1)^\alpha y^\beta + M \sum_{p=0}^{m} \left| \binom{\alpha + m - p}{m - p} \right| \sum_{q<m_0} |a_{p,q}| \leqq F(m) y^\beta,$$

say, where F depends upon m but not upon y. On the other hand,

$$m^{-\alpha} C_p(m) = o(1)$$

as $m \to \infty$, for $p = 0, 1, \cdots, m_0 - 1$. Accordingly, Lemma 4 is applicable, and it follows that

$$(4.13) \qquad m^{-\alpha} y^{-\beta} I_1(m, y) = o(1) \quad \text{as} \quad (m, y) \to (\infty, \infty).$$

From (4.13), (4.11) now follows. By (4.12) and (4.13), we have

$$\limsup_{(m, y) \to (\infty, \infty)} |m^{-\alpha} y^{-\beta} P(m, y)| \leqq \epsilon,$$

and since ϵ was arbitrary, this implies (4.11).

It remains to show that (4.11) with our hypotheses implies

$$(4.14) \qquad S_{\alpha,\beta}(m, n) = o(m^\alpha n^\beta) \quad \text{as} \quad (m, n) \to (\infty, \infty).$$

Letting ϵ and M_0 have the same significance as above, we select an M and a positive integer m_0 so that

$$|S_{\alpha,\beta}(m, n)| \leqq M m^\alpha n^\beta, \qquad M_0 |P(m, y)| \leqq \epsilon m^\alpha y^\beta,$$

the former for $m_0 \leqq m$, $m_0 \leqq n$ and the latter for $m_0 \leqq m$, $m_0 \leqq y$. Then we apply Lemma 3 again. We have, for $m_0 \leqq n$,

$$S_{\alpha,\beta}(m, n) = \sum_{q<m_0} C_q'(n) \sum_{p=0}^{m} \binom{\alpha + m - p}{m - p} a_{p,q} + I'(m, n)$$
$$= I_1'(m, n) + I',$$

say, where $C_q'(n)$ is bounded for $q < m_0 \leqq n$, and

$$|I'| \leqq M_0 \max_{m_0 \leqq y \leqq n+1} |P(m, y)| \leqq \epsilon m^\alpha (n+1)^\beta$$

for $m_0 \leqq m$, $m_0 \leqq n$. Now,

$$|I_1'| \leqq |S_{\alpha,\beta}| + |I'| \leqq M m^\alpha (n+1)^\beta$$

for $m_0 \leqq m$, $m_0 \leqq n$. Hence, by Lemma 4, since

$$n^{-\beta} C_q'(n) = o(1)$$

as $n\to\infty$, for $q=0, 1, \cdots, m_0-1$, we have

$$n^{-\alpha}n^{-\beta}I_1'(m, n) = o(1)$$

as $(m, n)\to(\infty, \infty)$. We conclude that

$$\limsup_{(m,n)\to(\infty,\infty)} \left| m^{-\alpha}n^{-\beta}S_{\alpha,\beta}(m, n)\right| \leqq \epsilon,$$

and accordingly, that (4.14) holds.

4.2. **Proof of Theorem II.** We have only to prove (a). We first observe that

$$(4.21) \qquad \left| S_{\alpha,\beta}(m, n)\right| \leqq M \max_{x\leqq m+1}\left| \sum_{p<x}(x-p)^{\alpha}\sum_{q=0}^{n}\binom{\beta+n-q}{n-q}a_{p,q}\right|.$$

This is a consequence of Lemma 2 if $0<\alpha$ and is trivial otherwise. We next observe that, similarly,

$$\left|\sum_{p<x}(x-p)^{\alpha}\sum_{q=0}^{n}\binom{\beta+n-q}{n-q}a_{p,q}\right| \leqq M\max_{y\leqq n+1}\left|\sigma_{\alpha,\beta}(x, y)\right|.$$

Hence, by (4.21), if the series (2.11) is bounded $(R; \alpha, \beta)$, we have

$$\left| S_{\alpha,\beta}(m, n)\right| \leqq M(m+1)^{\alpha}(n+1)^{\beta}$$

from which it follows that it is bounded $(C; \alpha, \beta)$.

In the same way, if the series is bounded $(C; \alpha, \beta)$, we have

$$\left|\sigma_{\alpha,\beta}(x, y)\right| \leqq M(x+1)^{\alpha}(y+1)^{\beta}.$$

It follows that $R_{\alpha,\beta}(x, y)$ is bounded for $1\leqq x$, $1\leqq y$. But we have, for $x<1$,

$$R_{\alpha,\beta}(x, y) = R_{\alpha,\beta}(1, y).$$

Hence, as a similar identity holds for $y<1$, we conclude the truth of (a).

Part II

5.1. **Extension of Hardy and Littlewood's Theorem.** We consider here a function $f(u, v)$ which is integrable* over the square $(0, 0; \pi, \pi)$ and is even and periodic with period 2π in each variable. We restrict our attention to the behavior of f and the Fourier series of f,

$$f(u, v) \sim \sum_{m,n=0}^{\infty} a_{m,n}\cos mu \cos nv,$$

at the origin. This restriction and that as to f being even-even do not of course limit the generality of our results. The series with whose summability we are concerned is then the series (2.11) where now

* All our integrals are understood to be taken in the sense of Lebesgue.

$$a_{m,n} = \lambda_m\lambda_n \int_0^\pi \cos mu\,du \int_0^\pi \cos nv\, f(u, v)dv$$

$$[\lambda_0 = 1/\pi,\ \lambda_1 = \lambda_2 = \cdots = 2/\pi].$$

To extend the Hardy and Littlewood theorem we need a definition of fractional integration and continuity in the mean for functions of two variables. These are direct generalizations of those for functions of a single variable. Let $\phi(u, v)$ be defined for almost all (u, v) in the quarter-plane $T: (0 < u, 0 < v)$. Let $0 < a$, $0 < b$. Then we define $\phi_{a,b}(x, y)$, $\phi_{a,0}$, $\phi_{0,b}$ as

$$\Gamma(a)\Gamma(b)\phi_{a,b}(x, y) = \int_{(0,0)}^{(x,y)} (x - u)^{a-1}(y - v)^{b-1}\phi(u, v)d(u, v),$$

$$\Gamma(a)\phi_{a,0}(x, y) = \int_0^x (x - u)^{a-1}\phi(u, y)du,$$

$$\Gamma(b)\phi_{0,b}(x, y) = \int_0^y (y - v)^{b-1}\phi(x, v)dv$$

provided the corresponding integral exists, and as ∞ otherwise. We define $\phi_{0,0}(x, y)$ as

$$\phi_{0,0}(x, y) = \phi(x, y)$$

where ϕ is defined, and as ∞ otherwise. We call $\phi_{a,b}(x, y)$, where $0 \leqq a$, $0 \leqq b$, the fractional integral of order (a, b) of ϕ at (x, y). We say that ϕ is continuous $(C; a, b)$ at the origin, or, more briefly, continuous $(C; a, b)$, with limit s, if $\Gamma(a+1)\Gamma(b+1)x^{-a}y^{-b}\phi_{a,b}(x, y) \to s$ as $(x, y) \to (+0, +0)$. In addition we say that ϕ is almost continuous $(C; a, b)$ with limit s if $\Gamma(a+1)\Gamma(b+1)x^{-a}y^{-b} \cdot \phi_{a,b}(x, y)$ coincides, except possibly on a set of measure 0, with a function $\Phi(x, y)$ which tends to s when $(x, y) \to (+0, +0)$.

In regard to these integrals we prove in §6.1 the following theorem:*

THEOREM III. *Suppose that* $0 \leqq a$, $0 \leqq b$ *and that* $\phi(u, v)$ *is integrable over every rectangle* $(0, 0; x, y)$. *Then* (a) $\phi_{a,b}(u, v)$ *is integrable over every such rectangle. In addition,* (b) *if* $a \leqq \alpha$, $b \leqq \beta$ *and if* $\phi_{\alpha,\beta}(x, y)$ *is finite, then* $\phi_{\alpha,\beta}(x, y)$ *is equal to the fractional integral of order* $(\alpha - a, \beta - b)$ *of* $\phi_{a,b}$ *at* (x, y).

Our principal extension of the Hardy and Littlewood theorem involves the idea of boundedness in much the same way as Theorem II. We say that f is bounded [almost bounded] $(C; a, b)$ in a domain D if $x^{-a}y^{-b}f_{a,b}$ is bounded [almost bounded] in D. The extension is then as follows:

* For a theorem of this type on fractional integrals of functions of a single variable, see Tonelli, **17**, p. 185.

THEOREM IV. (a) *Let*

$$(5.11) \qquad 0 \leqq a < \alpha, \qquad 0 \leqq b < \beta.$$

Then, if for some positive δ, f *is almost bounded* $(C;\ a,\ b)$ *in the infinite rectangles* $(0, 0;\ \infty,\ \delta)$, $(0, 0;\ \delta,\ \infty)$, *it follows that the series* (2.11) *is bounded* $(C;\ \alpha,\ \beta)$. *If, in addition,* f *is almost continuous* $(C;\ a,\ b)$ *with limit* s, *then the series is summable* $(C;\ \alpha,\ \beta)$ *to sum* s. (b) *On the other hand, suppose that*

$$(5.12) \qquad 0 \leqq \alpha < a - 1, \qquad 0 \leqq \beta < b - 1.$$

Then, if the series (2.11) *is bounded* $(C;\alpha,\beta)$, *it follows that* f *is bounded* $(C;\ a,\ b)$ *in the quarter-plane* T. *If, in addition, the series is summable* $(C;\ \alpha,\ \beta)$ *to sum* s, *then* f *is continuous* $(C;\ a,\ b)$ *with limit* s.*

The proof of this theorem is given in §§7.1 to 10.1. We were much influenced in our procedure by the work of Bosanquet and Paley previously cited.† We do not however follow one or the other of these authors completely. The proof of part (b) especially seems to involve new difficulties. In using this method of proof it is natural that we obtain relations between the order of summability and continuity analogous to those of Bosanquet and Paley.

Part (a) of the theorem is not entirely satisfactory. Conditions depending explicitly on f in the fundamental square $(0, 0;\ \pi,\ \pi)$ would be more desirable. Conditions of this type are given in the following theorem. It might be noted in particular that, when $a \leqq 1$, $b \leqq 1$, the boundedness condition reduces simply to f being almost bounded $(C;\ a,\ b)$ on $(0, 0;\ \pi,\ \delta)$ and $(0, 0;\ \delta,\ \pi)$. Part (b) we add for the sake of completeness.

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* This theorem contains as a particular case a result given independently by Moore, **13**, p. 96, and Young, **20**, p. 181, namely, that the series (2.11) is summable $(C;\ 1,1)$ to sum s if the conditions in (a) hold for $a=b=0$.

Another summability criterion which might be mentioned is one due to Tonelli, **16**, p. 490. Tonelli shows that the series (2.11) is summable $(C;\ 1,\ 1)$ if

$$\int_0^x du \int_0^y \left| f(u, v) - s \right| dv = o(x, y) \qquad \text{as } (x, y) \to (+0, +0)$$

and if the integrals $\int_0^\pi \left| f(x, v) \right| dv$, $\left| \int_0^\pi f(u, y) \right| du$ are almost bounded on $(0, \pi)$. This result does not seem to be contained in Theorem III. It seems likely that an extension of Tonelli's theorem, similar to the extension of Lebesgue's theorem by Hardy for simple series, can be obtained by means of the formula in Lemma 9. For references to Hardy's theorem and to Lebesgue's theorem and to similar theorems, see Kogbetliantz, **9**, p. 64.

The problem of extending Hardy and Littlewood's theorem to double series was first considered by Merriman, **12**.

† While this paper was being prepared for publication a second paper on summability was published by Bosanquet. This paper, **4**, contains a proof of the essential Lemma 9 below. In his first treatment of the problem Bosanquet used a somewhat different method.

THEOREM V. (a) *Part* (a) *of Theorem* IV *holds if we replace the rectangles* $(0, 0; \infty, \delta)$, $(0, 0; \delta, \infty)$ *by* $(0, 0; \pi, \delta)$, $(0, 0; \delta, \pi)$ *and assume in addition that* $y^{-b}f_{m,b}(\pi, y)$ $[x^{-a}f_{a,m}(x, \pi)]$ *is almost bounded on* $(0, \delta)$ *for each positive odd integer m less than a* $[b]$. (b) *On the other hand, if* (5.12) *holds and if the series* (2.11) *is bounded* $(C; \alpha, \beta)$ *it follows that* $y^{-b}f_{m,b}(\pi, y)$ $[x^{-a}f_{a,m}(x, \pi)]$ *is bounded for all y* $[x]$ *for each positive odd integer m less than a* $[b]$.

This proof is found in §§11.1 to 12.2. In §13.1 we obtain with the help of the previous Lemma 4 and the lemma of §9.5 a third extension of the Hardy and Littlewood theorem. This result is of the same type as that of Theorem I. Whether a corresponding result holds when the roles of summability and continuity are interchanged we are unable to say.

THEOREM VI. *Suppose that*

$$0 \leqq \alpha < a - 2, \qquad 0 \leqq \beta < b - 2,$$

and that the series (2.11) *is summable either* $(C; \alpha, \beta)$ *or* $(R; \alpha, \beta)$ *to sum s. Suppose also that, for a positive* δ, *f is bounded* $(C; a, b)$ *in the square* $(+0, +0; \delta, \delta)$. *Then f is continuous* $(C; a, b)$ *with limit s.*

6.1. **Proof of Theorem III.** Part (a) is trivial for $a = b = 0$, part (b), for $a = \alpha$, $b = \beta$. We shall assume in (a) that $0 < a$, $0 < b$, and in (b), that $0 < a < \alpha$, $0 < b < \beta$. The other cases can be handled by similar arguments.

Let (x, y) be fixed. Then the function $(x-\xi)^a(y-\eta)^b\phi(\xi, \eta)$ is integrable over $0 \leqq \xi \leqq x$, $0 \leqq \eta \leqq y$. Moreover,

$$(x - \xi)^a = a\int_\xi^x (u - \xi)^{a-1}du, \qquad (y - \eta)^b = b\int_\xi^y (v - \eta)^{b-1}dv.$$

Hence, by Fubini's theorem,* it is plain that the integral

$$\int_0^x d\xi \int_0^y d\eta \int_\xi^x du \int_\eta^y (u - \xi)^{a-1}(v - \eta)^{b-1} \left| \phi(\xi, \eta) \right| dv$$

exists. Thus, since the integrand here is measurable over the domain $V: (0 \leqq \xi \leqq u \leqq x, 0 \leqq \eta \leqq v \leqq y)$, the integral

$$\int_V (u - \xi)^{a-1}(v - \eta)^{b-1}\phi(\xi, \eta)d(\xi, \eta, u, v)$$

exists by a theorem due to Hobson.† Accordingly, the integral

$$\int_{(0,0)}^{(x,y)} d(u, v) \int_{(0,0)}^{(u,v)} (u - \xi)^{a-1}(v - \eta)^{b-1}\phi(\xi, \eta)d(\xi, \eta)$$

* See, for example, Hobson, 7, p. 630.

† See Hobson, 7, p. 631.

exists. We conclude that (a) holds.

The proof of (b) is similar. We have, assuming $\phi_{\alpha,\beta}(x, y)$ finite,

$$E\phi_{\alpha,\beta}(x, y)$$
$$= \int_0^x d\xi \int_0^y d\eta \int_\xi^x du \int_\eta^y (x-u)^{\alpha-a-1}(y-v)^{\beta-b-1}(u-\xi)^{a-1}(v-\eta)^{b-1}\phi(\xi,\eta)dv,$$

where $E = \Gamma(\alpha - a)\Gamma(\beta - b)\Gamma(a)\Gamma(b)$. Now, noting that $\psi_{\alpha,\beta}(x, y)$, where $\psi = |\phi|$, is likewise finite, we see that we can integrate first with respect to (ξ, η). We thus get

$$E\phi_{\alpha,\beta}(x, y)$$
$$= \int_{(0,0)}^{(x,y)} (x-u)^{\alpha-a-1}(y-v)^{\beta-b-1}d(u,v) \int_{(0,0)}^{(u,v)} (u-\xi)^{a-1}(v-\eta)^{b-1}\phi d(\xi,\eta),$$

and this completes the proof.

7.1. **Lemma on Young's functions.** We divide the proof of Theorem IV into several parts, considering first a lemma in connection with the function

$$\gamma_\eta(u) \equiv \gamma_\eta^{(0)}(u) \equiv (1/\eta)[1 - u^2/\{(\eta+1)(\eta+2)\}$$
$$+ u^4/\{(\eta+1)(\eta+2)(\eta+3)(\eta+4)\} - \cdots]$$

and its derivatives $\gamma_\eta^{(1)}, \gamma_\eta^{(2)}, \cdots$. For $0 < u$, we have

$$\gamma_\eta(u) = \Gamma(\eta)u^{-\eta}C_\eta(u), \tag{7.11}$$

where

$$C_\eta(u) = (1/\Gamma(\eta))\int_0^u (u-t)^{\eta-1}C_0(t)dt \qquad [C_0(u) = \cos u] \tag{7.12}$$

is Young's function.*

Lemma 5. *Let $0 < \eta$ and let $m < \eta + 1$. Then $\gamma_\eta^{(m)}(u)$ is continuous for all u, and*

$$\gamma_\eta^{(m)}(u) = O(u^{-\eta} + u^{-m-2}) \quad as \quad u \to \infty. \tag{7.13}$$

The conclusions of this lemma are familiar.† First, it is plain that $\gamma_\eta^{(m)}$ is continuous for all u. Next, Young‡ has proved that, if $0 \leq \xi$, then

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* Young, **21.**

† Hardy and Littlewood, **6,** p. 217, state without proof that, as $u \to \infty$,

$$\gamma_{1+\eta}(u) = Au^{-1-\eta}\sin(u - \pi\eta/2) + Bu^{-2} + O(u^{-2-\eta}) + O(u^{-3}),$$

where A and B are constants, and that asymptotic formulas for the derivatives of $\gamma_{1+\eta}$ are given by the formal derivatives of this expression. Using the method employed above, Young, **21,** Hobson, **8,** p. 566, and Pollard, **15,** p. 212, obtain (7.13) for various cases. None of these authors treat the case $\eta < m < \eta + 1$ however.

‡ Young, **21,** or see Hobson, **8,** p. 565.

$$C_\xi(u) = O(1 + u^{\xi-2}) \quad \text{as} \quad u \to \infty. \tag{7.14}$$

Using this result, (7.11), and the fact that

$$D_u C_\xi = C_{\xi-1} \tag{7.15}$$

for $0<u$, and $1 \leqq \xi$, we see that (7.13) holds if $m \leqq \eta$.

Suppose then that $\eta < m < \eta+1$; we have, by (7.11), (7.14) and (7.15),

$$\gamma_\eta^{(m)}(u) = O(u^{-\eta-1} + u^{-m-2} + u^{-\eta} \,|\, D_u C_{\eta+1-m} \,|\,)$$

as $u \to \infty$. But, replacing t by $u-t$ in (7.15), we get

$$\Gamma(\eta + 1 - m) C_{\eta+1-m}(u) = \cos u \int_0^u t^{\eta-m} \cos t \, dt + \sin u \int_0^u t^{\eta-m} \sin t \, dt.$$

Hence,

$$D_u C_{\eta+1-m} = O\left\{ u^{\eta-m} + \left| \int_0^u t^{\eta-m} \cos t \, dt \right| + \left| \int_0^u t^{\eta-m} \sin t \, dt \right| \right\} = O(1).$$

We conclude that (7.13) holds in this case. This completes the proof.

7.2. **Lemmas for part (a) of Theorem IV.** In these lemmas and in the proof of part (a) we suppose that a, α, b, β satisfy (5.11). We denote by h the integral part of a. Then it is readily seen, on applying Lemma 5, that $|\gamma_{\alpha+1}^{(h+1)}(tu)(t-1)^{h-a}|$ is integrable in t over $(1, \infty)$ for $0<u$. We set

$$H(u) = u^{h+1-a} \int_1^\infty \gamma_{\alpha+1}^{(h+1)}(tu)(t - 1)^{h-a} dt.$$

Similarly, we write k for the integral part of b, and set

$$K(v) = v^{k+1-b} \int_1^\infty \gamma_{\beta+1}^{(k+1)}(tv)(t - 1)^{k-b} dt.$$

We put

$$\psi_a = 2(-1)^{h+1}/\{\pi\Gamma(h + 1 - a)\}, \psi_b = 2(-1)^{k+1}/\{\pi\Gamma(k + 1 - b)\}, \psi = \psi_a\psi_b.$$

We consider first $H(u)$. We have

Lemma 6. *The function $H(u)$ is bounded and measurable over $0<u$. Moreover,* as $u \to \infty$,*

$$H(u) = O(u^{-\alpha-1} + u^{-h-2}).$$

* Bosanquet, **4**, p. 19, obtains an asymptotic formula for H by means of Cauchy's theorem.

Since, for each u, H is the limit as $n\to\infty$ of the sequence of continuous functions

$$\left\{u^{h+1-a}\int_{1+1/n}^{n}\gamma_{\alpha+1}^{(h+1)}(tu)(t-1)^{h-a}dt\right\},$$

we have immediately that H is measurable over $0<u$.

Now, for $0<u$,

$$\begin{aligned}|H| &= \left|u^{h+1-a}\left\{\int_{1}^{1+1/u}+\int_{1+1/u}^{\infty}\right\}\gamma_{\alpha+1}^{(h+1)}(tu)(t-1)^{h-a}dt\right| \\ &\leqq (h+1-a)^{-1}\max_{u\leqq t}\left|\gamma_{\alpha+1}^{(h+1)}(t)\right| + 2\max_{u\leqq t}\left|\gamma_{\alpha+1}^{(h)}(t)\right|.\end{aligned}$$

It follows that H is bounded for $0<u$, and that, as $u\to\alpha$,

$$\begin{aligned}H &= O(u^{-\alpha-1}+u^{-h-3}+u^{-\alpha-1}+u^{-h-2}) \\ &= O(u^{-\alpha-1}+u^{-h-2}).\end{aligned}$$

This proves the lemma.

7.3. We have next

LEMMA 7. *Let $\phi(u)$ be integrable over $(0,\pi)$, even, and periodic with period 2π. For $0\leqq\eta$ let $\phi_\eta(u)$ be the fractional integral of order η of ϕ. Then, if $0<\delta$, we have*

$$(7.31)\qquad Q\equiv\int_{\delta}^{\infty}\left|H(xu)\phi_a(u)\right|du\leqq M(x^{-\alpha-1}+x^{-h-2})\int_0^{\pi}|\phi|\,du\ \textit{for}\ 1\leqq x,$$

where M is independent of ϕ as well as x. In addition,

$$(7.32)\qquad \int_0^{\infty}\left|H(xu)\phi_a(u)\right|du\leqq N\int_0^{\pi}|\phi|\,du,$$

where N is independent of ϕ.

The proofs of (7.31) and (7.32) are much the same. We consider that of the former. We denote by M a number independent of u, t, x and ϕ for $0<u$, $0<t$, $1\leqq x$.

Now, if $1\leqq a$, then

$$|\phi_a(u)|\leqq M\int_0^{u}(u-t)^{a-1}|\phi|\,dt\leqq M(u+1)^aS,\quad\text{where } S=\int_0^{\pi}|\phi|\,du.$$

Hence, if $a=0$ or if $1\leqq a$,

$$Q \leqq MS\left(x^{-\alpha-1}\sum_{m=1}^{\infty} m^{a-\alpha-1} + x^{-h-2}\sum_{m=1}^{\infty} m^{a-h-2}\right) \leqq MS(x^{-\alpha-1} + x^{-h-2})$$

as a consequence of Lemma 6. Thus, (7.31) holds in this case.

Suppose then that $0 < a < 1$. We have

$$Q \leqq M \int_\delta^\infty |H| \, du \int_0^u (u-t)^{a-1} |\phi| \, dt$$

$$= M \int_0^\delta |\phi| \, dt \int_\delta^\infty |H| (u-t)^{a-1} du + M \int_\delta^\infty |\phi| \, dt \int_t^\infty |H| (u-t)^{a-1} du.$$

But

$$\int_t^\infty |H| (u-t)^{a-1} du \leqq Mx^{-\alpha-1} \int_t^\infty u^{-\alpha-1}(u-t)^{a-1} du$$

$$+ Mx^{-h-2} \int_t^\infty u^{-h-2}(u-t)^{a-1} du$$

$$\leqq Mx^{-\alpha-1} t^{a-\alpha-1} + Mx^{-h-2} t^{a-h-2}.$$

Hence,

$$Q \leqq Mx^{-\alpha-1}\left\{\int_0^\delta |\phi| \, dt + \int_\delta^\infty |\phi| \, t^{a-\alpha-1} dt\right\}$$

$$+ Mx^{-h-2}\left\{\int_0^\delta |\phi| \, dt + \int_\delta^\infty |\phi| \, t^{a-h-2} dt\right\}.$$

The lemma follows.

7.4. We have thirdly

LEMMA 8. *Let the hypotheses of Lemma 7 hold; and let the series* (3.11) *be the Fourier series of* ϕ *at the origin, so that*

$$A_m = \lambda_m \int_0^\pi \phi(u) \cos mu \, du.$$

Then

$$\psi_a x^{\alpha+a+1} \int_0^\infty H(xu)\phi_a(u) du = \sigma_\alpha(x).$$

This lemma is contained in one of Bosanquet's theorems, **4**, p. 22. The proof is analogous to that of Lemma 11 below. In this case the starting point is the well known formula

$$2x^{\alpha+1} \int_0^\infty \gamma_{\alpha+1}(xu)\phi(u) du = \pi\sigma_\alpha(x),$$

and integration by parts is carried out $(h+1)$ times.

7.5. We have finally

LEMMA 9. *Under the hypotheses that f is even-even, periodic with period 2π in each variable, and integrable over $(0, 0; \pi, \pi)$, the function $|H(xu)K(yv)f_{a,b}(u, v)|$ is integrable over T and*

$$\psi x^{a+1}y^{b+1}\int_T H(xu)K(yv)f_{a,b}(u, v)dT = R_{\alpha,\beta}(x, y).$$

Applying Lemmas 7 and 8, with $\phi(u)=f(u, v)$, we have, for almost all v on $(0, \pi)$,

$$\int_0^\infty |Hf_{a,0}|\,du \leqq N\int_0^\pi |f|\,du, \tag{7.51}$$

where N is independent of v, and

$$\psi_a x^{\alpha+a+1}\int_0^\infty Hf_{a,0}du = \sum_{p<x}(x-p)^\alpha\lambda_p\int_0^\pi f\cos pu\,du. \tag{7.52}$$

From (7.51), the measurability of $Hf_{a,0}$ over $(0, 0; \infty, \pi)$, and the integrability of f we deduce the existence of the integral*

$$I_0 = \int_0^\infty |H|\,du\int_0^\pi |f_{a,0}|\,dv.$$

In addition, we see that the integrals

$$I_q = \int_0^\infty |H|\,du\int_0^\pi |f_{a,0}\cos qv|\,dv \qquad (q = 1, 2, \cdots)$$

exist.

Now let E be the set of values u on $(0, \infty)$ such that $f_{a,b}$ is integrable in v over every finite interval $(0, |z|)$. Then, applying Theorem III, the complement of E relative to $(0, \infty)$ is of measure 0, and, for u on E,

$$\phi(v) = f_{a,0}(u, |v|)$$

satisfies the conditions of Lemma 7 and

$$\phi_b(v) = f_{a,b}(u, v)$$

for almost all v on $(0, \infty)$. Hence, for u on E,

$$\int_0^\infty |Kf_{a,b}|\,dv \leqq N\int_0^\pi |f_{a,0}|\,dv,$$

* See, for example, Hobson, **8**, p. 346.

where N is independent of u, and

$$\psi_b y^{\beta+b+1}\int_0^\infty K f_{a,b}dv = \sum_{q<y}(y-q)^\beta\lambda_q\int_0^\pi f_{a,0}\cos qv\,dv.$$

From the existence of I_0 we then conclude that $|HKf_{a,b}|$ is integrable over T, and from (7.52) and the existence of $I_0, I_1, \cdots$, that

$$\begin{aligned}\psi x^{a+1}y^{b+1}\int_T HKf_{a,b}dT &= \psi x^{a+1}y^{b+1}\int_0^\infty H du\int_0^\infty K f_{a,b}dv\\ &= \psi_a x^{a+1}y^{-\beta}\sum_{q<y}(y-q)^\beta\lambda_q\int_0^\pi \cos qv\,dv\int_0^\infty H f_{a,0}du\\ &= R_{\alpha,\beta}(x,y).\end{aligned}$$

8.1. **Proof of part** (a) **of Theorem IV.** We note that in the proof of the second part we can assume that $s=0$. For, on the one hand, $f-s$ satisfies the conditions imposed upon f with s replaced by 0; and, on the other hand, the series (2.11) is summable $(C;\alpha,\beta)$ to sum s if the Fourier series of $f-s$ at the origin is summable $(C;\alpha,\beta)$ to sum 0.

Suppose then that, corresponding to some ϵ, there is a positive d and an M such that

$$(8.11)\qquad |f_{a,b}(u,v)| \leqq \epsilon u^a v^b,\qquad |f_{a,b}(u,v)| \leqq M u^a v^b,$$

the former for almost all (u, v) in $(0, 0; d, d)$, and the latter for almost all (u, v) in $(0, 0; \infty, d)$ and in $(0, 0; d, \infty)$. Writing

$$\begin{aligned}\int_T H(xu)K(yv)f_{a,b}dT &= \left[\int_{(0,0)}^{(d,d)} + \int_{(d,0)}^{(\infty,d)} + \int_{(0,d)}^{(d,\infty)} + \int_{(d,d)}^{(\infty,\infty)}\right]HKf_{a,b}d(u,v)\\ &= F_1+F_2+F_3+F_4,\end{aligned}$$

say, we have

$$|F_1| \leqq \epsilon x^{-a-1}y^{-b-1}\int_0^\infty |H(u)|u^a du\int_0^\infty |K(v)|v^b dv \leqq M x^{-a-1}y^{-b-1}$$

as a consequence of Lemma 6,

$$\begin{aligned}|F_2| &\leqq M y^{-b-1}\int_d^\infty \{(xu)^{-a-1}+(xu)^{-h-2}\}u^a du\int_0^\infty |K(v)|v^b dv\\ &\leqq M(x^{-a-1}+x^{-h-2})y^{-b-1},\\ |F_3| &\leqq M x^{-a-1}(y^{-b-1}+y^{-k-2}),\end{aligned}$$

and, by Lemma 7, for $1\leqq x$, $1\leqq y$,

$$|F_4| \leqq M(y^{-\beta-1} + y^{-k-2}) \int_d^\infty |H(xu)|\, dv \int_0^\pi |f_{a,0}|\, dv$$
$$\leqq M(x^{-\alpha-1} + x^{-h-2})(y^{-\beta-1} + y^{-k-2}).$$

We conclude that

$$|R_{\alpha,\beta}(x, y)| \leqq M(1 + x^{a-\alpha} + x^{a-h-1})(1 + y^{b-\beta} + y^{b-k-1})$$

for $1 \leqq x$, $1 \leqq y$, and that

$$\limsup_{(x,\, y)\to(\infty,\,\infty)} |R_{\alpha,\beta}(x, y)| \leqq \epsilon \int_0^\infty |H(u)|\, u^a du \int_0^\infty |K(v)|\, v^b dv.$$

The proof now follows. If f is almost bounded $(C;\, a,\, b)$ in $(0,\, 0;\, \infty,\, \delta)$ and in $(0,\, 0;\, \delta,\, \infty)$, then (8.11) holds for some ϵ, and the first part of (a) follows from the former of these inequalities. If, in addition, f is almost continuous $(C;\, a,\, b)$ with limit 0, then (8.11) holds for each arbitrarily small ϵ, and the second part follows from the latter.

9.1. **Lemmas for part** (b) **of Theorem IV and Theorem VI.** In these lemmas we suppose that α, a, β, b satisfy (5.12). We denote by h the integral part of α, and by k the integral part of β. We set

$$\psi_\alpha = (-1)^h / \{\Gamma(\alpha + 1)\Gamma(h + 1 - \alpha)\Gamma(a)\},$$
$$\psi_\beta = (-1)^k / \{\Gamma(\beta + 1)\Gamma(k + 1 - \beta)\Gamma(b)\}, \qquad \psi = \psi_\alpha\psi_\beta,$$
$$H(u) = u^{h+1-\alpha} \int_1^\infty \gamma_a^{(h+2)}(tu)(t-1)^{h-\alpha} dt, \quad K(v) = v^{k+1-\beta} \int_1^\infty \gamma_b^{(k+2)}(tv)(t-1)^{k-\beta} dt.$$

The functions H, K exist for all positive values of their arguments.

9.2. In regard to $H(u)$ we have

Lemma 10. *The function $H(u)$ is bounded and measurable for $0 < u$. Moreover, as $u \to \infty$,*

$$H(u) = O(u^{-a} + u^{-h-3}).$$

The proof here is practically the same as that of Lemma 6 and can be omitted.

9.3. We have next

Lemma 11.* *Suppose that $\phi(u)$, $\phi_\eta(u)$, $0 \leqq \eta$, are the functions of Lemma 7. Suppose that the series (3.11) is the Fourier series of ϕ at the origin, and that, for some δ satisfying $0 \leqq \delta < a - 1$, $\delta < h + 2$,*

* The proof of this lemma is closely analogous to a proof given by Bosanquet, **3**, pp. 157–161, concerning, not the summability of a series, but the summability of an integral. In treating the series in (9.33) Bosanquet uses partial summation throughout rather than partial integration and partial summation.

$$\sigma_\alpha(u) = O(u^\delta) \quad as \quad u \to \infty. \tag{9.31}$$

Then the function $|H(xu)\sigma_\alpha(u)|$ *is integrable over* $(0, \infty)$ *and*

$$\psi_a x^{a+\alpha+1} \int_0^\infty H(xu)\sigma_\alpha(u)du = \phi_a(x). \tag{9.32}$$

It is plain that $|H\sigma_\alpha|$ is integrable over $(0, \infty)$. Consider then (9.32). We have*

$$x^a \sum_{m=0}^\infty A_m\gamma_a(mx) = \Gamma(a)\phi_a(x). \tag{9.33}$$

Now, for $0<t$,

$$\int_0^t \sigma_m(u)du = \sigma_{m+1}(t)/(m+1).$$

Hence, denoting by z a positive integer, using Abel's formula, and integrating by parts $(h+1)$ times, we have

$$\begin{aligned}\sum_{m=0}^z A_m\gamma_a(mx) &= \left\{\gamma_a(xz)\sigma_0(z+1) + \sum_{m=1}^{h+1}(-x)^m\gamma_a^{(m)}(xz)\sigma_m(z)/\Gamma(m+1)\right\}\\ &\quad + [(-x)^{h+2}/\Gamma(h+2)]\int_0^z \gamma_a^{(h+2)}(xt)\sigma_{h+1}(t)dt\\ &= I_1 + I_2,\end{aligned}$$

say. But, since $|A_m| \leqq M$, we have, as $z\to\infty$,

$$\sigma_m(z) = O\left\{\sum_{n<z}(z-n)^m\right\} = O(z^{m+1}).$$

Moreover, since

$$\sigma_{h+1}(t) = (h+1)\binom{h}{\alpha}\int_0^t (t-u)^{h-\alpha}\sigma_\alpha(u)du \text{ for } 0 < t, \tag{9.34}$$

we have, by (9.31),

$$\sigma_{h+1}(z) = O\left\{\int_0^z (z-u)^{h-\alpha}u^\delta du\right\} = O(z^{h+\delta+1-\alpha}).$$

Thus,

$$I_1 = O\left\{\sum_{m=0}^h (z^{-a} + z^{-m-2})z^{m+1} + (z^{-a} + z^{-h-3})z^{h+\delta+1-\alpha}\right\} = o(1).$$

* See, for example, Paley, **14**, p. 190.

It is sufficient then to show that

$$(9.35)\qquad I_2 = \Gamma(a)\psi_\alpha x^{\alpha+1}\int_0^z H(xu)\sigma_\alpha du + o(1) \text{ as } z\to\infty.$$

Making the substitution (9.34) in I_2,

$$\begin{aligned} I_2 &= \Gamma(a)\psi^\alpha x^{h+2}\int_0^z \gamma_\alpha^{(h+2)}(xt)dt\int_0^t (t-u)^{h-\alpha}\sigma_\alpha(u)du \\ &= \Gamma(a)\psi_\alpha x^{h+2}\int_0^z \sigma_\alpha(u)u^{h+1-\alpha}du\int_1^{z/u}\gamma_\alpha^{(h+2)}(xut)(t-1)^{h-\alpha}dt \\ &= \Gamma(a)\psi_\alpha x^{\alpha+1}\left\{\int_0^z H(xu)\sigma_\alpha(u)du - \int_0^z H_1(x,u,z)\sigma_\alpha(u)du\right\}, \end{aligned}$$

say, where

$$H_1 = (xu)^{h+1-\alpha}\int_{z/u}^\infty \gamma_\alpha^{(h+2)}(xut)(t-1)^{h-\alpha}dt.$$

Now, for $0<u<z$,

$$|H_1| \leqq 2x^{h-\alpha}(z-u)^{h-\alpha}\max_{xz\leqq t}|\gamma_\alpha^{(h+1)}(t)|.$$

Hence,

$$\int_0^z H_1\sigma_\alpha du = O\left\{(z^{-\alpha}+z^{-h-3})\int_0^z (z-u)^{h-\alpha}u^\delta du\right\} = o(1).$$

We conclude that (9.35) and, accordingly, that (9.32) holds.

9.4. We have thirdly

LEMMA 12. *Let the series* (2.11) *be bounded* $(C;\alpha,\beta)$. *Then, for any* n,

$$\zeta(u,n) \equiv \sum_{m<u}(u-m)^\alpha a_{m,n} = O(u^\alpha)$$

as $u\to\infty$.

We note first that, as $u\to\infty$,

$$\zeta = O\left\{\max_{m\leqq u}\left|\sum_{p=0}^m\binom{\alpha+m-p}{m-p}a_{p,n}\right|\right\}.$$

If $0<\alpha$, this follows from Lemma 2. If $\alpha=0$, it is trivial. Now,*

$$\sum_{p=0}^m\binom{\alpha+m-p}{m-p}a_{p,n} = \sum_{q=0}^n(-1)^q\binom{\beta+1}{q}S_{\alpha,\beta}(m,n-q),$$

* See Hobson, 8, p. 71, (4).

where we have set

$$\binom{\beta+1}{q} = 0 \text{ if } \beta + 1 - q = -1, -2, \cdots .$$

Thus, making use of our hypothesis,

$$\left| \sum_{p=0}^{m} \binom{\alpha + m - p}{m - p} a_{p,n} \right| \leqq M(m+1)^{\alpha}(n+1)^{\beta} \sum_{q=0}^{n} \left| \binom{\beta+1}{q} \right| = O(m^{\alpha})$$

as $m \to \infty$. The lemma follows.

9.5. We have finally

LEMMA 13. *If either* (a) *the series* (2.11) *is bounded* $(C; \alpha, \beta)$, *or* (b) $\alpha + 2 < a$, $\beta + 2 < b$, *then* $|H(xu)K(yv)\sigma_{\alpha,\beta}|$ *is integrable over* T, *and*

$$\psi x^{a+\alpha+1} y^{b+\beta+1} \int_T H(xu)K(yv)\sigma_{\alpha,\beta}(u, v)dT = f_{a,b}(x, y). \tag{9.51}$$

We note first that $|HK\sigma_{\alpha,\beta}|$ is integrable over T in either case. If (a) holds this follows from Theorem II and Lemma 10; and if (b) holds it follows on observing that, in general,

$$|\sigma_{\alpha,\beta}(x, y)| \leqq M x^{\alpha+1} y^{\beta+1}.$$

Consider then (9.51). In either case, since $1 < a$, the function

$$\Phi(v) \equiv f_{a,0}(x, |v|)$$

is integrable in v over $(0, \pi)$. Moreover, Φ is even and periodic with period 2π. Its Fourier series at the origin is

$$\sum_{n=0}^{\infty} F_a^{(n)}(x), \qquad \text{where} \quad F^{(n)}(u) = \lambda_n \int_0^{\pi} \cos nv\, f(u, v)dv.$$

Now, for a fixed n, $F^{(n)}$ is integrable over $(0, \pi)$, even, and periodic with period 2π. Its Fourier series at the origin is

$$\sum_{m=0}^{\infty} a_{m,n}.$$

But, if (a) holds, then

$$\zeta(u, n) = \sum_{m<u} (u - m)^{\alpha} a_{m,n} = O(u^{\alpha})$$

as $u \to \infty$. Hence in this case, by Lemma 11, with $\phi = F^{(n)}$ and $\delta = \alpha$,

$$F_a^{(n)}(x) = \psi_a x^{a+\alpha+1} \int_0^{\infty} H(xu)\zeta(u, n)dn. \tag{9.52}$$

On the other hand, in general,

$$\zeta(u, n) = O(u^{\alpha+1});$$

so that, if (b) holds, we have on applying Lemma 11 with $\delta=\alpha+1$ the result (9.52) again. Accordingly, in either case the Fourier series of $\Phi(v)$ at the origin is

$$\sum_{n=0}^{\infty} \psi_\alpha x^{a+\alpha+1} \int_0^\infty H(xu)\zeta(u, n)du.$$

The lemma now follows. We have

$$\sum_{n<v} (v-n)^\beta \int_0^\infty H\zeta du = \int_0^\infty H\sigma_{\alpha,\beta}(u, v)du.$$

If (a) holds this is

$$= O\left(v^\beta \int_0^\infty |H(xu)| u^\alpha du\right) = O(v^\beta)$$

as $v\to\infty$, and if (b) holds it is

$$= O\left(v^{\beta+1} \int_0^\infty |H| u^{\alpha+1}du\right) = O(v^{\beta+1}).$$

Accordingly, noting that since $1<a$, $1<b$, $f_{a,b}(x, y)$ is finite, we have, by Theorem III and Lemma 11,

$$f_{a,b}(x, y) = \Phi_b(y) = \psi x^{a+\alpha+1}y^{b+\beta+1} \int_0^\infty K(yv)\left\{\sum_{n<v} (v-n)^\beta \int_0^\infty H\zeta du\right\}dv$$

$$= \psi x^{a+\alpha+1}y^{b+\beta+1} \int_T HK\sigma_{\alpha,\beta}dT.$$

10.1. **Proof of part** (b) **of Theorem IV.** Let us assume first that the series (2.11) is bounded $(C; \alpha, \beta)$. Then we have, by Lemma 13,

$$f_{a,b}(x, y) = \psi x^{a+\alpha+1}y^{b+\beta+1} \int_T H(xu)K(yv)\sigma_{\alpha,b}dT$$

$$= \psi x^a y^b \int_T H(u)K(v)\sigma_{\alpha,b}(u/x, v/y)dT;$$

and, for all (u, v) in T,

$$|H(u)K(v)x^\alpha y^\beta \sigma_{\alpha,\beta}(u/x, v/y)| \leqq Mu^\alpha |H(u)| v^\beta |K(v)|,$$

where M is independent of (u, v) as well as (x, y). As the function on the right here is integrable over T, we conclude that f is bounded $(C; a, b)$ in T.

Now let us assume in addition that the series is summable $(C; \alpha, \beta)$ to sum s. Then we have

$$\lim_{(x,\, y)\to(+0,\, +0)} \{H(u)K(v)x^{\alpha}y^{b}\sigma_{\alpha,\beta}(u/x, v/y)\} = u^{\alpha}H(u)v^{\beta}K(v)$$

for every (u, v) in T. We conclude from Lebesgue's theorem that

$$\lim_{(x,\, y)\to(+0,\, +0)} \{f_{a,b}(x, y)/(x^{a}y^{b})\} = \psi s \int_T u^{\alpha}H(u)v^{\beta}K(v)dT.$$

Applying Lemma 13 to the function $f\equiv 1$, we get

$$\psi\Gamma(a+1)\Gamma(b+1)\int_T u^{\alpha}H(u)v^{\beta}K(v)dT = 1.$$

Thus,

$$\lim_{(x,\, y)\to(+0,\, +0)} \{f_{a,b}(x, y)/(x^{a}y^{b})\} = s/\{\Gamma(a+1)\Gamma(b+1)\}.$$

This completes the proof.

11.1. **Proof of part** (a) **of Theorem** V. We first prove

LEMMA 14. *Let $\phi(u), \phi_\eta(u), 0\leqq\eta$, be the functions of Lemma 7. Suppose that $0\leqq a<\alpha$ and that, for some fixed number K, $|\phi_a(x)|\leqq Kx^a$ for almost all x on $(0, \pi)$,*

(11.11) $\quad |\phi_m(\pi)|\leqq K$ *for each positive odd integer* $m<a$.

Then there exists a number N, independent of x, ϕ, K, such that

(11.12) $\quad |\phi_\alpha(x)|\leqq NKx^\alpha$ *if* $\phi_\alpha(x)$ *is finite.*

We see that, if a is a positive odd integer, then, by the continuity of ϕ_a at $x=\pi$, ϕ_a satisfies (11.11) with K replaced by $K\pi^a$. We see also that, if $a<a'$, then

$$\begin{aligned}|\phi_{a'}(x)| &\leqq (1/\Gamma(a'-a))\int_0^x (x-u)^{a'-a-1}|\phi_a(u)|\,du\\ &\leqq Kx^{a'}\Gamma(a+1)/\Gamma(a'+1)\end{aligned}$$

for almost all x on $(0, \pi)$. On the other hand, if $\alpha<\alpha'$ and the conclusion holds, then

$$|\phi_{\alpha'}(x)| \leqq NKx^{\alpha'}\Gamma(\alpha+1)/\Gamma(\alpha'+1)$$

wherever $\phi_{\alpha'}$ is finite. Accordingly, it is enough to prove the lemma with $h<a<\alpha<h+1$, where h is the integral part of a.

We denote by N a number independent of x, y, ϕ, K. Then

$$|\phi_{h+1}(x)| \leqq NK$$

for $x\leqq\pi$, and

$$\begin{aligned}
|\phi_{h+1}(y) - \phi_{h+1}(x)| &\leqq N\int_0^x |(y-u)^{h-a} - (x-u)^{h-a}|\,|\phi_a|\,du \\
&\quad + N\int_x^y (y-u)^{h-a}|\phi_a|\,du \\
&\leqq NK\int_x^y dt\int_{-\infty}^x (t-u)^{h-a-1}du + NK\int_x^y (y-u)^{h-a}du \\
&\leqq NK(y-x)^{h+1-a}
\end{aligned}$$

for $x<y\leqq\pi$.

From these inequalities we conclude that

$$\left|\int_0^\pi (x-u)^{\alpha-h-1}\phi_h du\right| \leqq NKx^{\alpha-1} \text{ for } \pi < x, \tag{11.13}$$

$$\limsup_{y\to x+0}\left|\int_y^\pi (u-x)^{\alpha-h-1}\phi_h du\right| \leqq NK \qquad \text{for } 0 \leqq x < \pi, \tag{11.14}$$

$$\left|\int_0^\pi (x+u)^{\alpha-h-1}\phi_h du\right| \leqq NK(x+\pi)^{\alpha-1}. \tag{11.15}$$

We have, for $\pi<x$,

$$\begin{aligned}
\left|\int_0^\pi (x-u)^{\alpha-h-1}\phi_h du\right| &\leqq x^{\alpha-h-1}|\phi_{h+1}(\pi)| \\
&\quad + N\int_0^\pi (x-u)^{\alpha-h-2}|\phi_{h+1}(\pi)-\phi_{h+1}(u)|\,du \\
&\leqq NKx^{\alpha-1} + NK\int_0^\pi (x-u)^{\alpha-h-2}(\pi-u)^{h+1-a}du \leqq NKx^{\alpha-1}
\end{aligned}$$

as can be seen by considering separately the cases $\pi<x<2\pi$, $2\pi\leqq x$. This is (11.13).

Next, for $0\leqq x<y<\pi$, we have

$$\begin{aligned}
\left|\int_y^\pi (u-x)^{\alpha-h-1}\phi_h du\right| &\leqq (\pi-x)^{\alpha-h-1}|\phi_{h+1}(\pi)-\phi_{h+1}(y)| \\
&\quad + N\int_y^\pi (u-x)^{\alpha-h-2}|\phi_{h+1}(u)-\phi_{h+1}(y)|\,du \\
&\leqq NK + NK\int_y^\pi (u-y)^{\alpha-a-1}du \leqq NK.
\end{aligned}$$

Hence, (11.14) holds.

Finally,

$$\left|\int_0^\pi (x+u)^{\alpha-h-1}\phi_h du\right| \leqq (x+\pi)^{\alpha-h-1}|\phi_{h+1}(\pi)| + N\int_0^\pi (x+u)^{\alpha-h-2}|\phi_{h+1}|du$$
$$\leqq K(x+\pi)^{\alpha-1} + NK\int_0^\pi (x+u)^{\alpha-h-2}u^{h+1}du$$
$$\leqq NK(x+\pi)^{\alpha-1},$$

as may be seen by considering the cases $x<\pi$, $\pi\leqq x$. This is (11.15).

Consider now (11.12). It is plain that our conclusion holds if we restrict ourselves to values of $x\leqq\pi$. We show now that it holds for $\pi<x\leqq 2\pi$. If $\phi_\alpha(x)$ is finite then the integral

$$\Psi(x) = \int_{2\pi-x}^\pi [u-(2\pi-x)]^{\alpha-1}\phi du$$

exists, and

$$\Gamma(\alpha)\phi_\alpha(x) = \int_0^\pi (x-u)^{\alpha-1}\phi du + \Psi(x).$$

If $h=0$ our conclusion then follows from (11.13) and (11.14). If $1\leqq h$ we integrate by parts h times. We get

$$\phi_\alpha(x) = \sum_{p<h/2} M(x-\pi)^{\alpha-2p-1}\phi_{2p+1}(\pi) + N\int_0^\pi (x-u)^{\alpha-h-1}\phi_h du$$
$$+ N\int_{2\pi-x}^\pi [u-(2\pi-x)]^{\alpha-h-1}\phi_h du.$$

Applying (11.11), (11.13) and (11.14) we get the desired conclusion.

Suppose now that $2\pi<x$. We write $x=2n\pi+\xi$, where n is a positive integer and $0<\xi\leqq 2\pi$. Then, if $\phi_\alpha(x)$ is finite, so also is $\phi_\alpha(\xi)$, and

$$\Gamma(\alpha)\phi_\alpha(x) = \sum_{q=0}^{n-1}\int_0^\pi \{(x-2q\pi-u)^{\alpha-1} + [x-2(q+1)\pi+u]^{\alpha-1}\}\phi du$$
$$+ \Gamma(\alpha)\phi_\alpha(\xi).$$

If $h=0$, we have, by (11.13) and (11.15),

$$\sum_{q=0}^{n-1}\left|\int_0^\pi \{(x-2q\pi-u)^{\alpha-1} + [x-2(q+1)\pi+u]^{\alpha-1}\}\phi du\right|$$
$$\leqq NK\sum_{q=0}^{n-1}[x-(2q+1)\pi]^{\alpha-1} \leqq NKx^\alpha,$$

so that the lemma follows in this case. If $1\leqq h$ we integrate by parts. We get, for $2\pi<x$,

$$\left|\int_0^\pi \{(x-u)^{\alpha-1}+(x-2\pi+u)^{\alpha-1}\}\phi du\right| \leqq N\sum_{p<h/2}(x-\pi)^{\alpha-2p-1}\left|\phi_{2p+1}(\pi)\right|$$

$$+N\left|\int_0^\pi (x-u)^{\alpha-h-1}\phi_h du\right| + N\left|\int_0^\pi (x-2\pi+u)^{\alpha-h-1}\phi_h du\right|$$

$$\leqq NKx^{\alpha-1}.$$

Hence,

$$\left|\phi_\alpha(x)\right| \leqq NK\sum_{q=0}^{n-1}(x-2q\pi)^{\alpha-1}+NK \leqq NKx^\alpha,$$

and this proves the lemma.

11.2. Turning now to the proof of (a), we select α_0, β_0 so that $a<\alpha_0<\alpha$, $b<\beta_0<\beta$, and note that, as a consequence of Theorem III, f is almost continuous $(C;\alpha_0,\beta_0)$ with limit s. Hence, by Theorem IV, it is enough to prove that f is almost bounded $(C;\alpha_0,\beta_0)$ on $(0, 0; \infty, \delta)$ and $(0, 0; \delta, \infty)$.

Let D denote the rectangle $(0, 0; \infty, \delta)$. We shall show that f is almost bounded $(C;\alpha_0,\beta_0)$ on D. We can assume that f is almost bounded $(C;a,\beta_0)$ on $(0, 0; \pi, \delta)$, and, since

$$\int_0^y (y-v)^{\beta_0-1}dv\int_0^\pi (\pi-u)^{m-1}|f|\,du \qquad (m=1,2,\cdots)$$

is finite for almost all y on d: $(y\leqq\delta)$, that $y^{-\beta_0}f_{m,\beta_0}(\pi,y)$ is almost bounded on d for each positive odd integer $m<a$.

Now there is a number M_0 and a set e, of measure δ, of values y on d, such that $f_{0,\beta_0}(u,y)$ is integrable over $(0,\pi)$, and

$$\left|f_{a,\beta_0}(x,y)\right| \leqq M_0x^ay^{\beta_0}, \qquad \left|f_{m,\beta_0}(\pi,y)\right| \leqq M_0y^{\beta_0},$$

the first for almost all x on $(0,\pi)$, the second for each positive odd integer $m<a$. Let E denote the set of points (x, y) such that y belongs to e and $f_{\alpha_0,\beta_0}(x,y)$ is finite. Then the complement of E relative to D is of measure 0. To prove the theorem we show that f is bounded $(C;\alpha_0,\beta_0)$ on E.

Consider any fixed y on d. The function

$$\phi(u) = f_{0,\beta_0}(|u|,y)$$

is integrable over $(0,\pi)$, even, and periodic with period 2π. In addition ϕ fulfils the remaining conditions of the lemma with $K=M_0y^{\beta_0}$. In fact, for almost all x on $(0,\pi)$, $f_{a,\beta_0}(x,y)$ is finite. At these points f_{a,β_0} is equal to the fractional integral of order $(a, 0)$ of f_{0,β_0}. Thus,

$$\left|\phi_a(x)\right| = \left|f_{a,\beta_0}(x,y)\right| \leqq M_0x^ay^{\beta_0}$$

for almost all x on $(0, \pi)$. Moreover, $f_{m,\beta_0}(\pi, y)$ being finite,

$$|\phi_m(\pi)| = |f_{m,\beta_0}(\pi, y)| \leqq M_0 y^{\beta_0}$$

for each positive odd integer $m < a$. The lemma can then be applied. We get, if $\phi_{\alpha_0}(x)$ is finite,

$$|\phi_{\alpha_0}(x)| \leqq N M_0 x^{\alpha_0} y^{\beta_0},$$

where N is independent of x and y. In particular then, if (x, y) is a point of E, we have

$$|f_{\alpha_0,\beta_0}(x, y)| = |\phi_{\alpha_0}(x)| \leqq N M_0 x^{\alpha_0} y^{\beta_0}.$$

This completes the proof.

12.1. **Proof of part (b) of Theorem V.** This proof depends on

LEMMA 15. *Let $\phi(u)$, $\phi_\eta(u)$, $0 \leqq \eta$, be the functions of Lemma* 7. *Suppose that* $1 < a$, *and that, for a fixed K,*

$$|\phi_a(x)| \leqq K x^a$$

for $x < 2\pi$. Then, for each positive odd integer $m < a$,

$$|\phi_m(\pi)| \leqq NK$$

where N is independent of K and ϕ.

We can suppose that $a = 2h+1$, where h is a positive integer. Then, for $\pi < x < 2\pi$, we have

$$\phi_a(x) = 2\sum_{p=0}^{h-1} \alpha_{2p+1}(x-\pi)^{2(h-p)}\phi_{2p+1}(\pi) + 2\alpha_a\phi_a(\pi) - \alpha_a\phi_a(2\pi - x),$$

where

$$\alpha_{2p+1} = (2h)!/[2(h-p)]!.$$

Taking $x = \pi + x_0^{1/2}, \pi + x_1^{1/2}, \cdots, \pi + x_{h-1}^{1/2}$ successively, where $\pi^2 < x_0 < x_1 < \cdots < x_{h-1} < 4\pi^2$, we arrive at the set of equations in the ϕ's

$$\sum_{p=0}^{h-1} \alpha_{2p+1} x_n^{h-p} \phi_{2p+1}(\pi) = \tau_n \qquad (n = 0, 1, \cdots, h-1),$$

where

$$|\tau_n| \leqq 2\alpha_a K(2\pi)^a.$$

As the α's are independent of ϕ and K, and as the determinant of the ϕ's is a non-zero multiple of the Vandermonde formed with the numbers $x_0, x_1, \cdots, x_{h-1}$, we see that our conclusion holds.

12.2. Consider now the proof of (b). As a consequence of Theorem IV there is an M_0 such that, for $x<2\pi$ and all y,

$$|f_{a,b}(x, y)| \leqq M_0 x^a y^b.$$

For a fixed y, let

$$\phi(u) = f_{0,b}(|u|, y).$$

Then ϕ satisfies the conditions of the lemma with $K=M_0y^b$. Hence,

$$|f_{m,b}(\pi, y)| = |\phi_m(u)| \leqq My^\beta$$

for each positive odd integer $m<a$. As the situation is symmetrical in a and b, the theorem follows.

13.1. **Proof of Theorem VI.** It is enough to prove the theorem when $s=0$. In fact, $f-s$ and its Fourier series satisfy the conditions of the theorem for $s=0$. Thus, the truth of the theorem in this case implies that $f-s$ is continuous $(C; a, b)$ with limit 0. But this implies that f is continuous $(C; a, b)$ with limit s.

We shall consider the case of Cesàro summability. The proof for Riesz summability follows the same lines but requires one less step. We shall assume that $0<\alpha$, $0<\beta$. The other cases can be treated in a similar fashion. The proof in all cases rests on the formula

$$(9.51) \qquad f_{a,b}(x, y) = \psi x^{a+\alpha+1} y^{b+\beta+1} \int_T H(xu)K(yv)\sigma_{\alpha,\beta}(u, v)dT.$$

We reduce this formula in the case at hand to one involving Cesàro means by an application of (3.33). We have

$$\sigma_{\alpha,\beta}(x, y) = \sum_{m<x} B(x-m) \sum_{n<y} C(y-n) S_{\alpha,\beta}(m, n),$$

where

$$\sum_{m<x} |B(x-m)| \leqq L, \qquad \sum_{n<y} |C(y-n)| \leqq L,$$

L being a suitably chosen constant independent of x, y.

Let $0<\epsilon$ be given. We select a positive integer m_0 so that

$$|S_{\alpha,\beta}(m, n)| \leqq \epsilon m^\alpha n^\beta$$

for $m_0 \leqq m$, $m_0 \leqq n$. We denote by M a number independent of x, y, m, n, for $x \leqq \delta$, $y \leqq \delta$, $m_0 \leqq m$, $m_0 \leqq n$.

We now write

$$\sigma_{\alpha,\beta}(u, v) = \Big\{-\sum_{m<m_0}\sum_{n<m_0} + \sum_{m<m_0}\sum_{n<v} + \sum_{m<u}\sum_{n<m_0} + \sum_{m_0\leqq m<u}\sum_{n_0\leqq n<v}\Big\}$$
$$\cdot B(u-m)C(v-n)S_{\alpha,\beta}(m, n)$$
$$= Q_1 + Q_2 + Q_3 + Q_4,$$

say, and

$$\int_T H(xu)K(yv)\sigma_{\alpha,\beta}dT = \Big\{\int_{(0,0)}^{(m_0,m_0)}\sigma_{\alpha,\beta} + \int_{(m_0,m_0)}^{(\infty,\infty)}Q_1\Big\}HKd(u, v)$$
$$+ \Big\{\int_{(0,m_0)}^{(m_0,\infty)}\sigma_{\alpha,\beta} + \int_{(m_0,m_0)}^{(\infty,\infty)}Q_2\Big\}HKd(u, v)$$
$$+ \Big\{\int_{(m_0,0)}^{(\infty,m_0)}\sigma_{\alpha,\beta} + \int_{(m_0,m_0)}^{(\infty,\infty)}Q_3\Big\}HKd(u, v)$$
$$+ \int_{(m_0,m_0)}^{(\infty,\infty)}HKQ_4d(u, v) = V_1 + V_2 + V_3 + V_4,$$

say. Then

$$|V_1| \leqq M + M\int_{m_0}^{\infty}|H(xu)|du\int_{m_0}^{\infty}|K(yv)|dv \leqq Mx^{-1}y^{-1},$$

$$|V_2| \leqq M\int_{m_0}^{\infty}|K|v^{\beta+1}dv + M\int_{m_0}^{\infty}|H|du\int_{m_0}^{\infty}|K|v^{\beta+1}dv \leqq Mx^{-1}y^{-\beta-2},$$

$$|V_3| \leqq Mx^{-\alpha-2}y^{-1},$$

$$|V_4| \leqq \epsilon L\int_{m_0}^{\infty}|H|u^{\alpha}du\int_{m_0}^{\infty}|K|v^{\beta}dv$$
$$\leqq \epsilon Lx^{-\alpha-1}y^{-\beta-1}\int_0^{\infty}|H(u)|u^{\alpha}du\int_0^{\infty}|K(v)|v^{\beta}dv.$$

In addition,

$$V_2 = \sum_{m<m_0}\int_m^{m_0}H(xu)(u-m)^{\alpha}du\cdot\int_{m_0}^{\infty}K(yv)\sum_{n<v}(v-n)^{\beta}a_{m,n}dv$$
$$+ \sum_{m<m_0}\int_{m_0}^{\infty}HB(u-m)du\cdot\int_{m_0}^{\infty}K\sum_{n<v}C(v-n)S_{\alpha,\beta}(m, n)dv$$
$$= \sum_{p<2m_0}g_p(x)G_p(y),$$

say, where, for $p=0, 1, \cdots, 2m_0-1$,

$$|g_p(x)| \leqq Mx^{-1}.$$

Consider V_2. We have

$$\begin{aligned} | x^{\alpha+1}y^{\beta+1}V_2 | &\leqq Mx^{-a}y^{-b} | f_{a,b} | + x^{\alpha+1}y^{\beta+1}(| V_1 | + | V_3 | + | V_4 |) \\ &\leqq M(1 + x^\alpha y^\beta + x^{-1}y^\beta + 1) \leqq F(x), \end{aligned}$$

where F depends upon x but not upon y. Hence, since

$$x^{\alpha+1}g_p(x) = O(x^\alpha) = o(1)$$

as $x \to +0$, for $p = 0, 1, \cdots, 2m_0 - 2$, we see, on replacing x by $1/x$ and y by $1/y$ in Lemma 4, that

$$(13.11) \qquad V_2 = o(x^{-\alpha-1}y^{-\beta-1}) \quad \text{as} \quad (x, y) \to (+0, +0).$$

In the same way we see that V_3 satisfies (13.11). On the other hand so also does V_1. Hence,

$$\begin{aligned} \limsup_{(x,y)\to(+0,+0)} | x^{-a}y^{-b}f_{a,b}(x, y) | &= \limsup_{(x,y)\to(+0,+0)} | x^{\alpha+1}y^{\beta+1}V_4 | \\ &\leqq \epsilon L \int_0^\infty | H(u) | u^\alpha du \int_0^\infty | K(v) | v^\beta dv. \end{aligned}$$

Since ϵ was arbitrary the theorem follows.

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ON IDEALS IN GENERALIZED QUATERNION ALGEBRAS AND HERMITIAN FORMS*

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1. Introduction. Let $\mathfrak{A}$ be a generalized quaternion algebra. The elements of $\mathfrak{A}$ may be written $X=x+Ey$, where x, y are numbers in a quadratic algebraic field F, $E^2=\alpha$, a rational integer, and $Ey=y'E$, y' being the conjugate of y with respect to F. The conjugate of X is $X'=x'-Ey$ and the norm of X is $N(X)=X'X=xx'-\alpha yy'$. It is well known that if X, Y are in $\mathfrak{A}$, $N(XY)=N(X)N(Y)$ and $(XY)'=Y'X'$. We shall assume that $\alpha\neq 0$.

Let $\mathfrak{G}$ be the ring consisting of all elements of $\mathfrak{A}$ in the form $x+Ey$, where x, y are in the set, G, of all integral algebraic numbers in F. We shall show that there is a one-to-one correspondence between certain classes of left ideals in $\mathfrak{G}$, which we call regular classes, and those classes of binary Hermitian forms in G, of determinant α, which represent positive integers. It will be shown that every ideal in a regular class contains two elements which form a basis with respect to G. The correspondence is then proved by a method which is similar to a method, due to Dickson,† of proving the well known correspondence between the classes of ideals in a quadratic algebraic field and certain classes of binary quadratic forms.

We also prove a theorem on the existence of a g.c.d. and the factorization of elements in $\mathfrak{G}$ under the assumption that all the ideals in a regular class are principal. Applications are made to a number of special quaternion algebras. Some of the results thus obtained have been previously proved by other methods, some are new. In particular, we obtain for an infinitude of algebras the same results on the existence of a g.c.d. and on factorization as were obtained by Dickson for the Lipschitz integral quaternions.

2. Ideals in $\mathfrak{G}$ and component ideals in G. An element in $\mathfrak{G}$ is said to be singular or non-singular according as its norm is or is not zero. An ideal $\mathfrak{L}$ in $\mathfrak{G}$ is defined as a set of elements in $\mathfrak{G}$, not all singular, such that if ξ_1, ξ_2 are in $\mathfrak{G}$ and η_1, η_2 are in $\mathfrak{L}$, then $\xi_1\eta_1+\xi_2\eta_2$ is in $\mathfrak{L}$.‡ If η is a non-singular element in $\mathfrak{L}$, $\eta'\eta=N(\eta)$ is in $\mathfrak{L}$. Hence $\mathfrak{L}$ contains elements in G, not zero. Those elements of $\mathfrak{L}$ which are in G form an ideal in G which we shall call the first

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† This was given in lectures at the University of Chicago in the spring of 1921.

‡ According to MacDuffee's definition, $\mathfrak{L}$ is a non-singular left ideal. See his *An introduction to the theory of ideals* etc., these Transactions, vol. 31 (1929), pp. 71–90. Since we shall not consider any other kind of ideal, we employ the briefer terminology.

component of $\mathfrak{L}$. If $X=x+Ey$ ranges over all the elements of $\mathfrak{L}$, y ranges over all the elements of an ideal in G which we shall call the second component of $\mathfrak{L}$. If an ideal $\mathfrak{p}$ in G has a basis ζ_1, ζ_2, we shall write $\mathfrak{p}=[\zeta_1, \zeta_2]$. A principal ideal in G defined by ρ will be written $\{\rho\}$. We shall now prove

LEMMA 1. *Let* $\mathfrak{a}=[\omega_1, \omega_2]$, $\mathfrak{b}=[\lambda_1, \lambda_2]$ *be the first and second components respectively of an ideal* $\mathfrak{L}$ *in* $\mathfrak{G}$. *Then* $\omega_1, \omega_2, \omega_3=b_1+E\lambda_1, \omega_4=b_2+E\lambda_2$ *form a basis of* $\mathfrak{L}$, *where* b_1, b_2 *are properly chosen numbers in* $\mathfrak{b}$.

By the definition of $\mathfrak{b}$, $\mathfrak{L}$ contains elements $\omega_3\equiv b_1+E\lambda_1$, $\omega_4\equiv b_2+E\lambda_2$, where the b's are in G. Then every element of $\mathfrak{L}$ may be written in the form $X=t+x_3\omega_3+x_4\omega_4$, where the x's are rational integers and t is in G. But $t=X-x_3\omega_3-x_4\omega_4$ is in $\mathfrak{L}$. Hence t is in $\mathfrak{a}$ and $t=x_1\omega_1+x_2\omega_2$, where the x's are rational integers. Since $E\omega_3=\alpha\lambda_1+Eb_1$, $E\omega_4=\alpha\lambda_2+Eb_2$, the b's belong to $\mathfrak{b}$. This proves the lemma.

We shall write $\mathfrak{L}=[\zeta_1, \zeta_2, \zeta_3, \zeta_4]$ if the ζ's form a basis of $\mathfrak{L}$. If ξ is a non-singular element of $\mathfrak{G}$, the product $\mathfrak{L}\xi$ is defined as the set of all elements $\eta\xi$, where η ranges over all the elements of $\mathfrak{L}$. Then $\mathfrak{L}\xi=[\zeta_1\xi, \zeta_2\xi, \zeta_3\xi, \zeta_4\xi]$. We shall now prove

LEMMA 2. *Let* $\mathfrak{a}$, $\mathfrak{b}$ *be the first and second components respectively of an ideal* $\mathfrak{L}$ *in* $\mathfrak{G}$ *and let* Δ *be the discriminant of G. Then* $\mathfrak{a}=a\mathfrak{b}\mathfrak{d}$, *where a is a positive rational integer and* $\mathfrak{d}$ *is an ideal, without a rational prime factor, which is either the unit ideal or a product of prime ideal divisors of* $\alpha\Delta$.

If u is in $\mathfrak{a}$, Eu is in $\mathfrak{L}$ and hence u is in $\mathfrak{b}$. Therefore $\mathfrak{b}$ contains $\mathfrak{a}$ and $\mathfrak{a}=a\mathfrak{b}\mathfrak{d}$ where a is a positive rational integer and $\mathfrak{d}$ contains no rational prime factor. It remains to show that every prime ideal divisor of $\mathfrak{d}$ divides $\alpha\Delta$.

$\mathfrak{b}$ is narrowly equivalent to an ideal $\mathfrak{b}_1$ which is prime to $\mathfrak{d}\mathfrak{d}'$, where $\mathfrak{d}'$ is the conjugate of $\mathfrak{d}$.* Then $\mathfrak{b}t=\mathfrak{b}_1t_1$ where t, t_1 are in G and $N(t)N(t_1)>0$. By Lemma 1, $\mathfrak{L}=[a\omega_1, a\omega_2, b_1+E\lambda_1, b_2+E\lambda_2]$, where $\mathfrak{b}\mathfrak{d}=[\omega_1, \omega_2]$, $\mathfrak{b}=[\lambda_1, \lambda_2]$, and the b's are in $\mathfrak{b}$. It may then be shown that $\mathfrak{L}t=\mathfrak{L}_1t_1$, where the first and second components of $\mathfrak{L}_1$ are $a\mathfrak{b}_1\mathfrak{d}$ and $\mathfrak{b}_1$ respectively. Therefore we may assume, without loss of generality, that $\mathfrak{b}$ is prime to $\mathfrak{d}\mathfrak{d}'$.

The rational integers $(b_i'-E\lambda_i)(b_i+E\lambda_i)=b_ib_i'-\alpha\lambda_i\lambda_i'$ $(i=1, 2)$ are in $\mathfrak{L}$ and therefore

$$b_ib_i' - \alpha\lambda_i\lambda_i' \equiv 0 \pmod{\mathfrak{a} = a\mathfrak{b}\mathfrak{d}} \qquad (i = 1, 2). \tag{1}$$

Let $\mathfrak{d}=[\mu_1, \mu_2]$. Then each $a\lambda_i\mu_j (i, j=1, 2)$ belongs to $\mathfrak{a}$ and hence each of

$$a\mu_j'(b_i + E\lambda_i) - Ea\lambda_i\mu_j = a\mu_j'b_i \qquad (i, j = 1, 2)$$

is in $\mathfrak{L}$. Therefore

* Bachmann, *Allgemeine Arithmetik der Zahlenkörper*, p. 373.

$$b_i \mathfrak{d}' \equiv 0 \pmod{\mathfrak{d}} \qquad (i = 1, 2). \tag{2}$$

Let $\mathfrak{d}_1$ be a prime ideal divisor of $\mathfrak{d}$ which is prime to Δ. Since $\mathfrak{b}$ is prime to $\mathfrak{b}\mathfrak{b}'$, we may assume that the λ_i are prime to $\mathfrak{d}_1\mathfrak{d}_1'$. Since $\mathfrak{d}'$ contains no rational prime factor, and $\mathfrak{d}_1$ is prime to Δ, $\mathfrak{d}_1$ is prime to $\mathfrak{d}'$. Then by (2) each $b_i \equiv 0 \pmod{\mathfrak{d}_1}$ and by (1), each $\alpha\lambda_i\lambda_i' \equiv 0 \pmod{\mathfrak{d}_1}$. But the λ_i are prime to $\mathfrak{d}_1\mathfrak{d}_1'$ and hence the same is true of the λ_i'. Therefore $\alpha \equiv 0 \pmod{\mathfrak{d}_1}$ and the lemma is proved.

3. **Classes of ideals in $\mathfrak{G}$; reduced ideals.** Two ideals $\mathfrak{L}$ and $\mathfrak{L}_1$ will be said to be equivalent if there are elements ξ, ξ_1 in $\mathfrak{G}$ such that $\mathfrak{L}\xi = \mathfrak{L}_1\xi_1$ and $N(\xi)N(\xi_1) > 0$. After multiplying both sides of the last equation on the right by ξ', we may assume that ξ is a rational integer and $N(\xi_1) > 0$. It may then be shown that equivalence is transitive. All the ideals equivalent to a given ideal are said to form a class. An ideal in $\mathfrak{G}$ will be called a reduced ideal if its second component is the unit ideal.

LEMMA 3. *Let $\mathfrak{L}$ be an ideal in $\mathfrak{G}$ whose first component is $a\mathfrak{b}\mathfrak{d}$ as in Lemma 2. Then $\mathfrak{L}$ is equivalent to a reduced ideal whose first component is $a_1\mathfrak{d}$, where a_1 is a rational integer.*

Since equivalence is a transitive property, by our proof of Lemma 2, we may assume that the second component $\mathfrak{b}$ of $\mathfrak{L}$ contains no rational prime factor and is prime to $a\alpha\Delta$. By Lemma 1, $\mathfrak{L} = [a\omega_1, a\omega_2, b_1+E\lambda_1, b_2+E\lambda_2]$, where $\mathfrak{b}\mathfrak{d} = [\omega_1, \omega_2]$, $\mathfrak{b} = [\lambda_1, \lambda_2]$ and the b's are in $\mathfrak{b}$. Since $\mathfrak{b}$ contains no rational prime factor, we may assume that $\lambda_1 = N(\mathfrak{b}) \equiv B$, where $N(\mathfrak{b})$ is the norm of $\mathfrak{b}$. Then B is prime to $a\alpha\Delta$ and there is a number k in G such that

$$Bk + b_1' \equiv 1 \pmod{a\alpha\Delta}. \tag{3}$$

We shall assume without loss of generality that k is prime to B and that $N(\rho \equiv k+E) = kk' - \alpha > 0$. Then $\mathfrak{L}$ is equivalent to $\mathfrak{L}_1 \equiv \mathfrak{L}\rho$. $\mathfrak{L}_1$ contains

$$\begin{aligned} a\omega_1\rho &= a\omega_1 k + E(a\omega_1'), \\ a\omega_2\rho &= a\omega_2 k + E(a\omega_2'), \\ (b_1 + EB)\rho &= b_1 k + \alpha B + E(Bk + b_1'), \\ (b_2 + E\lambda_2)\rho &= b_2 k + \alpha\lambda_2' + E(\lambda_2 k + b_2'). \end{aligned} \tag{4}$$

Suppose the second component of $\mathfrak{L}_1$ has a prime ideal divisor $\mathfrak{p}$. Since $\mathfrak{b}'\mathfrak{d}' = [\omega_1', \omega_2']$, by (4_1) and (4_2), $\mathfrak{p}$ divides $a\mathfrak{b}'\mathfrak{d}'$. By (4_3) $\mathfrak{p}$ divides $Bk+b_1'$. If $\mathfrak{p}$ divided $a\mathfrak{d}'$, it would divide $a\alpha\Delta$ and then by (3) it would divide 1. Hence $\mathfrak{p}$ is prime to $a\mathfrak{d}'$ and divides $\mathfrak{b}'$. By Lemma 1, $\mathfrak{b}'$ divides b_1', b_2'. Then by (4_3) and (4_4), $\mathfrak{p}$ divides $\mathfrak{b}k = [Bk, \lambda_2 k]$. But k is prime to $\{B\} = \mathfrak{b}\mathfrak{b}'$ and hence k is prime to $\mathfrak{p}$. Therefore $\mathfrak{p}$ divides $\mathfrak{b}$. But we have seen that $\mathfrak{p}$ divides $\mathfrak{b}'$, which is prime to Δ. Hence $\mathfrak{b}$ is divisible by $\mathfrak{p}\mathfrak{p}'$, contrary to our

hypothesis that $\mathfrak{b}$ has no rational prime factor. Therefore the second component of $\mathfrak{L}_1$ has no prime ideal divisor and $\mathfrak{L}_1$ is a reduced ideal.

Consider the first component $\mathfrak{a}_1$, of $\mathfrak{L}_1$. Every element of $\mathfrak{L}_1$ may be written in the form $(u+Ev)\rho = ku+\alpha v'+E(u'+kv)$, where $u+Ev$ is in $\mathfrak{L}$. Hence if $X=u+Ev$ is in $\mathfrak{L}$, $X\rho$ is in $\mathfrak{a}_1$ if and only if $u'=-kv$. Then $X=-v'(k'-E)$ $=-v'\rho'$ and the corresponding element in $\mathfrak{a}_1$ is $-v'\rho'\rho=-v'N(\rho)$. Let $\mathfrak{q}$ be the set of all elements v of G such that $-k'v'+Ev=-v'\rho'$ is in $\mathfrak{L}$. $\mathfrak{q}$ is an ideal in G and $\mathfrak{a}_1=\mathfrak{q}'N(\rho)$. Let $\mathfrak{b}=[\zeta_1, \zeta_2]$. Then $a\zeta_i(Bk'+b_1)$ is in $a\mathfrak{b}\mathfrak{b}$ and therefore $-aB\zeta_i\rho'=a\zeta_i(b_1+EB)-a\zeta_i(Bk'+b_1)$ is in $\mathfrak{L}$ $(i=1, 2)$. It follows from the definition of $\mathfrak{q}$ that each $aB\zeta_i'$ is in $\mathfrak{q}$. Hence $\mathfrak{q}$ divides $aB\mathfrak{b}'$ and $\mathfrak{a}_1$ $=\mathfrak{q}'N(\rho)$ divides $aBN(\rho)\mathfrak{b}$.

By Lemma 1, the norm, $n(\mathfrak{R})$, of an ideal $\mathfrak{R}$, according to MacDuffee's definition, is the product of the norms of its components.* Then

$$n(\mathfrak{L})=N(a\mathfrak{b}\mathfrak{b})N(\mathfrak{b})=a^2B^2N(\mathfrak{b}).$$

It will be found that the determinant of the second matrix of an element ξ in $\mathfrak{G}$ is $N^2(\xi)$. Then $n(\mathfrak{L}_1)=n(\mathfrak{L}\rho)=n(\mathfrak{L})N^2(\rho)=a^2B^2N^2(\rho)N(\mathfrak{b})$.† The second component of $\mathfrak{L}_1$ is the unit ideal and therefore $n(\mathfrak{L}_1)=N(\mathfrak{a}_1)$. But we have seen that $\mathfrak{a}_1$ divides $aBN(\rho)\mathfrak{b}$. It follows that $\mathfrak{a}_1=aBN(\rho)\mathfrak{b}$ and the lemma is proved.

4. **A basis of an ideal in $\mathfrak{G}$ with respect to G.** An ideal in $\mathfrak{G}$ may contain two elements $\omega_i=g_{i1}+g_{i2}E (i=1, 2)$, where the g's are in G, such that an element of $\mathfrak{G}$ is in $\mathfrak{L}$ if and only if it may be written $x\omega_1+y\omega_2$, where x, y are in G. Such a pair of elements will be called a basis of $\mathfrak{L}$ with respect to G and we shall write $\mathfrak{L}=[\omega_1, \omega_2]$. Let $1, \theta$ be a basis of G. Then $\mathfrak{L}=[\omega_1, \theta\omega_1, \omega_2, \theta\omega_2]$. Since $\mathfrak{L}$ contains a non-singular element, these four basal elements are linearly independent with respect to the rational field.‡ Hence ω_1, ω_2 are left linearly independent with respect to F. It will be understood hereafter when two elements are referred to as a basis of an ideal in $\mathfrak{G}$ that they form a basis with respect to G.

If the determinant $|g_{ij}|$ is a positive rational integer, the ω's will be said to form a proper basis of $\mathfrak{L}$. We then define the norm of $\mathfrak{L}$ as $N(\mathfrak{L})=|g_{ij}|$. If the ω's form a proper basis of $\mathfrak{L}$ and $\zeta_i=t_{i1}\omega_1+t_{i2}\omega_2$ are elements of $\mathfrak{L}$, it may be shown that they form a proper basis if and only if the determinant $|t_{ij}|=1$. It may also be shown that $N(\mathfrak{L})$ is independent of the particular proper basis employed. If $\xi=u+vE$ is in $\mathfrak{G}$ and $\omega_i\xi=h_{i1}+h_{i2}E$ $(i=1, 2)$, we find

* Loc. cit., p. 74.

† MacDuffee, loc. cit., p. 78, line 23.

‡ MacDuffee, loc. cit., Theorem 3, p. 74.

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} u & v \\ \alpha v' & u' \end{pmatrix}.$$

Taking determinants, we have $|h_{ij}| = N(\mathfrak{L})N(\xi)$. Since $\mathfrak{L}\xi = [\omega_1\xi, \omega_2\xi]$, it follows that if $N(\xi) > 0$, the $\omega_i\xi$ form a proper basis of $\mathfrak{L}\xi$ and $N(\mathfrak{L}\xi) = N(\mathfrak{L})N(\xi)$.

LEMMA 4. *If an ideal has a proper basis, every ideal in the same class has a proper basis.*

Let $\mathfrak{L} = [\omega_1, \omega_2]$, the indicated basis being proper, and let $\mathfrak{L}_1$ be an ideal in the same class. Then $\mathfrak{L}\xi = \mathfrak{L}_1\xi_1$ where $N(\xi)N(\xi_1) > 0$. $\mathfrak{L}_1$ contains elements ζ_1, ζ_2, such that $\omega_i\xi = \zeta_i\xi_1 (i = 1, 2)$ and $\mathfrak{L}_1 = [\zeta_1, \zeta_2]$. To show that the ζ's form a proper basis, let $\xi = u + vE$, $\xi_1 = u_1 + v_1E$, $\zeta_i = h_{i1} + h_{i2}E$ $(i = 1, 2)$. Then from $\omega_i\xi = \zeta_i\xi_1$ we have

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ \alpha v_1' & u_1' \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} u & v \\ \alpha v' & u' \end{pmatrix}.$$

Hence $|h_{ij}| N(\xi_1) = N(\mathfrak{L})N(\xi)$. But $N(\mathfrak{L})$ and $N(\xi)N(\xi_1)$ are positive integers and $|h_{ij}|$ is an integral algebraic number. Hence $|h_{ij}|$ is a positive rational integer and the ζ's form a proper basis of $\mathfrak{L}_1$. This proves the lemma.

An ideal $\mathfrak{L}$ in $\mathfrak{G}$ will be called a regular ideal if the corresponding ideal $\mathfrak{b}$ of Lemma 2 is the unit ideal. We shall now prove

THEOREM 1. *An ideal in $\mathfrak{G}$ has a proper basis if and only if it is a regular ideal.*

Suppose $\mathfrak{L}$ is a regular ideal. By Lemma 3, $\mathfrak{L}$ is equivalent to a reduced ideal $\mathfrak{L}_1$ whose first component is the principal ideal defined by a positive rational integer a. Then by Lemma 1, $\mathfrak{L}_1 = [a, a\theta, b_1 + E, b_2 + E\theta]$ where the b's are in G. Since $\theta'(b_1 + E) - (b_2 + E\theta) = \theta' b_1 - b_2$ is in $\mathfrak{L}_1$, $b_2 \equiv \theta' b_1 \pmod{a}$. Hence we may assume that $b_2 = \theta' b_1$. Since $1, \theta'$ also form a basis of G, it follows that $\mathfrak{L}_1 = [a, b_1 + E]$. The indicated basis of $\mathfrak{L}_1$ is proper and therefore by Lemma 4, $\mathfrak{L}$ has a proper basis.

Suppose $\mathfrak{L}$ has a proper basis and let $a\mathfrak{b}\mathfrak{b}$ and $\mathfrak{b}$ be the first and second components respectively of $\mathfrak{L}$, as in Lemma 2. By Lemmas 3 and 1, $\mathfrak{L}$ is equivalent to an ideal $\mathfrak{L}_1 = [a_1\omega_1, a_1\omega_2, b_1 + E, b_2 + E\theta]$ where a_1 is a positive rational integer, $\mathfrak{b} = [\omega_1, \omega_2]$, and the b's are in G. Since $\mathfrak{L}$ has a proper basis, by Lemma 4, $\mathfrak{L}_1$ has a proper basis $\mu_i = g_{i1} + g_{i2}\, E (i = 1, 2)$ and $N(\mathfrak{L}_1) = |g_{ij}|$. $\mathfrak{L}_1$ contains $b_1 + E$ and therefore for properly chosen numbers t_1, t_2 in G, $t_1\mu_1 + t_2\mu_2 = b_1 + E$. Then $t_1 g_{12} + t_2 g_{22} = 1$ and

$$\zeta_1 = g_{22}\mu_1 - g_{12}\mu_2 = N(\mathfrak{L}_1),$$

$$\zeta_2 = t_1\mu_1 + t_2\mu_2$$

form a proper basis of $\mathfrak{L}_1$. Since ζ_1 is a rational integer, ζ_2 is not in G. Therefore the first component of $\mathfrak{L}_1$ is the principal ideal defined by ζ_1. But the first component of $\mathfrak{L}_1$ is $a_1\mathfrak{b}$ and $\mathfrak{b}$ contains no rational prime factor. Hence $\mathfrak{b}=\{1\}$ and $\mathfrak{L}$ is a regular ideal. This proves the theorem.

A class of ideals which contains a regular ideal will be called a regular class. By Lemma 4 and Theorem 1, every ideal in a regular class is regular.*

5. The class of forms corresponding to a regular ideal. If a, c are rational integers, b is in G, x and y range over all the numbers of G, and b', x', y' are the conjugates of b, x, y respectively, then

$$f(x, y) = axx' + bx'y + b'xy' + cyy' \tag{5}$$

will be said to be an Hermitian form in G of determinant $bb'-ac$. If $f_1(x_1, y_1)$ is obtained from f by a linear homogeneous transformation on x, y of determinant unity, with coefficients in G, f and f_1 will be said to be equivalent. f_1 is an Hermitian form of determinant $bb'-ac$. All the forms equivalent to a given form will be said to form a class.

Let $\mathfrak{L}$ be a regular ideal. By Theorem 1, it has a proper basis $\omega_i=g_{i1}+g_{i2}E(i=1, 2)$ and $N(\mathfrak{L})=|g_{ij}|$. Since each $E\omega_i$ belongs to $\mathfrak{L}$, we have

$$E\omega_i = b_{i1}\omega_1 + b_{i2}\omega_2 \qquad (i = 1, 2), \tag{6}$$

where the b's are in G. The general element of $\mathfrak{L}$ is X as written below, where x, y range over all the numbers of G:

$$X = x\omega_1 + y\omega_2 = (g_{11}x + g_{21}y) + (g_{12}x + g_{22}y)E,$$

$$EX = l_1\omega_1 + l_2\omega_2 = (g_{11}l_1 + g_{21}l_2) + (g_{12}l_1 + g_{22}l_2)E,$$

where $l_i=b_{1i}x'+b_{2i}y'$ $(i=1, 2)$. Then

$$N(X) = \begin{vmatrix} g_{11}x + g_{21}y & g_{12}x + g_{22}y \\ g_{11}l_1 + g_{21}l_2 & g_{12}l_1 + g_{22}l_2 \end{vmatrix} = \begin{vmatrix} x & y \\ l_1 & l_2 \end{vmatrix} \cdot \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = N(\mathfrak{L})f(x, y)$$

where

$$f(x, y) = \begin{vmatrix} x & y \\ l_1 & l_2 \end{vmatrix} = b_{12}xx' - b_{11}x'y + b_{22}xy' - b_{21}yy'. \tag{7}$$

Since $f(x, y)$ is rational and is in G for every x, y in G, it is a rational integer for every such x, y. It may then be shown that b_{12}, b_{21} are rational integers and $b_{11}=-b'_{22}$. Hence f is an Hermitian form in G. We shall see later that the determinant of f is α. f will be said to correspond to the proper basis, ω_1, ω_2, of $\mathfrak{L}$.

* It may be shown that for a regular ideal $\mathfrak{L}$, $n(\mathfrak{L})=N^2(\mathfrak{L})$.

We have seen that $\zeta_i = t_{i1}\omega_1 + t_{i2}\omega_2 (i=1, 2)$ form a proper basis if and only if the t's are in G and $|t_{ij}| = 1$. The form corresponding to such a basis is $f_1(x_1, y_1) = N(x_1\zeta_1 + y_1\zeta_2)/N(\mathfrak{L})$. Hence f is transformed into f_1 by the transformation

$$(8) \qquad x = t_{11}x_1 + t_{21}y_1, \qquad y = t_{12}x_1 + t_{22}y_1,$$

and f is equivalent to f_1. Conversely if f is transformed into f_1 by (8), the t's being in G and $|t_{ij}| = 1$, then f_1 is the form corresponding to the proper basis $\zeta_i = t_{i1}\omega_1 + t_{i2}\omega_2$ $(i=1, 2)$. Hence there is a one-to-one correspondence between the proper bases of $\mathfrak{L}$ and the forms in the class C, containing f. We shall say that C corresponds to $\mathfrak{L}$.

THEOREM 2. *If C and C_1 are the classes of Hermitian forms in G which correspond to the regular ideals $\mathfrak{L}$ and $\mathfrak{L}_1$ respectively, then $C = C_1$ if and only if $\mathfrak{L}$ and $\mathfrak{L}_1$ are equivalent.*

Let $f(x, y)$ of (5) be a form in C. We may assume, without loss of generality, that $a \neq 0$. Suppose $C = C_1$. Then f corresponds to a proper basis ω_1, ω_2 of $\mathfrak{L}$ and to a proper basis ζ_1, ζ_2 of $\mathfrak{L}_1$. From (5), (6), and (7) we have

$$E\omega_1 = -b\omega_1 + a\omega_2, \qquad E\zeta_1 = -b\zeta_1 + a\zeta_2,$$
$$E\omega_2 = -c\omega_1 + b'\omega_2, \qquad E\zeta_2 = -c\zeta_1 + b'\zeta_2,$$

and $(b+E)\omega_1 = a\omega_2$, $(b+E)\zeta_1 = a\zeta_2$. From $N(x\omega_1 + y\omega_2) = N(\mathfrak{L})f(x, y)$, it follows that $N(\omega_1) = aN(\mathfrak{L}) \neq 0$. Similarly, $N(\zeta_1) = aN(\mathfrak{L}_1)$. Then $N(\omega_1)N(\zeta_1) > 0$. We have

$$\mathfrak{L}a\omega_1' = [a\omega_1, a\omega_2]\omega_1' = [a\omega_1, (b+E)\omega_1]\omega_1' = [a, b+E]N(\omega_1).$$

Similarly, $\mathfrak{L}_1 a\zeta_1' = [a, b+E]N(\zeta_1)$. Since $a \neq 0$, $\mathfrak{L}\omega_1' N(\zeta_1) = \mathfrak{L}_1\zeta_1' N(\omega_1)$ and $\mathfrak{L}_1$ and $\mathfrak{L}$ are equivalent.

Conversely, suppose $\mathfrak{L}$ and $\mathfrak{L}_1$ are equivalent. Let ω_1, ω_2 form a proper basis of $\mathfrak{L}$. As in the proof of Lemma 4, $\omega_i\xi = \zeta_i\xi_1$ $(i=1, 2)$, where $N(\xi)N(\xi_1) > 0$ and the ζ's form a proper basis of $\mathfrak{L}_1$. Let f of (7) be the form in C corresponding to the above basis of $\mathfrak{L}$. The coefficients of f are defined by (6). But from the last equations and $N(\xi_1) \neq 0$, it follows that each ω_i in (6) may be replaced by the corresponding ζ_i. Hence f is also the form in C_1 corresponding to the above basis of $\mathfrak{L}_1$. The theorem follows.

6. The correspondence between regular classes of ideals and classes of forms. We shall prove

THEOREM 3. *There is a one-to-one correspondence between the regular classes of ideals in $\mathfrak{G}$ and the classes of Hermitian forms in G, of determinant α, which represent positive integers.*

By Theorem 2, for every regular class of ideals there is a uniquely determined class of Hermitian forms in G. Also no class corresponds to two classes of ideals. To prove the above theorem, it is therefore sufficient to show that (a) if C is a class of forms corresponding to a class of ideals, then C contains a form which represents a positive integer and is of determinant α, and (b) every class of Hermitian forms in G of determinant α, which represent a positive integer, corresponds to a regular class of ideals in $\mathfrak{G}$.

By Lemmas 3 and 4 and Theorem 1, every regular class of ideals contains an ideal $\mathfrak{L}=[a, b+E]$, where a is a positive integer and b is in G. The indicated basis of $\mathfrak{L}$ is proper, $N(\mathfrak{L})=a$, and the form corresponding to this basis is $N[ax+y(b+E)]/a=f(x, y)$ where f is given by (5) and $c=(bb'-\alpha)/a$. Then f represents the positive integer a, the determinant of f is $bb'-ac=\alpha$ and the class containing f corresponds to the regular class containing $\mathfrak{L}$. This proves (a).

Let C be a class of Hermitian forms in G of determinant α, which represent a positive integer, and let f of (5) be a form in C. We may assume, without loss of generality, that $a\neq 0$. Since $bb'-ac=\alpha$, it is readily shown that there is an ideal $\mathfrak{L}=[a, b+E]$. If $X=ax+y(b+E)$ is the general element in $\mathfrak{L}$, $N(X)=af(x, y)$. If $a>0$, the above basis of $\mathfrak{L}$ is proper, $N(\mathfrak{L})=a$, and C corresponds to the class of ideals containing $\mathfrak{L}$. Suppose $a<0$. From $af(x, y)=N(X)$ and our hypothesis that f represents a positive integer, it follows that $\mathfrak{G}$ contains an element ξ, of negative norm. Then $\mathfrak{L}\xi=[a\xi, (b+E)\xi]$, the indicated basis of $\mathfrak{L}\xi$ is proper, $N(\mathfrak{L}\xi)=aN(\xi)$, $N[xa\xi+y(b+E)\xi]=aN(\xi)f(x, y)$, and C corresponds to the class of ideals containing $\mathfrak{L}\xi$. This completes the proof of the theorem.

7. **Principal ideals.** If $\eta_1, \eta_2, \cdots, \eta_r$ are elements in $\mathfrak{G}$ not all singular, the set of all elements $\sum\xi_i\eta_i$, where the ξ's are in $\mathfrak{G}$, form an ideal which will be written $\mathfrak{L}=\{\eta_1, \eta_2, \cdots, \eta_r\}$. If $r=1$, $\mathfrak{L}$ will be called a principal ideal. It will be observed that a principal ideal $\{\eta\}$ has a proper basis $\pm\eta$, $E\eta$ and hence by Theorem 1 it is a regular ideal. It may be shown that if $\mathfrak{L}$ is a principal ideal and $\mathfrak{L}\xi=\mathfrak{L}_1\xi_1$ where $N(\xi)N(\xi_1)\neq 0$, then $\mathfrak{L}_1$ is a principal ideal.

If $\lambda=\lambda_1\delta\neq 0$, where $\lambda, \lambda_1, \delta$ are in $\mathfrak{G}$, δ is said to be a right divisor of λ. If δ is also a right divisor of an element μ in $\mathfrak{G}$ and if every common right divisor of λ, μ is a right divisor of δ, then δ is said to be a greatest common right divisor, or g.c.r.d., of λ, μ. An element of $\mathfrak{G}$ of norm ± 1 is said to be a unit. Let α_1 be the product of the rational prime divisors of α which are divisible by prime ideals of the first degree in G or let $\alpha_1=1$ if α has no such divisors. Then every prime ideal divisor of $\mathfrak{d}$ of Lemma 2 is a divisor of $\alpha_1\Delta$. We shall now prove

THEOREM 4. *Let every regular ideal in* $\mathfrak{G}$ *be principal. Let* λ, μ *be elements in* $\mathfrak{G}$ *and assume that* $N(\lambda)\neq 0$. *If* $\mathfrak{G}$ *contains a non-regular ideal, assume that* $N(\lambda)$ *is prime to* $\alpha_1\Delta$. *Then* λ, μ *have a* g.c.r.d., δ, *which is uniquely determined apart from a unit left factor, and* $\delta=\xi\lambda+\eta\mu$, *where* ξ, η *are in* $\mathfrak{G}$. *If* λ *has no rational prime factor and* $N(\lambda)=\pm p_1\cdot p_2\cdot\cdots\cdot p_r$, *where the p's are rational primes arranged in an arbitrary but fixed order, then* $\lambda=\pi_1\cdot\pi_2\cdot\cdots\cdot\pi_r$ *where* $N(\pi_i)=\pm p_i$ $(i=1, 2, \cdots, r)$ *and each* π_i *is uniquely determined apart from a unit left factor.*

Every rational integer in an ideal is divisible by the first component of the ideal. Therefore by Lemma 2 and the definition of α_1, an ideal is regular if it contains a rational integer prime to $\alpha_1\Delta$. Consider the ideal in $\mathfrak{G}$, $\mathfrak{L}=\{\lambda, \mu\}$. If $\mathfrak{G}$ contains a non-regular ideal, by hypothesis $\mathfrak{L}$ contains a rational integer, $\lambda'\lambda=N(\lambda)$, which is prime to $\alpha_1\Delta$. Hence in every case $\mathfrak{L}$ is a principal ideal $\{\lambda, \mu\}=\{\delta\}$, where δ is in $\mathfrak{G}$. Then $\lambda=\lambda_1\delta$, $\mu=\mu_1\delta$, $\delta=\xi\lambda+\eta\mu$, where λ_1, μ_1, ξ, η are in $\mathfrak{G}$. If ζ is a common right divisor of λ and μ, by the last equation it is a right divisor of δ, and $\delta=\epsilon_1\zeta$ where ϵ_1 is in $\mathfrak{G}$. Then δ is a g.c.r.d. of λ and μ. Suppose ζ is also a g.c.r.d. of λ and μ. Then $\zeta=\epsilon_2\delta$ where ϵ_2 is in $\mathfrak{G}$. λ is non-singular and therefore δ is non-singular. It follows that $\epsilon_1\epsilon_2=1$ and the ϵ's are units in $\mathfrak{G}$. This proves the first part of the theorem.

To prove the second part, consider the ideal $\mathfrak{L}=\{p_r, \lambda\}$. As before $\mathfrak{L}=\{\pi_r\}$, $\lambda=\lambda_1\pi_r$, $p_r=\nu_r\pi_r$ where λ_1, π_r, ν_r are in $\mathfrak{G}$. Dropping the subscripts r, we have $p^2=N(\nu)N(\pi)$. Suppose $N(\pi)=\pm 1$. Then $\mathfrak{L}$ is the unit ideal, and for properly chosen ξ, η in $\mathfrak{G}$, $\xi\lambda=1+\eta p$. Taking norms, we have $N(\xi)N(\lambda)\equiv 1 \pmod{p}$, whereas $N(\lambda)\equiv 0 \pmod{p}$. Suppose $N(\pi)=p^2$. Then $N(\nu)=\pm 1$, $\lambda=\lambda_1\pi=p\lambda_1\nu^{-1}$ and ν^{-1} is in $\mathfrak{G}$. Then p is a divisor of λ, contrary to hypothesis. Hence $N(\pi)=\pm p$. Employing the ideal $\{p_{r-1}, \lambda_1\}$, we find similarly $\lambda_1=\lambda_2\pi_{r-1}$ where λ_2 is in $\mathfrak{G}$ and $N(\pi_{r-1})=\pm p_{r-1}$. Continuing this process, we find $\lambda=\pi_1\cdot\pi_2\cdot\cdots\cdot\pi_r$ where $N(\pi_i)=\pm p_i$ $(i=1, 2, \cdots, r)$. By the first part of the theorem, these π's are uniquely determined, apart from unit left factors. This completes the proof of the theorem.

8. **Applications.** In this paragraph, we shall employ the foregoing results to determine a number of special quaternion algebras for which the conclusions of Theorem 4 are valid.

LEMMA 5. *If for every rational integer* $a>1$ *and for every number b in G such that* $N(b)-\alpha\equiv 0 \pmod{a}$, *there is a number* b_0 *in G such that* $b_0\equiv b \pmod{a}$ *and* $0<|N(b_0)-\alpha|<a^2$, *then every regular ideal in* $\mathfrak{G}$ *is principal.*

By Lemma 3, every regular ideal $\mathfrak{L}$ is equivalent to an ideal $\mathfrak{L}_1=[a, b+E]$, where a is a positive rational integer and b is in G. If $a=1$, $\mathfrak{L}_1=\{1\}$ and $\mathfrak{L}$ is principal. Suppose $a>1$. $\mathfrak{L}_1$ contains $(b'-E)(b+E)=bb'-\alpha\equiv 0 \pmod{a}$.

Then by hypothesis, $\mathfrak{L}_1=[a, b_0+E]$, $b_0b_0'-\alpha=ac$, $0<|c|<a$ and $\mathfrak{L}_1(b_0'-E)=[c, -b_0'+E]a$. If $|c|=1$, it follows as before that $\mathfrak{L}$ is principal. If $|c|>1$, repetition of the process leads to the case $a=1$. Hence $\mathfrak{L}$ is principal and the lemma is proved.

Let F be the field defined by $\tau^{1/2}$. It may be shown for each of the following cases that the hypothesis of Lemma 5 is valid. Hence the conclusions of Theorem 4 are valid for these cases.*

$$(9)\qquad \begin{aligned}(\tau, \alpha) = (-1, -1), (-1, 3), (-3, \mp 2), (-3, 5), (5, \pm 2), (5, \pm 3),\\ (5, \pm 7), (5, \pm 13), (-7, -1), (13, \pm 2), (13, \pm 5), (-3, -1).\end{aligned}$$

Consider the question of the existence of non-regular ideals in $\mathfrak{G}$. By Lemma 3, every non-regular ideal is equivalent to a reduced ideal $\mathfrak{L}$ whose first component is $a\mathfrak{d}$, where a is a positive rational integer, $\mathfrak{d}\neq\{1\}$, and every rational prime divisor of $N(\mathfrak{d})$ is a divisor of $\alpha_1\Delta$. Let $\mathfrak{d}=[\omega_1, \omega_2]$. Then $\mathfrak{L}=[a\omega_1, a\omega_2, b_1+E, b_2+E\theta]$ where the b's are in G. By (1),

$$(10)\qquad N(b_1)-\alpha\equiv 0 \pmod{\mathfrak{d}}.$$

Suppose now $\alpha_1=1$ and $\Delta\equiv 1 \pmod 4$. Then every rational prime divisor, p, of $N(\mathfrak{d})$ is a divisor of Δ, and by (10), $N(2b_1)\equiv u^2\equiv 4\alpha \pmod p$, where u is a rational integer. We have then

LEMMA 6. *If* $\Delta\equiv 1 \pmod 4$, $\alpha_1=1$ *and if* α *is a quadratic non-residue of every prime factor of* Δ, *then every ideal in* $\mathfrak{G}$ *is regular.*

It will be observed that, by this lemma, the conclusions of Theorem 4 are valid for each of the cases (9), except the first three, with no restrictions on $N(\lambda)$ except that $N(\lambda)\neq 0$.

Consider the case where $\alpha\equiv\tau\equiv 3 \pmod 4$, $\alpha>0$, $\tau<0$ and $\alpha\tau$ contains no square factor. It may be shown that if f of (5) is an Hermitian form in G of determinant α, then a and c are not both even and a, b, c have no rational prime factor in common. Hence f is a properly primitive form. By a result due to Humbert,† there is only one class of such forms. Hence by Theorem 3, every regular ideal in $\mathfrak{G}$ is principal and Theorem 4 is applicable. It will be noted that $\Delta=4\tau$.

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* For the case $(-1, -1)$, see Dickson, *Arithmetic of quaternions*, Proceedings of the London Mathematical Society, (2), vol. 20 (1922), pp. 225–232, Theorems 3, 8. For the cases $(-3, -1)$ and $(-7, -1)$, see Dickson, *Algebren und ihre Zahlentheorie*, pp. 163, 167, 193, 195. Several of the remaining cases above were treated by Griffiths, *Generalized quaternion algebras and the theory of numbers*, American Journal of Mathematics, vol. 50 (1928), pp. 303–314; in particular, see pp. 309–310.

† Humbert, *Sur le nombre des classes de formes à indéterminées conjuguées, indéfinies, de determinant donné*, Comptes Rendus, Paris, vol. 166 (1918), pp. 865–870; Dickson, *History of the Theory of Numbers*, vol. 3, p. 275.

Suppose, in addition to the above conditions on α and τ, that for every prime factor p of α and every prime factor q of τ, the Legendre symbols

$$\left(\frac{\tau}{p}\right)=\left(\frac{\alpha}{q}\right)=-1.$$

It may then be shown that in (10), $N(\mathfrak{d})$ has no odd prime divisor. Hence in this case every ideal containing an odd rational integer is a principal ideal and Theorem 4 is valid with $\alpha_1\Delta$ replaced by 2.

Griffiths showed that a certain condition was satisfied by each of the algebras she considered.* This condition is similar to our Lemma 5 in that it insures a certain descent. By employing our Lemma 3, it may be shown that if her Lemma 2 is valid for a given $\mathfrak{G}$, then every regular ideal in $\mathfrak{G}$ is principal and hence Theorem 4 is applicable.

Throughout this paper, we have considered only left ideals. It will be observed that if X, Y are in $\mathfrak{G}$, then $(X+Y)'=X'+Y'$ and $(XY)'=Y'X'$ are in $\mathfrak{G}$. Hence $\mathfrak{G}$ is reciprocal to itself and from each of our results we may obtain at once a parallel result for right ideals.

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* Loc. cit., Lemma 2, p. 305.

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THE DIOPHANTINE EQUATION $X^2 - DY^2 = Z^M$*

BY

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I. Introduction

It has been known since the time of Euler and Lagrange† that solutions of the diophantine equation

$$X^2 - DY^2 = Z^M \tag{1.1}$$

may be obtained by setting

$$X + D^{1/2}Y = (a + D^{1/2}b)^M, \qquad Z = a^2 - Db^2,$$

where a and b are any rational integers. In 1891, Pepin‡ claimed to prove that if M is odd, and prime to the class-number of the quadratic field $\mathfrak{K}(D^{1/2})$ while a and b are co-prime, *all* solutions of (1.1) in which X, Y and Z have no common factor—for short, "primitive" solutions of (1.1)—are given by the formulas above. Later, Pepin§ recognized that Z must be restricted to be odd, while Landau‖ has pointed out (for a special case of (1.1)) that if D is positive, the units of the quadratic field $\mathfrak{K}(D^{1/2})$ must be taken into account.

Consider for example the equation $X^2 - 5Y^2 = Z^3$ to which Pepin's procedure should apply, since M is odd and the class-number of $\mathfrak{K}(5^{1/2})$ is unity. This equation has the primitive solution $X = 2$, $Y = 1$, $Z = -1$. It should therefore be possible to choose rational integers a and b such that

$$2 + 5^{1/2} = (a + 5^{1/2}b)^3, \qquad -1 = a^2 - 5b^2.$$

From the second equation, $a + 5^{1/2}b$ is a unit of $\mathfrak{K}(5^{1/2})$ and hence some power of the fundamental unit η multiplied by plus or minus one. But since the fundamental unit is $2 + 5^{1/2}$, the first equation would imply that $2 + 5^{1/2}$ is a root of unity. To obtain this particular solution, it would suffice to multiply $(a + 5^{1/2}b)^3$ by η^{-2}. But it is not at all obvious that such a device will always prove successful.

In the second part of the paper I utilize the theory of ideals to obtain explicit formulas for all the primitive solutions of (1.1) under the restrictions given below.

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* Presented to the Society, December 2, 1933; received by the editors December 26, 1933.

† Dickson's *History*, vol. II, chapter XX.

‡ Memorie della Pontificia Accademia dei Nuovi Lincei, vol. 8 (1891), pp. 41–42.

§ Annales de la Société Scientifique de Bruxelles, vol. 27 (1909), pp. 121–170.

‖ L'Intermédiaire des Mathématiciens, vol. 8 (1901), pp. 145–147.

FUNDAMENTAL THEOREM. *Let D be square-free, not equal to -3 or -1,* and incongruent to 1 modulo 8, and let M be any positive integer greater than one, and prime to the class-number h of the quadratic field $\mathfrak{K}(D^{1/2})$, but not necessarily odd.*

Let a and b be rational integers such that $(a, Db)=1$, and of opposite parity unless the contrary is expressly stated. Define A_M and B_M by

$$(1.2)\qquad (a+bD^{1/2})^M = A_M + D^{1/2}B_M.$$

Let $1, \omega$ be the canonical basis of the field $\mathfrak{K}(D^{1/2})$, and if D is positive, let

$$(1.3)\qquad \eta = r + \omega s$$

be the fundamental unit of the field. Define U_T and V_T $(T=0, 1, \cdots, M)$ by

$$U_T + D^{1/2}V_T = \eta^T,\quad D \equiv 2, 3\ (4)\ or\ D \equiv 5\ (8),\quad h \equiv 0\ (3);$$

$$U_T + D^{1/2}V_T = 2\eta^T,\quad D \equiv 5\ (8),\quad h \not\equiv 0\ (3).$$

Then all primitive solutions, and only primitive solutions, of the diophantine equation

$$(1.1)\qquad X^2 - DY^2 = Z^M$$

are given by the following formulas.

(I) *D negative.*

$$X = \pm A_M,\qquad Y = \pm B_M,\qquad Z = \pm (a^2 - Db^2).$$

(II) *D positive and either congruent to 2, 3 (4) or congruent to 5 (8) with $h\equiv 0$ (3).*

$$X = \pm (A_M U_T + DB_M V_T),\quad Y = \pm (A_M V_T + B_M U_T),\quad Z = \pm (a^2 - Db^2)\quad (T = 0, 1, \cdots, M-1).$$

(III) *D positive, congruent to 5 (8), $h\not\equiv 0$ (3).*

$$2X = \pm (A_M U_{3T} + DB_M V_{3T}),\ 2Y = \pm (A_M V_{3T} + B_M U_{3T}),\ Z = \pm (a^2 - Db^2)\quad (T = 0, 1, \cdots, [(M-1)/3]).$$

$$2^{M+1}X = \pm (A_M U_T + DB_M V_T),\ 2^{M+1}Y = \pm (A_M V_T + B_M U_T),\ 4Z = a^2 - DB^2,$$

$$a, b\ \text{both odd}.\ M + T \equiv 0\ (3)\ if\ \frac{a+b}{2} + r \equiv 0\ (2),\ and$$

$$M - T \equiv 0\ (3)\ if\ \frac{a+b}{2} + r \equiv 1\ (2)$$

$$(T = 0, 1, \cdots, M-1).$$

* The solutions in the cases $D=-1$ or $D=-3$ are well known.

If $M=2$, we have in addition

$$2^MX = \pm (A_MU_T + DB_MV_T), \quad 2^MY = \pm (A_MV_T + B_MU_T), \quad 2Z = a^2 - Db^2,$$
$$a, b \text{ both odd}, T = 0 \text{ or } 1.$$

In the final part of the paper, these formulas are applied to discuss several allied diophantine equations; notably $X^2+D=Z^M$, $1+DY^2=Z^M$, $X^{2N}-DY^{2N}=Z^N$.

II. The primitive solutions of $X^2-DY^2=Z^M$

1. Let D be a square-free integer not equal to -1 or -3 and incongruent to 1 modulo 8, and let M be an integer $\geqq 2$ and prime to the class-number of the quadratic field $\mathfrak{K}=\mathfrak{K}(D^{1/2})$. A solution $X=A$, $Y=B$, $Z=C$ of the diophantine equation

$$X^2 - DY^2 = Z^M \tag{1.1}$$

will be said to be primitive if A, B, C are rational integers with no common factor save unity. For brevity, we shall speak of "the solution A, B, C."

We shall adhere to the notations of Landau's *Vorlesungen*; italic letters are reserved for rational integers, small Greek letters for integers of the field $\mathfrak{K}$, and small German letters for ideals of $\mathfrak{K}$. A square bracket enclosing a Greek letter denotes the corresponding principal ideal; thus $[\alpha]$, $[\beta]$, $\cdots$. Round parentheses enclosing two or more letters denote greatest common divisors, (a, b), $(\mathfrak{a}, \mathfrak{b})$, $\cdots$; enclosing a single letter, they denote that it is to be used as a modulus. The conjugate of a number α of $\mathfrak{K}$ is denoted by $\bar{\alpha}$.

The following three lemmas are easily proved.

Lemma 1.1. *If A, B, C is a primitive solution of the diophantine equation* (1.1), *then both A, B, C and A, D, C are relatively prime in pairs.*

Lemma 1.2. *If A, B, C is a primitive solution of the diophantine equation* (1.1), *then* (i) *if $M\geqq 3$, C must be odd unless $D\equiv 1$* (8); (ii) *if $M=2$, C must be odd unless $D\equiv 1$ or* 5 (8). *In the latter case, if C is even, $C/2$ must be odd.*

Lemma 1.3. *If M is prime to the class-number of the algebraic field $\mathfrak{K}$, and if $\mathfrak{a}$ is any ideal of $\mathfrak{K}$, then if $\mathfrak{a}^M$ is a principal ideal, $\mathfrak{a}$ is a principal ideal.*

Lemma 1.4. *If A, B, C is a primitive solution of the diophantine equation* (1.1) *and if C is odd, then the principal ideals $[A+D^{1/2}B]$ and $[A-D^{1/2}B]$ of the quadratic field $\mathfrak{K}$ are co-prime.*

For otherwise, there exists a prime ideal $\mathfrak{p}$ of $\mathfrak{K}$ such that

$$[A + D^{1/2}B] \equiv 0 \ (\mathfrak{p}), \qquad [A - D^{1/2}B] \equiv 0 \ (\mathfrak{p}).$$

Then $[C^M]=[A+D^{1/2}B]\,[A-D^{1/2}B]\equiv 0$ $(\mathfrak{p})$, so that

$$C \equiv 0\ (\mathfrak{p}),\ \text{and}\ ([2],\ \mathfrak{p}) = 1 \text{ since } C \text{ is odd}.$$

Since $\mathfrak{p}$ contains both $A+D^{1/2}B$ and $A-D^{1/2}B$, it contains their sum $2A$ and hence A itself. Therefore the rational prime which $\mathfrak{p}$ divides divides both A and C contrary to Lemma 1.1.

LEMMA 1.5. *If D is congruent to 5 modulo 8, and if 1, ω is the canonical basis for the integers of the field $\mathfrak{K}$, and if $(c+\omega d)^M = c'+\omega d'$, where c and d are rational integers, then if M is prime to three, d' is even when and only when d is even. If M is divisible by three, d' is always even.*

For $5 \equiv D = (2\omega+1)^2 \equiv 4\omega^2+4\omega+1$ (8), so that

$$\omega^2 \equiv \omega + 1\ (2).$$

If d is even, d' is obviously even for any value of M. If d is odd, we have either $c+\omega d \equiv 1+\omega$ (2) or $c+\omega d \equiv \omega$ (2). In the first case, $(c+\omega d)^2 \equiv \omega^2+1 \equiv \omega$ (2), $(c+\omega d)^3 \equiv \omega^2+\omega \equiv 1$ (2), $(c+\omega d)^4 \equiv c+\omega d$ (2). In the second case, $(c+\omega d)^2 \equiv \omega^2 \equiv 1+\omega$ (2), $(c+\omega d)^3 \equiv \omega^2+\omega \equiv 1$ (2), $(c+\omega d)^4 \equiv c+\omega d$ (2). Hence in either case, if $M \equiv N$ (3), $N = 0$, 1 or 2, $(c+\omega d)^M \equiv (c+\omega d)^N$ (2), from which the rest of the lemma easily follows.

LEMMA 1.6. *If D is congruent to 5 modulo 8, not equal to -3, and negative, the class-number of the quadratic field $\mathfrak{K}$ is always divisible by three.**

LEMMA 1.7. *If D is congruent to 5 modulo 8 and positive, and if*

$$\eta = r + \omega s \tag{1.3}$$

is the fundamental unit of the quadratic field $\mathfrak{K}$, then the class-number of $\mathfrak{K}$ is divisible by three when and only when the rational integer s is even.†

2. Let A, B, C be a primitive solution of (1.1). During the next three sections of the paper, we assume that $M \geqq 3$, so that C is necessarily odd.

If 1, ω is the canonical basis of the field $\mathfrak{K}$, we have

$$\begin{aligned} &\omega^2 = D^{1/2}, \quad \bar{\omega} = -\omega \text{ if } D \equiv 2, 3\ (4)\cdot \quad 2\omega + 1 = D^{1/2}, \\ &\bar{\omega} = -1 - \omega \text{ if } D \equiv 5\ (8). \end{aligned} \tag{2.1}$$

Let

$$\begin{aligned} &\kappa = A + \omega B, &&\lambda = \bar{\kappa} = A - \omega B \ \text{ if } D \equiv 2, 3\ (4), \\ &\kappa = A + B + 2\omega B, &&\lambda = \bar{\kappa} = A - B - 2\omega B \text{ if } D \equiv 5\ (8). \end{aligned} \tag{2.2}$$

Then in either case, κ and λ are integers of $\mathfrak{K}$, and $\kappa\lambda = A^2 - DB^2 = C^M$ or

$$[\kappa][\lambda] = [C]^M. \tag{2.3}$$

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* Dirichlet-Dedekind, *Zahlentheorie*, 4th edition, 1894, p. 244.

† Dirichlet-Dedekind, work cited, p. 250.

Since C is odd, the principal ideals $[\kappa]$ and $[\lambda]$ in (2.3) are co-prime by Lemma 1.4. Hence there exist two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathfrak{K}$ such that

$$[\kappa] = \mathfrak{a}^M, \qquad [\lambda] = \mathfrak{b}^M, \qquad [C] = \mathfrak{a}\mathfrak{b}, \qquad (\mathfrak{a}, \mathfrak{b}) = 1.$$

Since M is prime to the class-number of the field $\mathfrak{K}$, $\mathfrak{a}$ and $\mathfrak{b}$ are principal ideals of $\mathfrak{K}$ by Lemma 1.3. Denote them by $[\alpha]$ and $[\beta]$ respectively. Then

$$[\kappa] = [\alpha^M], \qquad [\lambda] = [\beta^M], \qquad [C] = [\alpha][\beta], \qquad ([\alpha], [\beta]) = 1.$$

Moreover, since λ is conjugate to κ, β is conjugate to α. Therefore there exist two units ϵ_1 and ϵ_2 of $\mathfrak{K}$ such that

$$\kappa = \epsilon_1\alpha^M, \qquad \lambda = \bar{\epsilon}_1\bar{\alpha}^M, \qquad C = \epsilon_2\alpha\bar{\alpha}, \qquad ([\alpha], [\bar{\alpha}]) = 1.$$

Since $\alpha\bar{\alpha} = N\alpha$ is a rational integer, $\epsilon_2 = \pm 1$. Let η be the fundamental unit of the field $\mathfrak{K}$. Then there exists an integer R such that $\epsilon_1 = \pm \eta^R$.

Divide R by M, and let the quotient and remainder be Q, T: $R = QM + T$, $0 \leqq T \leqq M-1$. Then if we write α' for $\eta^Q\alpha$, we have

$$\kappa = \pm\, \eta^T\alpha'^M, \quad \lambda = \pm\, \eta^{-T}\bar{\alpha}'^M, \quad C = \pm\, \alpha'\bar{\alpha}', \quad 0 \leqq T \leqq M-1, \tag{2.4}$$

$$([\alpha'], [\bar{\alpha}']) = 1. \tag{2.5}$$

If D is negative, the only units in $\mathfrak{K}$ are ± 1, since $D \neq -1, -3$, and (2.4) holds with $T = 0$. Henceforth we retain only the positive signs in (2.4).

3. If $D \equiv 2, 3$ (4), α' and $\bar{\alpha}'$ in (2.4) are of the form

$$\alpha' = a + \omega b, \qquad \bar{\alpha}' = a - \omega b, \qquad \omega^2 = D,$$

where a and b are rational integers. Then

$$(a, Db) = 1. \tag{3.1}$$

For otherwise, there exists a prime ideal $\mathfrak{p}$ of $\mathfrak{K}$ such that $\alpha \equiv 0$ $(\mathfrak{p})$, $Db = \omega^2 b \equiv 0$ $(\mathfrak{p})$, so that $a \equiv \omega b \equiv 0$ $(\mathfrak{p})$, $\alpha' \equiv \bar{\alpha}' \equiv 0$ $(\mathfrak{p})$ contradicting (2.5).

Since $C = a^2 - Db^2$ is odd, we must have

(3.2) a, b of opposite parity if D is odd, a odd if D is even.

Now $\alpha'^M = A_M + D^{1/2}B_M$, where

$$\begin{aligned} A_M &= a^M + \binom{M}{2} D a^{M-2} b^2 + \binom{M}{4} D^2 a^{M-4} b^4 + \cdots ; \\ B_M &= \binom{M}{1} a^{M-1} b + \binom{M}{3} D a^{M-3} b^3 + \cdots . \end{aligned} \tag{3.3}$$

If the fundamental unit η is $r + \omega s$ in the case when D is positive, we write $r = u_1$, $s = v_1$, $\omega = D^{1/2}$,

$$\eta^T = (r + \omega s)^T = U_T + D^{1/2}V_T \qquad (T = 0, 1, \cdots, M-1).$$

(2.4) then gives us our final formulas:

$$(3.4) \quad A = U_TA_M + DV_TB_M, \qquad B = U_TB_M + V_TA_M, \qquad C = a^2 - Db^2,$$

where if D is positive, T may have any integral value from 0 to $M-1$, but if D is negative, T is zero.

We have thus shown that in the case $D\equiv 2, 3$ (4), *every primitive solution* A, B, C *of* (1.1) *is of the form* (3.4). We shall now show that if a and b are rational integers subject to the conditions (3.1), (3.2), *the formulas* (3.4) *always give a primitive solution of* (1.1).

It is obvious the formulas always give a solution of (1.1), and that for such a solution, C is odd. To show that the solution is primitive, it suffices to prove that $(A, B)=1$.

If $(A, B)\neq 1$, there exists a prime ideal $\mathfrak{p}$ of $\mathfrak{K}$ such that $A\equiv B\equiv 0$ ($\mathfrak{p}$) so that $A\pm D^{1/2}B\equiv 0$ ($\mathfrak{p}$). Since $U_T\pm D^{1/2}V_T$ is a unit of $\mathfrak{K}$ and $A\pm D^{1/2}B =(U_T\pm D^{1/2}V_T)(A_M\pm D^{1/2}B_M)$, $A_M\pm D^{1/2}B_M=(a\pm D^{1/2}b)^M\equiv 0$ ($\mathfrak{p}$) or $a \pm D^{1/2}b\equiv 0$ ($\mathfrak{p}$). Therefore $2a\equiv 2D^{1/2}b\equiv 0$ ($\mathfrak{p}$); or since $(a, Db)=1$, $2\equiv 0$ ($\mathfrak{p}$), and $A\equiv B\equiv 0$ (2). But then $C^M=A^2-DB^2\equiv 0$ (2), so that C would be even.

4. If $D\equiv 5$ (8), α' and $\bar{\alpha}'$ in (2.4) are of the form

$$(4.1) \quad \alpha' = c + \omega d, \qquad \bar{\alpha}' = c - d - \omega d, \qquad (2\omega + 1)^2 = D,$$

where c and d are rational integers which are co-prime by (2.5). There are two cases according as D is negative or positive.

If D is negative, the class-number of $\mathfrak{K}$ is divisible by three by Lemma 1.6. Hence $(M, 3)=1$. Since 1 is the fundamental unit, we obtain from (2.4) $\alpha'^M=(c+\omega d)^M=\kappa=A+B+2B\omega$. Therefore, by Lemma 1.5, d is even. If we write $d=2b$, $c=a-b$, we have $\alpha'=a+D^{1/2}b$. Hence

$$\kappa = A + BD^{1/2} = (a + D^{1/2}b)^M = A_M + D^{1/2}B_M.$$

Thus we obtain as in the previous case D negative and congruent to 2 or 3 (4),

$$(4.2) \quad A = A_M, \qquad B = B_M, \qquad C = a^2 - Db^2\alpha, \qquad (a, Db) = 1.$$

Since $M\geqq 3$, C is odd by Lemma (1.2). Therefore a and b must be of opposite parity. A_M and B_M are as in (3.3).

Conversely, it may be shown as in §3 that if a and b are rational integers of opposite parity, the formulas (4.2) always give a primitive solution of (1.1).

Next, assume that D is positive, and denote the fundamental unit of the field $\mathfrak{K}$ by

(1.3) $$\eta = r + \omega s,$$

as in Lemma 1.7. Then if the class-number of $\mathfrak{K}$ is divisible by three, s is even. Writing $s = 2v$, $r = u - v$,

$$\eta = u + vD^{1/2}, \qquad \eta^T = U_T + V_T D^{1/2} \qquad (T = 0, 1, \cdots, M-1).$$

Then by (2.4), (4.1)

$$\alpha'^M = (c + \omega d)^M = \bar{\eta}^T \kappa = c' + \omega d'$$

where d' is even. Since $(M, 3) = 1$, d is therefore even by Lemma 1.5. On writing $d = 2b$, $c = a - b$, we obtain therefore

(4.3) $$A = U_T A_M + D V_T B_M, \qquad B = U_T B_M + V_T A_M, \qquad C = a^2 - Db^2.$$

a and b here are of opposite parity, and $(a, Db) = 1$. Conversely, we may show as in §3 that (4.3) always gives a primitive solution of (1.1).

If the class-number of $\mathfrak{K}$ is not divisible by three, the integer s in (1.3) is odd. We obtain therefore from (2.4) and (4.1)

$$(r + \omega s)^T (c + \omega d)^M = \kappa = c' + \omega d', \qquad d' \text{ even}.$$

Therefore if d is *even*, T must be divisible by three by Lemma 1.5. On the other hand, if d is *odd*, we have the following restrictions on T and M according to the parity of r in order that d' may be even.

If $r + \omega s \equiv 1 + \omega\ (2)$ and $c + \omega d \equiv 1 + \omega\ (2)$, then $T + M \equiv 0\ (3)$;
if $r + \omega s \equiv 1 + \omega\ (2)$ and $c + \omega d \equiv \omega\ (2)$, then $T - M \equiv 0\ (3)$;
if $r + \omega s \equiv \omega\ (2)$ and $c + \omega d \equiv \omega\ (2)$, then $T + M \equiv 0\ (3)$;
if $r + \omega s \equiv \omega\ (2)$ and $c + \omega d \equiv 1 + \omega\ (2)$, then $T - M \equiv 0\ (3)$.

Let us write

$$2\eta^T = U_T + V_T D^{1/2} \qquad (T = 0, 1, \cdots, M-1).$$

$\alpha' = a + D^{1/2} b$ if d is even; $2\alpha' = a + D^{1/2} b$ if d is odd, where, in the first case, $d = 2b$, $c = a - b$, and in the second case, $d = b$, $a = 2c - b$, so that a and b are both odd. The four cases above when s is odd may then be stated as follows:

(4.4) $$T + M \equiv 0\ (3) \text{ if } r - (a + b)/2 \equiv 0\ (2),$$
$$T - M \equiv 0\ (3) \text{ if } r - (a + b)/2 \equiv 1\ (2).$$

The solutions of (1.1) are given by the following formulas:

(4.5) $$2A = U_{3T} A_M + D V_{3T} B_M, \quad 2B = U_{3T} B_M + V_{3T} A_M, \quad C = a^2 - Db^2,$$
a, b of opposite parity, $(a, Db) = 1$, $0 \leqq T \leqq [(M-1)/3)]$,

or

(4.6) $2^{M+1}A = U_TA_M + DV_TB_M,\ 2^{M+1}B = U_TB_M + V_TA_M,\ 4C = a^2 - Db^2,$

a, b both odd, $(a, Db) = 1$, and T restricted by (4.4).

It is easily shown as before that both (4.5) and (4.6) give us primitive solutions of (1.1) with the specified restrictions on a, b and T.

The possibility of primitive solutions of (1.1) of the form (4.6) seems to have been overlooked heretofore. On taking $D=5$, $M=3$, $T=0$, $a=b=1$ in (4.6), we obtain the solution 2, 1, -1 of $X^2-5Y^2=Z^3$ discussed in the introduction.

5. The case when $M=2$, $D\equiv 5$ (8) requires separate discussion, as we see from Lemma 1.2 that primitive solutions of

$$X^2 - DY^2 = Z^2 \tag{5.1}$$

will exist of the form $X=A$, $Y=B$, $Z=C=2E$, where A, B, E are odd and co-prime. The other solutions with C odd may be obtained from our general formulas in §4. In the present case, we write

(5.2) $2\kappa = A + B + 2\omega B,\ \ 2\lambda = A - B - 2\omega B$ where as usual $2\omega + 1 = D^{1/2}$,

or letting $A+B=2G$,

$$\kappa = G + \omega B, \qquad \lambda = \bar{\kappa}, \qquad \kappa\lambda = E^2, \qquad [\kappa][\lambda] = [E]^2.$$

If we now apply the reasoning used in §2 to this ideal equation, we deduce that either

$$\begin{aligned} &\kappa = (c + \omega d)^2, && E = (c + \omega d)(c + \bar{\omega} d), \qquad \text{or} \\ &\kappa = (r + \omega s)(c + \omega d)^2, && E = (c + \omega d)(c + \bar{\omega} d), \end{aligned} \tag{5.3}$$

where $r+\omega s$ is the fundamental unit of the field $\mathfrak{K}$. To agree with our former notation, let $U_0=2$, $V_0=0$, $U_1=2r-s$, $V_1=s$, $2c-d=a$, $d=b$. Then

$$8\kappa = (U_T + D^{1/2}V_T)(a + D^{1/2}b)^2, \qquad 4E = a^2 - Db^2, \qquad T = 0, 1,$$

so that

$$\begin{aligned} &4A = U_T(a^2 + Db^2) + 2abDV_T, && 4B = V_T(a^2 + Db^2) + 2abU_T, \\ &2C = a^2 - Db^2, \qquad T = 0, 1, \end{aligned} \tag{5.4}$$

where a and b are both odd, and $(a, Db)=1$. As before, (5.4) always gives a primitive solution of (5.1) with Z even.

For the case $M=2$, it may be noted, a knowledge of all of the primitive solutions of (1.1) gives us immediately the most general solution of (1.1). On collecting all of our results, we obtain the fundamental theorem stated in the introduction.

III. Applications of the formulas

6. Consider the diophantine equation

$$X^2 - D = Z^M, \tag{6.1}$$

where D is square-free, negative, $\neq -1$ or -3, incongruent to 1 (8), while M is prime to the class-number of the quadratic field $\mathfrak{K}(D^{1/2})$. Then if $X=A$, $Z=C$ is a solution of (6.1), A, ± 1, C is a primitive solution of (1.1). Conversely, any primitive solution of (1.1) with $B=\pm 1$ gives a solution of (6.1). Accordingly, all solutions of (6.1) are obtainable by setting $Y=\pm 1$ in the formulas of case I of the fundamental theorem; thus

$$\pm 1 = \binom{M}{1} a^{M-1}b + \binom{M}{3} Da^{M-3}b^3 + \cdots . \tag{6.2}$$

If M is even, the last term on the right of (6.2) is $\binom{M}{M-1}D^{(M-2)/2}ab^{M-1}$. Since the numbers $\binom{M}{1}$, $\binom{M}{3}$, $\cdots$, $\binom{M}{M-1}$ are all even when M is even, (6.2) is impossible, so that (6.1) *has no solutions if M is even.*

If M is odd, the last term on the right of (6.2) is $D^{(M-1)/2}b^M$. Hence every term is divisible by b, so that $b=\pm 1$, and a must be a root of the equation

$$\binom{M}{1} x^{M-1} + \binom{M}{3} Dx^{M-3} + \cdots + D^{(M-1)/2} \mp 1 = 0. \tag{6.21}$$

For fixed D and M meeting our restrictions, the solution of (6.1) reduces then to finding all the integral roots of (6.2).

Under the same restrictions on D and M, we can obtain information about the diophantine equation

$$1 - DY^2 = Z^M. \tag{6.3}$$

We have in place of (6.2) the condition

$$\pm 1 = a^M + \binom{M}{2} a^{M-2}b^2 + \cdots . \tag{6.4}$$

If M is even, we obtain no direct information. But if M is odd, the right side of (6.4) is divisible by a, so that $a=\pm 1$, and b must be an integral root of the equation*

$$\begin{aligned}\binom{M}{M-1} D^{(M-3)/2}x^{M-3} + \binom{M}{M-3} D^{(M-5)/2}x^{M-5} + \cdots \\ + \binom{M}{4} Dx^2 + \binom{M}{2} = 0.\end{aligned} \tag{6.41}$$

* The conceivable case when $a=\mp 1$ and the left side of (6.4) is ± 1 is easily shown to be impossible.

To give a numerical example, consider the equation $X^2+42=Z^5$ to which the method is applicable since the class-number of $\mathfrak{K}(42^{1/2}i)$ is 4. If M is a prime,

$$D^{(M-1)/2} \equiv \left(\frac{D}{M}\right)(M),$$

while $\binom{M}{1}, \binom{M}{3}, \cdots, \binom{M}{M-2}$ are all divisible by M. We must therefore choose the ambiguous sign in (6.21) equal to $-\left(\frac{D}{M}\right)$, or $+1$ in this case. On dividing out $M=5$, (6.21) becomes

$$x^4 - 84x^2 + 353 = 0.$$

Since $84^2-4\cdot 353=5644$ is not a square, the initial diophantine equation has no solutions.

7. Consider now the diophantine equation

$$X^2 - 16DY^{2N} = Z^4. \tag{7.1}$$

We assume as before that D is square-free, negative, incongruent to 1 (8), and in addition, that the class-number of the quadratic field $\mathfrak{K}(D^{1/2})$ is *odd.**

Let A, B, C be a primitive solution of (7.1). Then $A, 4B^N, C$ is a primitive solution of

$$X^2 - DY^2 = Z^4. \tag{7.2}$$

Hence by case I of our fundamental theorem, there exist rational integers a and b such that $(a, Db)=1$, $a+b$ odd, and

$$A = a^4 + 6a^2b^2D + D^2b^4, \qquad 4B^N = 4ab(a^2 + Db^2), \qquad C = a^2 - Db^2.$$

From the expression for $4B^N$, we deduce that a, b, a^2+Db^2 are perfect Nth powers: $a=E^N$, $b=F^N$, $a^2+Db^2=G^N$ so that $X=E, Y=F, Z=G$ is a primitive solution of

$$X^{2N} + DY^{2N} = Z^N. \tag{7.3}$$

Conversely, a primitive solution of (7.3) gives us a primitive solution of (7.1). But it is easy to see that if (7.3) has any solutions whatever, it has primitive solutions. Therefore: *A necessary and sufficient condition that the diophantine equation* (7.3) *be solvable is that the diophantine equation* (7.1) *have a primitive solution.*

Assume next that D is negative, and congruent to 2 or 3 (4), and that the class-number of $\mathfrak{K}(D^{1/2})$ is prime to 3, while D is divisible by three. Consider

* This always occurs for example if D is a prime, $\equiv 5$ (8). See Dirichlet's Works, vol. I, 1889, pp. 357–370, or Crelle's Journal, vol. 18 (1838), pp. 259–274.

$$X^2 - 9DY^{2N} = Z^3. \tag{7.4}$$

A similar procedure to that given for (7.1) connects (7.4) with the diophantine equation

$$X^N + \frac{DY^N}{3} = Z^N, \tag{7.5}$$

and we have the theorem that *a necessary and sufficient condition that the diophantine equation* (7.5) *be solvable is that the diophantine equation* (7.4) *have a primitive solution.*

For example take $D = -21$. The class number of $\mathfrak{K}(21^{1/2}i)$ is four, and for $N = 7$,

$$X^7 - 7Y^7 = Z^7$$

is known to have no solutions.* Hence

$$X^2 + 189Y^{14} = Z^3$$

has no primitive solutions.

This result generalizes an interesting correspondence recently obtained by Kapferer† between the solutions of Fermat's equation and the primitive solutions of an equation of the form (7.4).

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* Maillet, Comptes Rendus, vol. 129 (1899), pp. 189–199.

† Sitzungsberichte, Heidelberg Akademie, 1933, part 2, pp. 32–37.

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INTERPOLATION IN REGULARLY DISTRIBUTED POINTS*

BY

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1. **Introduction.** Let G_n be a set of n distinct points chosen on the rectifiable Jordan curve C† in the complex z-plane, and let $\{G_n\}$ denote a sequence of such sets. This sequence may be written out in the following triangular array:

$$\begin{aligned} G_1&: \quad z_1^{(1)} \\ G_2&: \quad z_1^{(2)}, z_2^{(2)} \\ &\cdot \\ &\cdot \\ &\cdot \\ G_n&: \quad z_1^{(n)}, z_2^{(n)}, \cdots, z_n^{(n)}. \end{aligned}$$

Furthermore, let $f(z)$ be a function defined‡ and integrable in the sense of Riemann on the curve C; we shall say that such a function is integrable (R) on C. By $L_n(z)$ we shall denote the unique polynomial of degree at most $n-1$ which coincides with the function $f(z)$ in the points of the set G_n; we shall call it the Lagrange polynomial interpolating to $f(z)$ in the points G_n. We shall say that the sequence $\{G_n\}$ yields effective interpolation to the function $f(z)$ if the sequence $\{L_n(z)\}$ converges to the function $f_1(z)$ at every point of B, the region interior to the curve C,§ and uniformly for z on any closed point set of B, where

$$f_1(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,dt.$$

About half a century ago, Méray pointed out that if $f(z)=1/z$ and if the set G_n consists of the nth roots of unity, then since $L_n(z)=z^{n-1}$, the sequence $\{L_n(z)\}$ approaches zero for $|z|<1$.‖ (It will be noted that zero is the value of the function $f_1(z)$ in this case.) The following theorem, of comparatively recent origin, discloses the theory underlying this example:

* Presented to the Society, September 4, 1934; received by the editors, in revised form, April 3, 1935.

† We define a Jordan curve as a one-to-one continuous transform of a circumference.

‡ Infinity will not be admitted as a functional value in connection with the definition of functions other than the mapping function $\phi(w)$ introduced in §2.

§ By the interior of C we mean the region bounded by C which does not contain the point at infinity.

‖ Méray, Annales de l'École Normale Supérieure, (3), vol. 1 (1884), pp. 165–176.

THEOREM A.* *Let $f(z)$ be a function defined and integrable (R) on the unit circle, and let the set G_n be the nth roots of unity. Then the sequence $\{G_n\}$ yields effective interpolation to the function $f(z)$. But for a properly chosen function $f(z)$ a sub-sequence of the sequence $\{L_n(z)\}$ will diverge to infinity at points on the unit circle itself.*

The main results of the present paper arose from the suggestion made by Walsh that it would be of interest to extend the theorem to the consideration of curves other than the unit circle.† The extension will be derived first under the hypothesis that the function $f(z)$ is analytic on the curve C (as in Méray's example), and then under the hypothesis that the function is merely bounded in modulus and integrable (R) on C.‡ The theorems thus obtained will be supplemented by a study of the degree of convergence of the sequence $\{L_n(z)\}$. This study will result in equalities for z on the curve C as well as for z in the region B, and so will have an additional significance in that it will elucidate the statement in Theorem A concerning the possibility of divergence on C.

The paper concludes with a discussion of the results which arise from interpolation to more than one function defined on one or more Jordan curves.

2. **The choice of the points of interpolation.** An arbitrarily chosen sequence $\{G_n\}$ will not in general lead to effective interpolation, even if the function $f(z)$ is analytic in the closed region $B+C$ and if the points of the nth set G_n become everywhere dense on the curve C as n approaches infinity.§ Thus the proper choice of the set G_n is of fundamental importance in a generalization of Theorem A. We shall base our selection of the set G_n upon a notable precedent; namely, that of Fejér, who established an extension of Theorem A for functions analytic in the closed region $B+C$ by using a set G_n which he called a set of "regularly distributed" ("regelmässig verteilt") points on the curve C.‖ Fejér's set G_n may be defined in the following man-

* This theorem is due to Fejér and Walsh. Fejér, in a brilliant paper entitled *Interpolation und konforme Abbildung* which appeared in the Göttinger Nachrichten, 1918, pp. 319–331, proved the theorem for the case in which the function $f(z)$ is continuous on and within the unit circle and analytic within the circle, and he also gave an example of such a function for which the corresponding Lagrange polynomials diverge at a point of the circle. Walsh showed that the theorem is true for functions more general than those considered by Fejér; Bulletin of the American Mathematical Society, vol. 38 (1932), pp. 290–291.

† Walsh, loc. cit., p. 294.

‡ The methods of approach indicated for these two cases are entirely dissimilar; see §5.

§ For a simple illustration of this statement, see Walsh, loc. cit., p. 293.

‖ Fejér, loc. cit., pp. 324–327. Theorem A becomes a classical result due to Runge when the function $f(z)$ is assumed to be analytic in the closed region; see Runge, *Theorie und Praxis der Reihen*, Berlin, 1904, p. 137. In this case, and also in Fejér's extension, the convergence of the sequence $\{L_n(z)\}$ takes place in the closed region.

ner: Let the Jordan curve C lie in the z-plane and let the function $\phi(w)$ map the exterior of the unit circle in the w-plane onto the exterior of C in such a way that the points at infinity in the two planes correspond to each other.* The set G_n consists of those points on C into which the nth roots of unity are transformed by the equation $z=\phi(w)$.

Henceforth in this paper the symbol G_n, wherever it appears in connection with a curve C, will denote the nth set of Fejér's regularly distributed points on C.

3. **Restrictions on the curve.** It is assumed that the function $\phi(w)$, which we have just introduced, gives a conformal, one-to-one map of the exterior of the unit circle onto the exterior of the curve C, which means that with the exception of the point at infinity, $\phi(w)$ is analytic for $|w|>1$, univalent and continuous for $|w|\geqq 1$. The function generates a Laurent series of the following type:

$$\phi(w) \sim cw + c_0 + \frac{c_1}{w} + \frac{c_2}{w^2} + \cdots, c \neq 0,$$

which may be considered as a representation of the function for all $|w|\geqq 1$. In particular, we have

$$\phi'(\infty) = \lim_{w\to\infty} \frac{\phi(w)}{w} = c.$$

We shall denote the inverse of the function $\phi(w)$ by $\phi^{-1}(z)$.

We define a function $\Phi(z, w)$ by the following equations:

$$\Phi(z, w) = \log \frac{\phi(w) - z}{cw}, \quad z \text{ in } B, \ |w| \geqq 1,$$

$$\Phi(z, \infty) = 0;\dagger$$

and in the event that the function $\phi(w)$ possesses a non-vanishing first tangential derivative at every point of the circle $|w|=1$, we define also a function $\Psi(\overline{w}, w)$ as follows:

$$\left.\begin{aligned} \Psi(\overline{w}, w) &= \log \frac{\phi(\overline{w}) - \phi(w)}{c(\overline{w} - w)} \\ \Psi(\overline{w}, \overline{w}) &= \log \frac{\phi'(\overline{w})}{c} \end{aligned}\right\} |w| \geqq 1, \ |\overline{w}| \geqq 1,$$

$$\Psi(\overline{w}, \infty) = 0.\ddagger$$

* Hilbert first indicated the significance of this type of mapping function in the study of interpolation, Göttinger Nachrichten, 1897, pp. 63–70.

† This equation identifies the particular branch of the logarithmic function under consideration.

‡ See the preceding footnote.

(The variables z, w, and $\bar{w}$ are all supposed to be independent.) Both of these functions are analytic functions of w for $|w| \geqq 1$, provided that z is a point of the region B.

We shall now introduce a pair of conditions on the curve C which will be expressed in terms of these functions; the conditions play a central role in our generalization of Theorem A.

The curve C will be said to satisfy condition (a) if given an arbitrary closed point set S of the region B, there exist polynomials in $1/w$, $f_n(z, w)$, of respective degrees $n-1$, which satisfy the equation

$$\Phi(z, w) - f_n(z, w) = o\left(\frac{1}{n}\right)$$

uniformly for $|w| \geqq 1$ and for z on S.

The curve C will be said to satisfy condition (b) if the corresponding mapping function $\phi(w)$ possesses a non-vanishing first tangential derivative at every point of the circle $|w| = 1$, and if there exist polynomials in $1/w$, $F_n(\bar{w}, w)$, of respective degrees $n-1$, which satisfy the equation

$$\Psi(\bar{w}, w) - F_n(\bar{w}, w) = o\left(\frac{1}{n}\right),$$

uniformly for $|w| \geqq 1$ and $|\bar{w}| = 1$.*

A Jordan curve will satisfy condition (a) if the first tangential derivative of the corresponding mapping function $\phi(w)$ on the circle $|w| = 1$ exists and satisfies a Lipschitz condition with exponent $\alpha > 0$.† The curve C will also satisfy condition (b) if the second tangential derivative of the function $\phi(w)$ on the circle $|w| = 1$ exists and satisfies a Lipschitz condition with exponent $\alpha > 0$, and if the first tangential derivative does not vanish.

To prove that a curve of the first type satisfies condition (a), we observe that the first tangential partial derivative with respect to w of the function $\Phi(z, w)$ on the circle $|w| = 1$ satisfies a Lipschitz condition with exponent α and with a constant which is a uniformly bounded function of z for z on any closed point set of the region B. A similar assertion may also be made in connection with a curve of the second type concerning its function $\Psi(\bar{w}, w)$

* It is to be observed that no assumption is made as to the continuity of the functions $f_n(z, w)$ and $F_n(\bar{w}, w)$ in the variables z and $\bar{w}$ respectively.

† A function $f(z)$ is said to satisfy a Lipschitz condition on a curve C with exponent α and constant λ if $|f(x_1) - f(x_2)| \leqq \lambda |x_1 - x_2|^\alpha$ for all x_1 and x_2 on C.

for $|\bar{w}|=1$, $|w|=1$, although the proof is not as simple as in the first case.* The existence of the required polynomials in $1/w$ is now established by a theorem due to Sewell, which, for future reference, we shall call Theorem B.†

4. Products associated with the Lagrange polynomial. The Lagrange polynomial interpolating to the function $f(z)$ in the arbitrary set of distinct points z_k, $k=1, 2, \cdots, n$, may be written in this form:

$$L_n(z) = \sum_{k=1}^{n} \frac{f(z_k)}{z - z_k} \frac{\omega_n(z)}{\omega_n'(z_k)}, \tag{1}$$

where $\omega_n(z)$ is the following product:

$$\omega_n(z) = \prod_{k=1}^{n} (z - z_k).$$

If the set of points happens to be the nth roots of unity, $e^{2\pi ik/n}$, $k=1, 2, \cdots, n$, then

$$\omega_n(z) = z^n - 1,$$

and (1) becomes

$$L_n(z) = \sum_{k=1}^{n} f(e^{2\pi ik/n}) \frac{(z^n - 1)e^{2\pi ik/n}}{n(z - e^{2\pi ik/n})}.$$

In particular,

$$nL_n(0) = \sum_{k=1}^{n} f(e^{2\pi ik/n}); \tag{2}$$

hence if there exists an upper bound for the function $f(z)$ on the unit circle, and if this bound be denoted by f, then

$$|L_n(0)| \leqq f. \tag{3}$$

This fact leads at once to the proof of the first of three lemmas upon which our generalization of Theorem A will rest.

LEMMA I.‡ *Let the function $f(z)$ be defined for $|z| \leqq 1$, let $p_n(z)$ be a polynomial of degree $n-1$, and let ϵ_n be a positive number such that*

$$|f(z) - p_n(z)| \leqq \epsilon_n$$

for $|z|=1$ and for $z=0$.

* A proof can be given by taking the real and imaginary parts of the derivative $\Phi_w(\bar{w}, w)$ and then applying the integral form of the law of the mean.

† W. E. Sewell, Bulletin of the American Mathematical Society, vol. 41 (1935), p. 117.

‡ This lemma is a special case of Theorem IIIa.

Let the polynomial $L_n(z)$ interpolate to the function $f(z)$ in the set $e^{2\pi ik/n}$, $k=1, 2, \cdots, n$. Then

$$|f(0) - L_n(0)| \leqq 2\epsilon_n.$$

For let $\lambda_n(z)$ be the Lagrange polynomial interpolating to the function $p_n(z)-f(z)$ in the set $e^{2\pi ik/n}$. Then $f=\epsilon_n$, so $|\lambda_n(0)|\leqq\epsilon_n$, by (3). Therefore

$$|f(0) - p_n(0) + \lambda_n(0)| \leqq 2\epsilon_n.$$

But

$$p_n(e^{2\pi ik/n}) - \lambda_n(e^{2\pi ik/n}) = f(e^{2\pi ik/n}) \qquad (k = 1, 2, \cdots, n),$$

so $p_n(z)-\lambda_n(z)$ must be the unique polynomial of degree at most $n-1$ interpolating to the function $f(z)$ in the set $e^{2\pi ik/n}$, which is none other than $L_n(z)$ itself.

LEMMA II. *Let C be a curve which satisfies condition* (a) *and let the product $\omega_n(z)$ be formed for the corresponding set G_n. Then*

$$\frac{\omega_n(z)}{-c^n} \to 1$$

uniformly for z on any closed point set S of the region B.

To prove the assertion, we first write

$$\frac{\omega_n(z)}{-c^n} = -\prod_{k=1}^{n}\left[\frac{z-\phi(e^{2\pi ik/n})}{c}\right] = \prod_{k=1}^{n}\left[\frac{\phi(e^{2\pi ik/n})-z}{ce^{2\pi ik/n}}\right].$$

We shall compute the limit by studying that branch of the function $\log\,[\omega_n(z)/-c^n]$ which is identified by the following equation:

$$\log\frac{\omega_n(z)}{-c^n} = \sum_{k=1}^{n}\Phi(z, e^{2\pi ik/n}) = \sum_{k=1}^{n}\Phi(z, e^{-2\pi ik/n}). \tag{4}$$

By hypothesis, there exist polynomials in the variable $W=1/w$, $f_n(z, w)$, of respective degrees $n-1$, which satisfy the equation

$$\Phi(z, w) - f_n(z, w) = o\left(\frac{1}{n}\right)$$

uniformly for $|W|\leqq 1$ and for z on S. Therefore if we denote by $\Lambda_n(z, w)$ the Lagrange polynomial in W interpolating to the function $\Phi(z,w)$ in the points of the set $W=e^{2\pi ik/n}$, $k=1, 2, \cdots, n$, we may write, by Lemma I,

$$\Phi(z, \infty) - \Lambda_n(z, \infty) = o\left(\frac{1}{n}\right),$$

or

$$n\Phi(v, \infty) - n\Lambda_n(z, \infty) = o(1)$$

uniformly for z on S. But by (2) and (4),

$$n\Lambda_n(z, \infty) = \log \frac{\omega_n(z)}{-c^n},$$

and since $\Phi(z, \infty) = 0$, the proof is complete.

LEMMA III. *Let C be a curve which satisfies condition* (b) *and let the product $\omega_n(z)$ be formed for the corresponding set G_n. Let*

$$\pi_n(\bar{w}) = \frac{\omega_n[\phi(\bar{w})]}{c^n(\bar{w}^n - 1)}.$$

Then $\pi_n(\bar{w}) \to 1$ uniformly for $|\bar{w}| = 1$.

The proof is the same as that of the preceding lemma except for obvious changes in notation.

It is worth while noticing that the existence of this limit can also be proved for $\bar{w}$ on any closed point set lying exterior to the circle $|\bar{w}| = 1$ by modifying condition (b) accordingly. The modification would have the effect of lightening the restriction on the curve C, for the function $\Psi(\bar{w}, w)$ is an analytic function of the two variables $\bar{w}$ and w for $|\bar{w}| > 1$, $|w| > 1$. Thus, in particular, the new condition would be satisfied if the first tangential derivative of the function $\phi(w)$ existed on the circle $|w| = 1$ and satisfied a Lipschitz condition with positive exponent.*

5. **The convergence of sequences of Lagrange polynomials.** We now apply the foregoing results to the theory of interpolation. Let $f(z)$ be a function known to be analytic on the curve C, but not necessarily analytic at all points of the region B. Furthermore, let the curve C satisfy condition (a) and let the product $\omega_n(z)$ be formed for the corresponding set G_n. We determine two contours C_1 and C_2 with the following properties: (1) C_1 contains C in its interior and C contains C_2 in its interior; (2) the function $f(z)$ is analytic in the closed annular region bounded by C_1 and C_2. Then we may write the following formula for the Lagrange polynomial which interpolates to the function $f(z)$ in the points G_n:

$$L_n(z) = \frac{1}{2\pi i} \int_{C_1+C_2} \frac{f(t)}{t - z} \left[1 - \frac{\omega_n(z)}{\omega_n(t)}\right] dt.$$

* A number of writers have employed the limit, $\lim_{n\to\infty} |\pi_n(\bar{w})|^{1/n}$, $|\bar{w}| > 1$; see for example Fejér, loc. cit., pp. 322–324, and Kalmár, Mathematikai és Physikai Lapok, vol. 33 (1926), pp. 120–140.

(The integration over C_2 is taken in a sense which is positive with respect to the region exterior to C_2.) This is a slight extension of the Cauchy-Hermite form of the Lagrange polynomial; its validity may easily be checked by noting that both integrals represent polynomials in z of degree $n-1$, and that when $z=z_k^{(n)}$, we have

$$L_n(z_k^{(n)}) = \frac{1}{2\pi i}\int_{C_1+C_2} \frac{f(t)}{t - z_k^{(n)}}\,dt = f(z_k^{(n)}).$$

We may write

$$\begin{aligned} &| L_n(z) - f_1(z) | \\ &\leqq \frac{1}{2\pi}\int_{C_1}\left|\frac{f(t)}{t-z}\right|\left|\frac{\omega_n(z)}{\omega_n(t)}\right| |dt| + \frac{1}{2\pi}\int_{C_2}\left|\frac{f(t)}{t-z}\right|\left|1-\frac{\omega_n(z)}{\omega_n(t)}\right| |dt|, \end{aligned} \tag{5}$$

where

$$f_1(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,dt = \frac{1}{2\pi i}\int_{C_1}\frac{f(t)}{t-z}\,dt,$$

z being interior to the curve C. Fejér has shown that with the present choice of G_n and $\omega_n(z)$,

$$| \omega_n(z) |^{1/n} \to | c | \, | \phi^{-1}(z) | \tag{6}$$

for all z exterior to C and uniformly for z on any closed point set exterior to C.* On the other hand if z lies interior to C, Lemma II indicates that

$$| \omega_n(z) |^{1/n} \to | c | \tag{7}$$

and that

$$\left| 1 - \frac{\omega_n(z)}{\omega_n(t)} \right| \to 0.$$

Let z be any point on a Jordan curve C' lying between C and C_2 and containing C_2 in its interior. Then combining (6) and (7), we obtain

$$\frac{\omega_n(z)}{\omega_n(t)} \to 0$$

uniformly for z on C', t on C_1; so that inequality (5) implies that $L_n(z)\to f_1(z)$ uniformly for z on C'. The principle of the maximum then tells us that the sequence $\{L_n(z)\}$ approaches the same limit for z interior to C'. We have proved the following theorem:

* Fejér, loc. cit., pp. 322–324. See also the remark following Lemma III and the accompanying footnote.

THEOREM I. *Let C be a Jordan curve which satisfies condition* (a) *and let $f(z)$ be a function analytic on C. The sequence $\{G_n\}$ corresponding to C yields effective interpolation to the function $f(z)$.*

If we interpolate to a function $f(z)$ which is only known to be bounded in modulus and integrable (R) on the curve C, we can no longer use the convenient Cauchy-Hermite formula. To study the convergence in this case, we assume that the curve C satisfies both conditions (a) and (b) and that the tangential derivative of the corresponding mapping function on the circle $|w|=1$ is bounded in modulus and integrable (R).

Let S be an arbitrary closed point set of the region B, let $t=\phi(e^{i\theta})$ and let $z_k^{(n)}=\phi(e^{2\pi ik/n})$. Since

$$\pi_n(e^{i\theta}) = \frac{\omega_n(t)}{c^n(e^{in\theta}-1)},$$

we have

$$\frac{\omega_n(t)-\omega_n(z_k^{(n)})}{t-z_k^{(n)}} = \frac{\omega_n(t)}{t-z_k^{(n)}} = c^n \frac{e^{in\theta}-1}{e^{i\theta}-e^{2\pi ik/n}} \frac{e^{i\theta}-e^{2\pi ik/n}}{\phi(e^{i\theta})-\phi(e^{2\pi ik/n})}\pi_n(e^{i\theta}). \tag{8}$$

Therefore,

$$\frac{1}{\omega'(z_k^{(n)})} = \frac{1}{inc^n\pi_n(e^{2\pi ik/n})}\frac{d\phi(e^{2\pi ik/n})}{d\theta}. \tag{9}$$

We may now write

$$\begin{aligned} L_n(z) &= \sum_{k=1}^{n} \frac{f(z_k^{(n)})}{z-z_k^{(n)}} \frac{\omega_n(z)}{\omega_n'(z_k^{(n)})} \\ &= \frac{1}{2\pi i}\sum_{k=1}^{n} \frac{f[\phi(e^{2\pi ik/n})]}{[\phi(e^{2\pi ik/n})-z]} \frac{\omega_n(z)}{-c^n\pi_n(e^{2\pi ik/n})} \frac{d\phi(e^{2\pi ik/n})}{d\theta}\frac{2\pi}{n}. \end{aligned} \tag{10}$$

Lemmas II and III state that

$$\frac{\omega_n(z)}{-c^n\pi_n(e^{i\alpha})} \to 1$$

uniformly for z on S and for all real α. Therefore since both $f[\phi(e^{i\theta})]$ and $d\phi(e^{i\theta})/d\theta$ are integrable (R) and of bounded modulus, we have

$$L_n(z) \to \frac{1}{2\pi i}\int_0^{2\pi} \frac{f[\phi(e^{i\theta})]}{\phi(e^{i\theta})-z}\frac{d\phi(e^{i\theta})}{d\theta}d\theta = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}dt$$

uniformly for z on S. We have established the following theorem:

THEOREM II. *Let C be a Jordan curve which satisfies conditions* (a) *and* (b) *and for which the mapping function possesses a tangential derivative bounded in modulus and integrable* (R) *on the circle* $|w|=1$. *Let* $f(z)$ *be a function bounded in modulus and integrable* (R) *on* C. *The sequence* $\{G_n\}$ *corresponding to* C *yields effective interpolation to the function* $f(z)$.

6. **Degree of convergence; convergence on the boundary.** We shall now study the degree of convergence of the sequence $\{L_n(z)\}$ in comparison with that of any other given sequence of approximating polynomials. At the same time, we shall be able to obtain a result which casts some light on the question of convergence on the boundary C of the Jordan region under consideration.

THEOREM IIIa. *Let C be a Jordan curve which satisfies conditions* (a) *and* (b) *and for which the mapping function possesses a first tangential derivative bounded in modulus on the circle* $|w|=1$. *Let S be any point set interior to C. Let $f(z)$ be a function defined on C and S and let the polynomials $L_n(z)$ interpolate to $f(z)$ in the set G_n corresponding to C. If there exist positive numbers ϵ_n and polynomials $p_n(z)$ of respective degrees $n-1$ such that*

$$|f(z) - p_n(z)| \leqq \epsilon_n$$

for z on C and S, then

$$|f(z) - L_n(z)| \leqq K_1\epsilon_n$$

for z on S, where K_1 depends only on C and S.

THEOREM IIIb. *Let C be a Jordan curve which satisfies condition* (b) *and for which the mapping function possesses the following property*:

$$\left| \frac{d\phi(e^{i\theta})}{d\theta} \cdot \frac{e^{i\theta} - e^{i\alpha}}{\phi(e^{i\theta}) - \phi(e^{i\alpha})} \right| \leqq M, \text{ all } \theta \text{ and all } \alpha.$$

Let $f(z)$ be a function defined on C, and let the polynomials $L_n(z)$ interpolate to $f(z)$ in the set G_n corresponding to C. If there exist positive numbers ϵ_n and polynomials $p_n(z)$ of respective degrees $n-1$ such that

$$|f(z) - p_n(z)| \leqq \epsilon_n$$

for z on C, then

$$|f(z) - L_n(z)| \leqq K_2\epsilon_n \log n, \; n > 1,$$

for z on C, where K_2 depends only on C.

The restrictions on the curve C in both theorems are satisfied by a curve

for which the mapping function possesses a non-vanishing first tangential derivative on the circle $|w|=1$, and a second tangential derivative satisfying a Lipschitz condition with a positive exponent.

For the proofs of these theorems we first consider the polynomial $\Lambda_n(z)$ which interpolates in the set G_n to a function $F(z)$ of bounded modulus on the curve C. Let F be an upper bound to the modulus of this function. If the curve C satisfies the conditions of Theorem IIIa, we may conclude at once, by referring to (10) and the reasoning which accompanies that equation, that there exists a positive number K_1 such that $|\Lambda_n(z)| \leqq (K_1-1) F$ for all n and for all z on S. The number K_1 depends only on C and S.

If the curve C satisfies the conditions of Theorem IIIb, we proceed as follows. Using (8), (9), and (10), and setting $z=\phi(e^{i\theta})$, we write

$$\Lambda_n(z) = \frac{1}{in}\sum_{k=1}^{n} F\left[\phi(e^{2\pi ik/n})\right]\frac{d\phi(e^{2\pi ik/n})}{d\theta}\,\frac{e^{i\theta}-e^{2\pi ik/n}}{\phi(e^{i\theta})-\phi(e^{2\pi ik/n})}\,\frac{\pi_n(e^{i\theta})}{\pi_n(e^{2\pi ik/n})}\,\frac{e^{in\theta}-1}{e^{i\theta}-e^{2\pi ik/n}}.$$

Lemma III establishes the existence of a positive number M_1 such that

$$\left|\frac{\pi_n(e^{i\theta})}{\pi_n(e^{i\alpha})}\right| \leqq M_1$$

for all n and for all real θ and α. Also, it can be shown that for $n>1$,

$$\sum_{k=1}^{n}\left|\frac{e^{in\theta}-1}{e^{i\theta}-e^{2\pi ik/n}}\right| = \sum_{k=1}^{n}\left|\frac{\sin\frac{1}{2}n\left(\theta-\frac{2\pi k}{n}\right)}{\sin\frac{1}{2}\left(\theta-\frac{2\pi k}{n}\right)}\right| \leqq M_2\log n,$$

where M_2 is independent of θ and n.* We may therefore write

$$|\Lambda_n(z)| \leqq FMM_1M_2\log n \leqq (K_2\log n - 1)F$$

for all z on C and all $n>1$, where K_2 is independent of n.

The remaining steps in the proofs of the two theorems can now be given simultaneously. If we let $F(z)=p_n(z)-f(z)$, then $F=\epsilon_n$, and we have

$$|\Lambda_n(z)| \leqq \begin{cases}(K_1-1)\epsilon_n, & z \text{ on } S, \text{ Theorem IIIa},\\ (K_2\log n - 1)\epsilon_n, & z \text{ on } C, \text{ Theorem IIIb}, n>1.\end{cases}$$

Therefore

$$|f(z)-p_n(z)+\Lambda_n(z)| \leqq \begin{cases}K_1\epsilon_n, & z \text{ on } S, \text{ Theorem IIIa},\\ K_2\epsilon_n\log n, & z \text{ on } C, \text{ Theorem IIIb}, n>1.\end{cases}$$

* For the proof of this inequality, see Jackson, *The Theory of Approximation*, New York, 1930, p. 120.

But $p_n(z)-\Lambda_n(z)\equiv L_n(z)$, so the proofs are complete.

If $\epsilon_n \log n \to 0$, then we obtain convergence of the sequence $\{L_n(z)\}$ on the curve C in Theorem IIIb. There is no implication in either Theorem IIIa or Theorem IIIb, however, that the numbers ϵ_n tend to zero; they may be any positive numbers whatsoever.

The example given by Fejér to establish the possibility of divergence on the unit circle in Theorem A employed a function $f(z)$ which was analytic throughout the interior of the unit circle and continuous in the corresponding closed region. Theorem IIIb permits us to make the general assertion in connection with this example that if the function $f(z)$ is continuous in the closed region $B+C$ and analytic in the region B (where C satisfies the condition of the theorem), then $L_n(z)=o(\log n)$ for z on C; for by a theorem due to Walsh there exist polynomials such that the corresponding numbers ϵ_n tend to zero.* Moreover, if the curve C is analytic, if the function $f(z)$ is continuous in the closed region $B+C$ and analytic in the region B, and if the pth tangential derivative of $f(z)$ on C satisfies a Lipschitz condition with exponent $\alpha>0$, then by Theorem IIIa,

$$|f(z)-L_n(z)| \leqq \frac{M_3 \log n}{n^{p+\alpha}}, \qquad z \text{ on } S,$$

and by Theorem IIIb,

$$|f(z)-L_n(z)| \leqq \frac{M_4 (\log n)^2}{n^{p+\alpha}}, \quad z \text{ on } C,$$

where M_3 and M_4 are both independent of n and z. This result follows directly from Theorem B. In particular, the value zero is admissible for p in these inequalities, so a sufficient condition for the convergence of the sequence $\{L_n(z)\}$ in the closed region $B+C$ is that the function $f(z)$ be continuous in the closed region, analytic in the region B, and satisfy a Lipschitz condition with positive exponent on C.

7. **Simultaneous interpolation to several functions.** We now turn to certain immediate generalizations of the results of §5. First of all, it is natural to inquire whether Theorem II admits of some form of extension when the Lagrange polynomial is defined by the requirement of coinciding, not with one function $f(z)$ at points of C, but simultaneously with several distinct functions in several distinct sets of points on C. This problem may readily be attacked by the methods which we have previously developed, and the following theorem indicates the type of result to be expected.

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* Mathematische Annalen, vol. 96 (1926), pp. 430–436.

THEOREM IV. *Let C be subject to the restrictions of Theorem* II. *Let* $f_1(z)$, $f_2(z), \cdots, f_m(z)$ *be m functions which are bounded in modulus and integrable* (*R*) *on C. Let* $L_{mn}(z)$ *be the polynomial of degree at most* $mn-1$ *which interpolates to the function* $f_\mu(z)$ *in the points*

$$z_{\mu,k}^{(n)} = \phi(e^{2\pi i\mu/(mn)+2\pi ik/n}) \qquad (k = 1, 2, \cdots, n; \mu = 1, 2, \cdots, m).$$

Then

$$L_{mn}(z) \to \frac{1}{m}\sum_{\mu=1}^{m} \frac{1}{2\pi i}\int_C \frac{f_\mu(t)dt}{t-z}$$

uniformly for z on an arbitrary closed point set S interior to C.

The proof of this theorem is based on the fact that the nth polynomial under consideration may be written in the following manner:

$$L_{mn}(z) = \sum_{\mu=1}^{m}\sum_{k=1}^{n} \frac{f_\mu(z_{\mu,k}^{(n)})}{z - z_{\mu,k}^{(n)}} \frac{\Omega_n(z)}{\Omega_n'(z_{\mu,k}^{(n)})},$$

where

$$\Omega_n(z) = \prod_{\mu=1}^{m}\prod_{k=1}^{n}(z - z_{\mu,k}^{(n)}).$$

The rest of the proof follows the procedure used in that of Theorem II, with certain minor modifications.

The remainder of this section will be devoted to the discussion of two aspects of the problem of interpolation simultaneously to a finite number of functions defined respectively on the same number of Jordan curves. The first case is that in which the curves are all mutually exterior, and the second is that in which the curves lie one within another.

It is possible to generalize the theorem of Fejér mentioned in §2 to the case of a finite number of functions analytic on and within the same number of mutually exterior Jordan curves. The details have been carried through by Walsh,* who made use of the function $w = e^{G(x,y)+iH(x,y)}$, where $G(x, y)$ is the Green's function with pole at infinity for the region R exterior to the curves under consideration, and $H(x, y)$ is the harmonic conjugate of $G(x, y)$. This function maps R conformally, but not uniformly, onto the exterior of the unit circle in the w-plane so that the points at infinity in the two planes correspond.

But no similar extension of either Theorem I or Theorem II is possible with the use of this mapping function. First it should be noted that now cer-

* Unpublished.

tain of the points of the nth set G_n may coincide, because neither the function nor its inverse is single-valued if the region R is multiply connected. Thus we are no longer dealing with strictly the Lagrange type of polynomial, but rather with the Hermite type, and the existence of derivatives of the function to which we are interpolating must be postulated at the points of G_n. This fact alone precludes the possibility of generalizing Theorem II by the use of this mapping function. The Cauchy-Hermite formula used to prove Theorem I is applicable when some or all of the points of interpolation are coincident; nevertheless we shall be able to show by an example that Theorem I cannot be extended either.

The function

$$w = \frac{(z^2 - 1)^{1/2}}{\mu^{1/2}}, \qquad 0 < \mu < 1,$$

gives a map, of the type under consideration, of the region exterior to the lemniscate $|z^2-1| = \mu$ onto the exterior of the unit circle in the w-plane. This lemniscate consists of the two ovals of Cassini, and if we denote the two branches of the inverse function by

$$z = + (\mu w^2 + 1)^{1/2}$$

and

$$z = - (\mu w^2 + 1)^{1/2},$$

the right hand oval may be considered as the transform of the unit circle under the first branch, and the left hand oval, the transform under the second branch. We form the Hermite interpolation formula for the function $1/(z-1)$, using as the set G_n the following transforms of the roots of the equation $w^{2n}-1=0$:

$$\left.\begin{aligned} z_{1,k}^{(n)} &= + (\mu e^{2\pi i k/n} + 1)^{1/2} \\ z_{2,k}^{(n)} &= - (\mu e^{2\pi i k/n} + 1)^{1/2} \end{aligned}\right\} \qquad (k = 1, 2, \cdots, 2n).$$

Then

$$L_{4n}(z) = (z + 1)\frac{(z^2 - 1)^{n-1}}{\mu^n}\left[2 - \left(\frac{z^2 - 1}{\mu}\right)^n\right],$$

as the reader may verify directly. When the point z lies interior to either oval, then $|z^2-1| < \mu$, so $L_n(z) \to 0$ for all points z within the ovals. But in the left oval O_1 we are seeking convergence to the value

$$\frac{1}{2\pi i}\int_{O_1} \frac{1}{t-1}\,\frac{1}{t-z}\,dt = \frac{1}{z-1},$$

the function $1/(z-1)$ being analytic on and within this oval. Thus Theorem I fails to generalize under this type of map.*

If the curves upon which the functions are defined lie one within another, we obtain a class of results of which the following theorems may be considered typical. For the sake of simplicity we shall state the theorems for the case of only two curves.

THEOREM V. *Let C_1 and C_2 be two Jordan curves subject to the restrictions upon C in Theorem* II, *C_2 lying interior to C_1. Let $\phi_1(w)$ denote the function which maps the exterior of the circle $|w|=1$ onto the exterior of C_1 so that the points at infinity in the z-plane and the w-plane correspond, and let $\phi_2(w)$ denote the analogous function for C_2. Let $F(n)$ be a monotonically increasing function of n such that $F(n)\to\infty$. Let $f(z)$ be a function bounded in modulus and integrable (R) on C_2. Let $\{\nu_1^{(m)}, \nu_2^{(m)}, \cdots, \nu_m^{(m}\}$, $m=[F(n)]$,† denote a sequence of sets of m numbers which is subject to the restriction that no number shall exceed a given fixed number in modulus. Form the Lagrange polynomial $L_{n+m}(z)$ of degree at most $n+m-1$ which takes on the values $\nu_h^{(m)}$ in the points $\phi_1(e^{2\pi ih/m})$, $h=1, 2, \cdots, m$, and which coincides with $f(z)$ in the points $\phi_2(e^{2\pi ik/n})$, $k=1, 2, \cdots, n$. Then*

$$L_{n+m}(z) \to \frac{1}{2\pi i}\int_{C_2} \frac{f(t)}{t-z}\, dt$$

uniformly for z on any closed point set interior to C_2.

For the proof of the theorem we employ a process similar to that used in the proof of Theorem II. The details are left to the reader.

The parallel theorem for two functions respectively analytic on and within the two curves C_1 and C_2 permits greater freedom in the choice of the curves and the function $F(n)$:

THEOREM VI. *Let C_1 and C_2 be two arbitrary Jordan curves, C_2 lying interior to C_1. Let $f_1(z)$ be a function analytic on and within C_1, and let $f_2(z)$ be a function analytic on and within C_2. Let $\phi_1(w)$ and $\phi_2(w)$ denote the mapping functions corresponding to the curves C_1 and C_2 respectively. Consider as points of interpolation to $F_2(z)$ the points $\phi_2(e^{2\pi ik/n})$ and as points of interpolation to $f_1(z)$ the points $\phi_1 e^{2\pi ih/n}$; where $m=[F(n)]$, $F(n)$ being either a positive constant or a posi-*

* This is the mapping function which has been used most frequently in the generalization to several regions of theorems concerning approximation in the complex domain. See for example Walsh and Russell, these Transactions, vol. 36 (1934), pp. 13–28. The present writer has investigated the use of other mapping functions in extensions of Theorems I and II in this direction, but so far with only negative results.

† The symbol $[x]$ means the greatest integer not greater than x.

tive monotonically increasing or decreasing function of n. Then the sequence $\{L_{n+m}(z)\}$ *of corresponding Lagrange polynomials converges to* $f_2(z)$ *geometrically for z on and within* C_2.

This theorem may be proved by writing down the appropriate extension of the Cauchy-Hermite formula and then applying (6) and Lemma III.

Divergence to infinity is possible in the annular region between C_1 and C_2 in both Theorems V and VI, as can be shown by example. The restriction to only two curves is not important, as any finite number of curves may be considered; the result will always be convergence to the value to be expected from interpolation only to the function defined on the innermost curve, for the sequence of Lagrange polynomials will ignore interpolating values assigned to outer curves. The study of combinations of the two theorems yields similar results.

It is worth pointing out that although m may remain constant with respect to n in Theorem VI, it is necessary in Theorem V that m approach infinity in some manner with n, as the following example indicates: Interpolate to the function $1/z$ in the points $e^{2\pi ik/n}$, $k=1, 2, \cdots, n$, and also in the points $Re^{2\pi ih/m}$, $h=1, 2, \cdots, m$. The corresponding Lagrange polynomial is

$$L_{n+m}(z) = \frac{1}{z}\left[1 - \left(\frac{z^n - 1}{-1}\right)\left(\frac{z^m - R^m}{-R^m}\right)\right];$$

and if m remains finite as n approaches infinity, it is apparent that the sequence $\{L_{n+m}(z)\}$ will not approach the value

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{1}{t}\,\frac{1}{t-z}\,dt = 0$$

for z interior to the circle $|z| = 1$.

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ABSTRACT THEORY OF INVERSION OF FINITE SERIES*

BY

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1. **Introduction.** The summation of a number-theoretic function $f(n)$ over the divisors of n, and the inversion of a series of this type by means of Dedekind's inversion formula, occupy a prominent place in the elementary theory of numbers.† A similar inversion formula is valid in any system whose elements are commutative with respect to a multiplication operation with respect to which a unique factorization law holds, if every element has only a finite number of divisors: for example, primary polynomials in a field, and ideals of an algebraic field.

There are, however, systems for which a *divisor relation* may be properly defined, but for which no unique factorization law holds, and, indeed, in which no rule of multiplication may be defined, as the concept of a divisor is abstractly independent of that of multiplication. For a system of this character the extension of Dedekind's inversion formula is not obvious.

An important example is the class of all subgroups of a finite group, with "divisor" defined to mean "subgroup." The problem suggested by Dedekind's inversion formula may be stated as follows: Suppose we are given two group-theoretic functions $\alpha(G)$ and $\beta(G)$, such that

$$\beta(G) = \sum \alpha(D),$$

where D ranges over the subgroups of G. Can $\alpha(G)$ be expressed in terms of $\beta(G)$ by means of a generalized Dedekind inversion formula with the aid of a generalized Möbius function? One of the objects of this paper is to answer this question.

Instead of confining my attention to this particular question I have treated the subject abstractly, showing that an inversion formula exists in any *hierarchy* (a system satisfying the axioms of §2). A hierarchy is somewhat similar to what has been called a *dual group*,‡ an *A-Menge*,§ a

* Presented to the Society, February 23, 1935; received by the editors December 5, 1934.

† Dickson, *History of the Theory of Numbers*, vol. 1, chapter XIX.

‡ R. Dedekind, *Über Zerlegungen von Zahlen durch ihre grössten gemeinsamen Teiler*, Werke, vol. 2, p. 112; *Über die von drei Moduln erzeugte Dualgruppe*, Werke, vol. 2, p. 236.

§ Fritz Klein, *Zur Theorie der abstrakten Verknüpfungen*, Mathematische Annalen, vol. 105 (1931), p. 310.

*lattice** and a structure.† These are systems which are closed with respect to two operations defined abstractly so as to have the essential properties of a greatest common divisor and a least common multiple respectively, or of a logical product and a logical sum respectively. I mention them because many examples of hierarchies will be found among those of dual groups, etc. However, inversion formulas of the type referred to do not exist in the most general type of dual groups.

2. **Hierarchy axioms.** A class H, consisting of at least one element, is a *hierarchy* with respect to a relation / if the following axioms (in which a, $b, \cdots$ denote elements of H) are satisfied:

1. The relation / is reflexive: a/a.‡
2. The relation / is asymmetric: if a/b and b/a, then $a=b$.
3. The relation / is transitive: if a/b and b/c, then a/c.
4. For every pair of elements a and b of H an element d of H exists such that d/a and d/b; and such that if c is an element of H satisfying c/a and c/b, then c/d.
5. For every pair of elements a and b of H an element l of H exists such that a/l and b/l; and such that if c is an element of H satisfying a/c and b/c, then l/c.
6. For every pair of elements a and b of H only a finite number of elements x of H exist such that $a/x/b$.

A simple example of a hierarchy is the class of all positive integers with respect to the divisor relation, so that a/b means "a is a divisor of b."§ In view of this example and the previously described purpose of this paper, the notation a/b may be read "a is a divisor of b" for any abstract hierarchy, *divisor* being regarded as an undefined term subject to the hierarchy axioms.

The converse of the relation / will be denoted by \. Thus a/b and $b\backslash a$ are equivalent. The notation $b\backslash a$ may be read "b is a *multiple* of a."

To every term defined in terms of the relation / there corresponds a *dual*, obtained by replacing / by \ in the definition. For example, *divisor* and *multiple* are duals.

We shall call the elements d and l of Axioms 4 and 5 a *greatest common divisor* and a *least common multiple* respectively of a and b. (After proving their uniqueness, we shall call them *the* g.c.d. and *the* l.c.m. respectively.) These terms are duals.

* Garrett Birkhoff, *On the combination of subalgebras*, Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), p. 441; *On the lattice theory of ideals*, Bulletin of the American Mathematical Society, vol. 40 (1934), p. 613.

† O. Ore, *On the foundations of abstract algebra*, I, Annals of Mathematics, vol. 36 (1935), p. 408.

‡ The notation a/b means "a has the relation / to b." The notation $a/x/b$ means "a/x and x/b."

§ Other examples will be found among those given in the papers cited in §1.

If, in the hierarchy axioms, the symbol / is replaced by \, six theorems are obtained which are immediate consequences of the axioms. Hence: *A class which is a hierarchy with respect to a certain relation is also a hierarchy with respect to the converse relation.* It follows that *a true proposition is obtained on replacing each term by its dual in any theorem which is a consequence of the hierarchy axioms.* This is the principle of duality for hierarchies. For example, Axioms 4 and 5 are duals, while each of the other axioms is self-dual.

3. **The g.c.d. and l.c.m. of a set of elements.*** Let $a_1, \cdots, a_n (n \geqq 1)$ be a set of elements of a hierarchy H. If an element d of H exists such that

$$d/a_i \qquad (i = 1, \cdots, n),$$

and such that if c is an element of H satisfying

$$c/a_i \qquad (i = 1, \cdots, n),$$

then c/d, we shall call d a g.c.d. of $a_1, \cdots, a_n$. If an element l of H exists such that

$$a_i/l \qquad (i = 1, \cdots, n),$$

and such that if c is an element of H satisfying

$$a_i/c \qquad (i = 1, \cdots, n),$$

then l/c, we shall call l a l.c.m. of $a_1, \cdots, a_n$.

THEOREM 1. *A g.c.d. and a l.c.m. of any finite set of elements of a hierarchy exist and are unique elements of the hierarchy.*

In view of the principle of duality it is sufficient to prove the existence and uniqueness of a g.c.d.

The existence of a g.c.d. of a set consisting of only one element follows from Axiom 1: the element itself is a g.c.d. (as well as a l.c.m.). The existence of a g.c.d. of a set consisting of two elements is asserted by Axiom 4. We shall prove the theorem by complete induction, assuming that every set of $n-1$ $(n \geqq 3)$ elements of H has at least one g.c.d., and proving that the same is true of a given set of n elements $a_1, \cdots, a_n$.

By assumption, $a_1, \cdots, a_{n-1}$ have a g.c.d., δ. Let d be a g.c.d. of δ and a_n. As d/a_n and d/δ,

$$d/a_i \qquad (i = 1, \cdots, n),$$

by Axiom 3. Suppose that

$$c/a_i \qquad (i = 1, \cdots, n).$$

* No use is made of Axiom 6 in this section.

Writing these n statements in two parts

$$c/a_n, \quad c/a_i \qquad\qquad (i = 1, \cdots . n-1),$$

we infer that c/a_n and c/δ. Hence c/d. It follows from the definition that d is a g.c.d. of $a_1, \cdots, a_n$.

If d' is also a g.c.d. of $a_1, \cdots, a_n$, then d/d' and d'/d by the definition of g.c.d. Hence $d=d'$ by Axiom 2. The proof of the theorem is complete.

The notation $(a_1, \cdots, a_n)$ and $a_1 \wedge \cdots \wedge a_n$ will be employed for the g.c.d. and l.c.m. respectively of $a_1, \cdots, a_n$. The uniqueness part of Theorem 1 implis that the g.c.d. and l.c.m. of a set of elements are independent of the order in which these elements are taken. The following relations are readily established:

(1) $(a, a) = a \wedge a = a.$

(2) $(a_1, a_2) = (a_2, a_1), \; a_1 \wedge a_2 = a_2 \wedge a_1.$

(3) $((a_1, \cdots, a_n), (b_1, \cdots, b_m)) = (a_1, \cdots, a_n, b_1, \cdots, b_m),$
$(a_1 \wedge \cdots \wedge a_n) \wedge (b_1 \wedge \cdots \wedge b_m) = a_1 \wedge \cdots \wedge a_n \wedge b_1 \wedge \cdots \wedge b_m.$

(4) $a \wedge (a, b) = a, \; (a, a \wedge b) = a.$

(5) If c/a, then $(b, c)/(b, a)$ and $(b \wedge c)/(b \wedge a)$.

4. **Finite subhierarchies.** Let $\tau(x_1, x_2)$ be the number of divisors of x_2 that are multiplies of x_1. By Axiom 6, this number is finite. If x_1/x_2, we shall write $\tau(x_1/x_2)$ for $\tau(x_1, x_2)$. Evidently $\tau(x_1, x_2)=0$ if x_1 is not a divisor of x_2; $\tau(x/x)=1$; while $\tau(x_1/x_2) \geqq 2$ if $x_1 \neq x_2$.

A *finite* hierarchy is one which contains only a finite number elements. This number is the *order* of the hierarchy.

THEOREM 2. *If x_1/x_2, the class of all elements x of H which satisfy $x_1/x/x_2$ is a finite hierarchy, of order $\tau(x_1/x_2)$, with respect to the relation* /.

The proof is immediate, consisting principally in showing that the elements of H which satisfy $x_1/x/x_2$ verify the hierarchy axioms. We shall denote this *subhierarchy* of H by $H(x_1/x_2)$.

If x_1/x_2, but $x_1 \neq x_2$, x_1 is a *proper* divisor of x_2, and x_2 is a *proper* multiple of x_1. If x_1 is a proper divisor of x_2 and the order of the finite hierarchy $H(x_1/x_2)$ is 2, x_1 is a *maximal* divisor of x_2, and x_2 is a *minimal* multiple of x_1.

THEOREM 3. *If x_1 is a proper divisor of x_2, H contains at least one divisor of x_2 that is a minimal multiple of x_1; and H contains at least one multiple of x_1 that is a maximal divisor of x_2.*

If $\tau(x_1/x_2)=2$, x_2 is a minimal multiple of x_1. In the contrary case $H(x_1/x_2)$

contains at least one element x_3 different from x_1 and x_2. Evidently $\tau(x_1/x_2) > \tau(x_1/x_3) \geqq 2$. If $\tau(x_1/x_3) > 2$, the preceding argument is repeated for $H(x_1/x_3)$; etc. Finally an x_n is obtained such that $\tau(x_1/x_n) = 2$. This element x_n is a divisor of x_2 and a minimal multiple of x_1.

The second part of the theorem is the dual of the first.

5. **Functions of the elements of a hierarchy.** The symbol $f(x_1/x_2)$ (and similarly $g(x_1/x_2), \cdots$) denotes a single-valued function of two independent variables, defined for every pair of elements x_1 and x_2 of a hierarchy, subject to x_1/x_2, the values which the function assumes being elements of some module. Similarly $f(a/x)$ denotes a function of a single variable x, defined for every x which is a multiple of a *fixed* element a. Dually, we have $f(x/a)$. The functions $f(a/x)$ and $f(x/a)$ are not necessarily defined for *every* a. However, for every $f(x_1/x_2)$ we have an $f(a/x)$ and an $f(x/a)$, where a is any element of the hierarchy.

The symbol

$$\sum_{x_1/x_2/\cdots/x_{n-1}/x_n}$$

pertains to a summation extended over all elements $x_2, \cdots, x_{n-1}$ of a hierarchy H satisfying $x_1/x_2/ \cdots /x_{n-1}/x_n$, where x_1 and x_n are *fixed* elements of H. Hence $n \geqq 3$. In particular,

$$\sum_{a/d/b}$$

pertains to a summation extended over all elements d of H that are divisors of b and multiples of a; that is, over the elements of the finite hierarchy $H(a/b)$.

THEOREM 4. *If, for every multiple x of a,*

$$\sum_{a/d/x} f(a/d) = \sum_{a/d/x} g(a/d),$$

then $f(a/x) = g(a/x)$.

We shall prove the theorem by complete induction. For $x = a$ we have $f(a/a) = g(a/a)$. Now let b be a proper multiple of a. Suppose that we have verified that, for every multiple d of a that is a proper divisor of b, $f(a/d) = g(a/d)$. Then

$$\sum_{\substack{a/d/b \\ d \neq b}} f(a/d) = \sum_{\substack{a/d/b \\ d \neq b}} g(a/d).$$

By hypothesis,

$$\sum_{a/d/b} f(a/d) = \sum_{a/d/b} g(a/d).$$

Subtracting, we have $f(a/b) = g(a/b)$.

The dual of this theorem is

THEOREM 5. *If, for every divisor x of a,*

$$\sum_{x/d/a} f(d/a) = \sum_{x/d/a} g(d/b),$$

then $f(x/a) = g(x/a)$.

6. **The function $\mu(x_1/x_2)$ and related functions.** A *P-divisor* of an element x_2 of a hierarchy H is a divisor of x_2 that has the property P or the relation P to x_2. If x_1 is a P-divisor of x_2, x_2 is a *P'-multiple of* x_1. Examples: $P=P'$ = proper; P = maximal, P' = minimal.

Let $P(x_1/x_2)$ be the number of multiples of x_1 that are P-divisors of x_2; let $P'(x_1/x_2)$ be the number of divisors of x_2 that are P'-multiples of x_1. These functions are duals. For each integer $k \geqq 1$, let $Q_k(x_1/x_2)$ be the number of sets of k distinct elements of H that are P-divisors of x_2 and such that the g.c.d. of the elements of each set is x_1; let $Q_k'(x_1/x_2)$ be the number of sets of k distinct elements of H that are P'-multiples of x_1 and such that the l.c.m. of the elements of each set is x_2.

There are

$$\binom{P(x_1/x_2)}{k}$$

sets of k distinct elements of H that are multiples of x_1 and P-divisors of x_2. Form the g.c.d. of the elements of each set. The number of times that a particular element d of H, satisfying $x_1/d/x_2$, occurs among these g.c.d.'s is, by definition, $Q_k(d/x_2)$. Hence

$$\sum_{x_1/d/x_2} Q_k(d/x_2) = \binom{P(x_1/x_2)}{k} \qquad (k = 1, 2, \cdots). \tag{6}$$

Dualizing, we have

$$\sum_{x_1/d/x_2} Q_k'(x_1/d) = \binom{P'(x_1/x_2)}{k} \qquad (k = 1, 2, \cdots). \tag{7}$$

For the further development of the theory we find it necessary to restrict P so that

$$P(x/x) = P'(x/x) = 0, \tag{8}$$

$$P(x_1/x_2)P'(x_1/x_2) \neq 0 \qquad (x_1 \neq x_2). \tag{9}$$

These conditions are satisfied if P = proper, or P = maximal (Theorem 3). It follows from (8) that

(10) $$Q_k(x/x) = Q_k'(x/x) = 0 \qquad (k = 1, 2, \cdots).$$

The function $\mu(x_1/x_2)$ is defined by

(11) $$\mu(x/x) = 1,$$

(12) $$\mu(x_1/x_2) = \sum_{k=1}^{\infty} (-1)^k Q_k(x_1/x_2) \qquad (x_1 \neq x_2).$$

The series involves only a finite number of terms, as

(13) $$Q_k(x_1/x_2) = 0 \qquad (k > P(x_1/x_2)).$$

The dual function $\mu'(x_1/x_2)$ is defined by

(14) $$\mu'(x/x) = 1,$$

(15) $$\mu'(x_1/x_2) = \sum_{k=1}^{\infty} (-1)^k Q_k'(x_1/x_2) \qquad (x_1 \neq x_2).$$

It is noteworthy that $\mu(x_1/x_2)$ and $\mu'(x_1/x_2)$ *are independent of P if* (8) *and* (9) *are satisfied, and that* $\mu(x_1/x_2) = \mu'(x_1/x_2)$.* We proceed to prove these statements.

THEOREM 6.

$$\sum_{x_1/d/x_2} \mu(d/x_2) = \begin{cases} 1 \ \textit{if}\ x_1 = x_2, \\ 0 \ \textit{if}\ x_1 \neq x_2. \end{cases}$$

The theorem being obvious if $x_1 = x_2$, we suppose $x_1 \neq x_2$. By (12),

$$\begin{aligned}
\sum_{x_1/d/x_2} \mu(d/x_2) &= \mu(x_2/x_2) + \sum_{\substack{x_1/d/x_2 \\ d \neq x_2}} \sum_{k=1}^{\infty} (-1)^k Q_k(d/x_2) \\
&= 1 + \sum_{x_1/d/x_2} \sum_{k=1}^{\infty} (-1)^k Q_k(d/x_2) && \text{(by (11) and (10))} \\
&= 1 + \sum_{k=1}^{P(x_1/x_2)} (-1)^k \binom{P(x_1/x_2)}{k} && \text{(by (6))}, \\
&= (1-1)^{P(x_1/x_2)} = 0 && \text{(by (9))}.
\end{aligned}$$

Dualizing, we have

(16) $$\sum_{x_1/d/x_2} \mu'(x_1/d) = \begin{cases} 1 \ \text{if}\ x_1 = x_2, \\ 0 \ \text{if}\ x_1 \neq x_2. \end{cases}$$

* Consider, for example, the hierarchy, with respect to the subgroup relation, formed by the subgroups of a finite group G. If P = maximal, the equation $\mu(1/G) = \mu'(1/G)$, in which 1 stands for the identity group, embodies a relation between the maximal and the minimal subgroups of G, the minimal subgroups being those of prime order if the order of G is not a prime. This relation would be too cumbersome to be expressed in words.

It follows from Theorems 4 and 5 that the functions $\mu(x_1/x_2)$ *and* $\mu'(x_1/x_2)$ *are independent of P if P satisfies* (8) *and* (9).

Let

$$f(x_1/x_2) = \sum_{x_1/\delta/x_2} \mu(x_1/\delta).$$

Then

$$\sum_{x_1/d/x_2} f(d/x_2) = \sum_{x_1/d/\delta/x_2} \mu(d/\delta) = \sum_{x_1/\delta/x_2} \sum_{x_1/d/\delta} \mu(d/\delta).$$

Hence, by Theorem 6,

$$\sum_{x_1/d/x_2} f(d/x_2) = 1. \tag{17}$$

Let $g(x_1/x_2) = 1$ or 0 according as $x_1 = x_2$ or $x_1 \neq x_2$. Then

$$\sum_{x_1/d/x_2} g(d/x_2) = 1.$$

Comparing with (17), we have $f(x_1/x_2) = g(x_1/x_2)$ by Theorem 5. Hence, from the definitions of these functions, we have

THEOREM 7.

$$\sum_{x_1/d/x_2} \mu(x_1/d) = \begin{cases} 1 & \textit{if } x_1 = x_2, \\ 0 & \textit{if } x_1 \neq x_2. \end{cases}$$

Comparing with (16), we have

THEOREM 8. $\mu(x_1/x_2) = \mu'(x_1/x_2)$.

THEOREM 9. *If* $x_1/x_2/x_3$, *and* $x_2 \neq x_3$, *then*

$$\sum_{(d,x_2)=x_1} \mu(d/x_3) = 0,$$

where d ranges over all divisors of x_3 *that satisfy* $(d, x_2) = x_1$.

Separate the elements of the finite hierarchy $H(x_1/x_2)$ into classes, placing in the same class those elements which have the same g.c.d. with x_2. Each of these g.c.d.'s is an element of $H(x_1/x_2)$, and every element of $H(x_1/x_3)$ occurs in one and in only one of the classes. Hence, if

$$f(x_1/x_2/x_3) = \sum_{(d,x_2)=x_1} \mu(d/x_3),$$

then

$$\sum_{x_1/\delta/x_2} f(\delta/x_2/x_3) = \sum_{x_1/\delta/x_2} \sum_{(d,x_2)=\delta} \mu(d/x_3) = \sum_{x_1/d/x_3} \mu(d/x_3).$$

Hence, by Theorem 6, as $x_1 \neq x_3$,

$$\sum_{x_1/\delta/x_2} f(\delta/x_2/x_3) = 0.$$

From this equation it follows by induction, as in the proof of Theorem 4, that $f(x_1/x_2/x_3)=0$ if $x_2 \neq x_3$. The theorem follows from the definition of $f(x_1/x_2/x_3)$.

The dual of this theorem is

THEOREM 10. *If $x_1/x_2/x_3$, and $x_2 \neq x_1$, then*

$$\sum_{d \wedge x_2 = x_3} \mu(x_1/d) = 0,$$

where d ranges over all multiples of x_1 that satisfy $d \wedge x_2 = x_3$.

7. **Inversion formulas.** We proceed to answer in the affirmative the question raised in §1.

THEOREM 11. *If*

$$g(a/x) = \sum_{a/d/x} f(a/d),$$

then

$$f(a/x) = \sum_{a/d/x} \mu(d/x) g(a/d).$$

We have

$$\begin{aligned} \sum_{a/\delta/x} \mu(\delta/x) g(a/\delta) &= \sum_{a/d/\delta/x} \mu(\delta/x) f(a/d) \\ &= \sum_{a/d/x} \left(\sum_{d/\delta/x} \mu(\delta/x) \right) f(a/d) \\ &= f(a/x) \qquad \text{(by Theorem 6).} \end{aligned}$$

The dual of this theorem is

THEOREM 12. *If*

$$g(x/a) = \sum_{x/d/a} f(d/a),$$

then

$$f(x/a) = \sum_{x/d/a} \mu(x/d) g(d/a).$$

It is noteworthy that these inversion formulas are valid in any system S satisfying Axioms 1, 2, 3, and 6. In other words, there exists for such a system S a function $\mu(x_1/x_2)$ such that Theorems 11 and 12 are valid. The values assumed by this function may be calculated by induction with the aid of Theorems 6 and 7. This is clearly unsatisfactory if S is an infinite set. What is desired is a definition of the function $\mu(x_1/x_2)$ in terms of the internal struc-

ture of the system. I have been unable to provide a definition of this character without assuming Axioms 4 and 5. These axioms are verified in a sufficiently large number of important cases to warrant their inclusion in the present paper.

8. **Hierarchies containing a unit element.** *A unit element* of a hierarchy is an element which is a divisor of every element of the hierarchy. A hierarchy need not contain a unit element. For example, the class of all rational integers is a hierarchy with respect to the relation $\leqq$. The g.c.d. and l.c.m. of two elements x_1 and x_2 of this hierarchy are $\min(x_1, x_2)$ and $\max(x_1, x_2)$ respectively. The hierarchy clearly contains no unit element.

If a hierarchy contains a unit element, the number of divisors of each element of the hierarchy is finite by Axiom 6. Summations extended over all the divisors of an element are particularly important in a hierarchy having this property. Let $f(x)$ be defined for every element x of a hierarchy H containing a unit element u. Contrary to the notation of §5, we denote by

$$\sum_{d/x} f(d)$$

the sum of $f(d_1), \cdots, f(d_n)$, where $d_1, \cdots, d_n$ are the divisors of x. Define $f(u/x)$ by $f(u/x) = f(x)$. We have, by Theorem 11,

THEOREM 13. *If*

$$g(x) = \sum_{d/x} f(d),$$

then

$$f(x) = \sum_{d/x} \mu(d/x)g(d).$$

This theorem can be dualized only if H contains a *predominant* element: an element which is a multiple of every element of H and which is therefore the dual of the unit element. A hierarchy which contains a predominant element as well as a unit element is finite by Axiom 4.

For the hierarchy consisting of the positive integers, in which a/b has its usual meaning, Theorem 13 reduces to Dedekind's inversion formula; for it is readily proved from the definitions of §6 that, in this hierarchy,

$$u(x_1/x_2) = \mu\left(\frac{x_2}{x_1}\right),$$

the function in the right member being Möbius' function.

9. **An elementary application to the theory of groups.** The subgroups of a finite group G form a hierarchy with respect to the subgroup relation. In this

hierarchy D/G means that D is a subgroup of G. The g.c.d. and l.c.m. of two elements are their cross-cut and the group which they generate, respectively.

Let $\beta(\Gamma)$ be the number of subgroups of order n of the group Γ, where n is a fixed positive integer; and let $\alpha(\Gamma)$ be the number of pairs of distinct subgroups of order n of Γ that generate Γ. G contains exactly $\frac{1}{2}\beta(G)(\beta(G)-1)$ pairs of distinct subgroups of order n, and each pair generates some subgroup of G. Hence

$$\sum_{D/G} \alpha(D) = \tfrac{1}{2}\beta(G)(\beta(G) - 1).$$

By Theorem 13,

$$\alpha(G) = \frac{1}{2} \sum_{D/G} \mu(D/G)\beta(D)(\beta(D) - 1). \tag{18}$$

Now let $n = p^s$ be a prime-power integer. If p^s is not a divisor of the order of D, $\beta(D) = 0$; while if p^s is a divisor of the order of D, $\beta(D) \equiv 1 \pmod{p}$.* In either case,

$$\tfrac{1}{2}\beta(D)(\beta(D) - 1) \equiv 0 \pmod{p} \qquad (p > 2).$$

Hence $\alpha(G) \equiv 0 \pmod{p}$, by (18).

THEOREM 14. *If p^s $(p \neq 2)$ is a prime-power integer, the number of pairs of distinct subgroups of order p^s of a group G, that generate G, is either zero or a multiple of p.*

To obtain more important results, a detailed investigation must be made of the numerical properties of the function $\mu(D/G)$. I have completed this investigation for the case in which G is a prime-power group, obtaining the precise value of $\mu(D/G)$, and have deduced new and interesting properties of prime-power groups therefrom. These results will be communicated in a subsequent paper.

* G. Frobenius, *Verallgemeinerung des Sylowschen Satzes*, Berliner Sitzungsberichte, 1895, p. 989; Miller, Blichfeldt and Dickson, *Finite Groups*, 1916, p. 125.

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SOME PROPERTIES OF PRIME-POWER GROUPS*

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1. **Introduction.** I have shown in a recent paper† that inversion formulas, analogous to Dedekind's inversion formula, exist in any hierarchy. As the class of all subgroups of a finite group is a hierarchy, it is to be expected that the inversion formulas will prove useful in the theory of groups. The number of applications is at present limited because of insufficient knowledge of the generalized Möbius function, in terms of which the inversion formulas are expressed. The obstacles which present themselves in the general case do not arise in the case of prime-power groups. In the present paper I evaluate the generalized Möbius function for the hierarchy consisting of the subgroups of a prime-power group, and deduce some properties of these groups therefrom. The theorems derived, while of interest in themselves, serve to illustrate the usefulness of the inversion formulas, but by no means exhaust the list of possible applications.

2. **The inversion formulas.** Except for some obvious changes, made to conform to conventional notations of the theory of groups, I shall follow the notations of my earlier paper. For convenience of reference, I shall restate the inversion theorems and pertinent definitions.

For every pair of subgroups X_1 and X_2 of a finite group G, such that X_1 is a subgroup of X_2 (notation: X_1/X_2), the function $Q_k(X_1/X_2)$ ($k \geqq 1$) is defined as the number of sets of k distinct maximal subgroups of X_2, such that the cross-cut of each set is X_1. The function $\mu(X_1/X_2)$ is defined by

$$\mu(X_2/X_2) = 1, \qquad \mu(X_1/X_2) = \sum_k (-1)^k Q_k(X_1/X_2) \quad (X_1 \neq X_2). \tag{1}$$

The series terminates naturally. It is not difficult to prove that if X_2 is a cyclic group, and the orders of X_1 and X_2 are x_1 and x_2 respectively, then

$$\mu(X_1/X_2) = \mu(x_2 \div x_1),$$

the function in the right member being Möbius' function.

The function $\mu(X_1/X_2)$ has the following properties:

$$\sum_{X_1/D/X_2} \mu(D/X_2) = \begin{cases} 1 \text{ if } X_1 = X_2, \\ 0 \text{ if } X_1 \neq X_2. \end{cases} \tag{2}$$

* Presented to the Society, February 23, 1935; received by the editors February 3, 1935.
† In the present issue of these Transactions, 474–484.

$$\sum_{X_1/D/X_2} \mu(X_1/D) = \begin{cases} 1 \text{ if } X_1 = X_2, \\ 0 \text{ if } X_1 \neq X_2. \end{cases} \tag{3}$$

$$\sum_{(D,X_2)=X_1} \mu(D/X_3) = 0 \qquad (X_1/X_2/X_3;\ X_2 \neq X_3). \tag{4}$$

In (2) and (3) D ranges over all subgroups of X_2 that contain X_1 (including X_1 and X_2). In (4) D ranges over all subgroups of X_3 that satisfy $(D, X_2) = X_1$, where (D, X_2) denotes the cross-cut of D and X_2.

There are two inversion formulas:

I. If Γ is a subgroup of a group G and, for every subgroup X of G that contains Γ,

$$A'(\Gamma/X) = \sum_{\Gamma/D/X} A(\Gamma/D),$$

then

$$A(\Gamma/X) = \sum_{\Gamma/D/X} \mu(D/X)A'(\Gamma/D).$$

II. If Γ is a subgroup of a group G and, for every subgroup X of Γ,

$$B'(X/\Gamma) = \sum_{X/D/\Gamma} B(D/\Gamma),$$

then

$$B(X/\Gamma) = \sum_{X/D/\Gamma} \mu(X/D)B'(D/\Gamma).$$

In the first formula, $A(\Gamma/X)$ and $A'(\Gamma/X)$ are single-valued functions of Γ and X, defined for every subgroup X of G that contains Γ. The functions are not necessarily defined for *every* subgroup Γ of G. The symbols in the second formula have similar connotations. Finally we remark that $A(\Gamma/X)$ and $B(X/\Gamma)$ may be functions of X alone, in which case they may be denoted by $A(X)$ and $B(X)$ respectively; but that the same need not necessarily be the case of the corresponding functions $A'(\Gamma/X)$ and $B'(X/\Gamma)$.

While the groups considered in subsequent sections are prime-power groups, we note at this point the following general theorem which we shall find useful.

THEOREM 1. *If X_1 is an invariant subgroup of X_2, then*

$$\mu(X_1/X_2) = \mu\left(1 \Big/ \frac{X_2}{X_1}\right).$$

(Here and elsewhere 1 denotes the identity group.)

The theorem is an immediate consequence of the definition of the function $\mu(X_1/X_2)$ and the fact that there is a one-one correspondence between

the sets of maximal subgroups of X_2 whose cross-cut is X_1 and the sets of maximal subgroups of $X_2 \div X_1$ whose cross-cut is 1.

3. **Value of $\mu(X_1/X_2)$ for a prime-power group.** We begin with the case in which $X_1=1$ and $X_2=X$ is a group of order p^x (p prime). We shall write $\mu(X)$ for $\mu(1/X)$. We shall prove that

$$\mu(X) = (-1)^x p^{x(x-1)/2} \quad \text{or} \quad 0, \tag{5}$$

according as X is or is not an abelian group of type $(1, 1, 1, \cdots)$.

If X is not an abelian group of type $(1, 1, 1, \cdots)$, the cross-cut of *all* its maximal subgroups (the subgroups of index p) is not 1.* It follows from the definition that $\mu(X)=0$.

We now suppose that X is an abelian group of type $(1, 1, 1, \cdots)$. Because of the importance of the result, two proofs of (5) follow.

First proof. For the case in which $x=1$, (5) is an immediate consequence of

$$\sum_{D/X} \mu(D) = 0 \tag{6}$$

(see (3)), as this equation then involves only two terms and $\mu(1)=1$. We proceed to prove (5) by induction. Suppose we have verified that if D is an abelian group of order p^d $(d<x)$ and type $(1, 1, 1, \cdots)$, then $\mu(D)=(-1)^d p^{d(d-1)/2}$. An abelian group of order p^x and type $(1, 1, 1, \cdots)$ contains exactly

$$\frac{(p^x-1)\cdots(p^{x-d+1}-1)}{(p-1)\cdots(p^d-1)} \qquad (1 \leqq d \leqq x) \tag{7}$$

subgroups of order p^d, and each of them is an abelian group of type $(1, 1, 1, \cdots)$. Hence, by (6),

$$\mu(X) = -1 - \sum_{d=1}^{x-1} \frac{(p^x-1)\cdots(p^{x-d+1}-1)}{(p-1)\cdots(p^d-1)}(-1)^d p^{d(d-1)/2}. \tag{8}$$

Substituting $y=-1$ in *Cauchy's identity*†

$$\prod_{r=0}^{x-1}(1+p^r y) = 1 + \sum_{d=1}^{x} \frac{(p^x-1)\cdots(p^{x-d+1}-1)}{(p-1)\cdots(p^d-1)} p^{d(d-1)/2} y^d, \tag{9}$$

we obtain

$$0 = 1 + \sum_{d=1}^{x} \frac{(p^x-1)\cdots(p^{x-d+1}-1)}{(p-1)\cdots(p^d-1)}(-1)^d p^{d(d-1)/2}.$$

Comparing with (8), we have (5).

* Michael Bauer, *Note sur les groupes d'order p^α*, Nouvelles Annales de Mathématiques, vol. 19 (1900), p. 510. See also Miller, Blichfeldt and Dickson, *Finite Groups*, 1916, pp. 123, 127.

† A. L. Cauchy, Oeuvres, (1), vol. 8, p. 50. The identity is valid if p is an indeterminate.

Second proof. As already noted, (5) is verified for $x=1$. We shall suppose that $x\geqq 2$. Let Y be any subgroup of order p^{x-1} of X. Taking $X_1=1$, $X_2=Y$, $X_3=X$ in (4), we have

$$\sum_{(D,Y)=1} \mu(D/X) = 0. \tag{10}$$

Aside from $D=1$, the only subgroups of X that satisfy $(D, Y)=1$ are those subgroups of order p of X that are not contained in Y. This follows from the theorem that the order of the group generated by two permutable groups equals the product of their orders divided by the order of their cross-cut. Now the number of subgroups of order p of X that are not contained in Y is

$$\frac{p^x - p^{x-1}}{p - 1} = p^{x-1}.$$

Applying Theorem 1, we have by (10),

$$\mu(X) = - p^{x-1}\mu(A_{x-1}),$$

where A_{x-1} is an abelian group of order p^{x-1} and type $(1, 1, 1, \cdots)$. Again,

$$\mu(A_{x-1}) = - p^{x-2}\mu(A_{x-2}), \quad \mu(A_{x-2}) = - p^{x-2}\mu(A_{x-3}), \cdots ,$$

where A_k is an abelian group of order p^k and type $(1, 1, 1, \cdots)$. Hence, as $\mu(A_1)=-1$,

$$\mu(X) = (- p^{x-1})(- p^{x-2}) \cdots (- p)(- 1) = (- 1)^x p^{x(x-1)/2}.$$

THEOREM 2. *Let X_1 be a subgroup of order p^{x_1} of a group X_2 of order p^{x_2} ($0\leqq x_1<x_2$; p prime). If X_1 is not an invariant subgroup of X_2, $\mu(X_1/X_2)=0$. If X_1 is an invariant subgroup of X_2,*

$$\mu(X_1/X_2) = (- 1)^{x_2-x_1}p^{(x_2-x_1)(x_2-x_1-1)/2} \quad \textit{or} \quad 0,$$

according as $X_2\div X_1$ is or is not an abelian group of type $(1, 1, 1, \cdots)$.

The maximal subgroups of X_2 are those of index p. They are all invariant in X_2. Hence, if X_1 is not invariant in X_2, X_1 cannot be the cross-cut of a set of maximal subgroups of X_2. It follows from the definition that $\mu(X_1/X_2)=0$.

If X_1 is an invariant subgroup of X_2, $\mu(X_1/X_2)=\mu(X_2\div X_1)$ by Theorem 1. The value of $\mu(X_2\div X_1)$ is given by (5), with x replaced by x_2-x_1.

4. **Explicit forms of the inversion formulas.** The inversion formulas of §2 may now be stated as follows:

I. If Γ is a subgroup of a group G of order p^g and, for every subgroup X of G that contains Γ,

$$A'(\Gamma/X) = \sum_{\Gamma/D/X} A(\Gamma/D),$$

then

$$A(\Gamma/X) = \sum_{r=0}^{x} (-1)^r p^{r(r-1)/2} \sum A'(\Gamma/X_{x-r}),$$

where p^x is the order of X and, in $\sum A'(\Gamma/X_i)$, X_i ranges over all invariant subgroups of order p^i of X such that $X \div X_i$ is an abelian group of type $(1, 1, 1, \cdots)$, the identity group being regarded as a limiting case of a group of this type.

II. If Γ is a subgroup of order p^γ of a group G of order p^g and, for every subgroup X of Γ,

$$B'(X/\Gamma) = \sum_{X/D/\Gamma} B(D/\Gamma),$$

then

$$B(X/\Gamma) = \sum_{r=0}^{\gamma-x} (-1)^r p^{r(r-1)/2} \sum B'(X_{x+r}/\Gamma),$$

where p^x is the order of X and, in $\sum B'(X_i/\Gamma)$, X_i ranges over all subgroups of order p^i of Γ of which X is an invariant subgroup such that $X_i \div X$ is an abelian group of type $(1, 1, 1, \cdots)$.

5. **Number of subgroups having certain properties.** We proceed to give a few applications of the inversion formulas.

THEOREM 3. *The number of subgroups of order p^s of a group of order p^g that contain a particular subgroup of order p^h is* $\equiv 1 \pmod p$ $(0 \leqq h \leqq s \leqq g)$.*

Let $B(X) = 1$ or 0 according as X is or is not of order p^s. Then

$$B'(X/G) = \sum_{X/D/G} B(D)$$

is the number of subgroups of order p^s of G that contain X. By the second inversion formula,

$$B(X) \equiv B'(X/G) - \sum B'(X_{x+1}/G) \pmod p.$$

As the theorem is trivial if $s = h$, we suppose $s > h$. Taking $X = H$ (the particular subgroup of order p^h) we have, as $B(H) = 0$,

$$B'(H/G) \equiv \sum B'(H_{h+1}/G) \pmod p, \tag{11}$$

where H_{h+1} ranges over all subgroups of order p^{h+1} of G that contain H. The

* When $h=0$, the theorem reduces to a well known theorem of Frobenius.

number of these subgroups is known to be $\equiv 1 \pmod{p}$; that is, the theorem is verified for $s=h+1$.

We proceed to prove the theorem by induction on $s-h$, where s is fixed; that is, we assume that $B'(K/G)\equiv 1 \pmod{p}$ if K is a subgroup of G whose order p^k satisfies $p^s>p^k>p^h$, so that $1\leqq s-k<s-h$; and infer that $B'(H/G)\equiv 1 \pmod{p}$.

By assumption, each term of the right member of (11) is $\equiv 1 \pmod{p}$. We have seen that the number of terms is $\equiv 1 \pmod{p}$. We conclude that $B'(H/G)\equiv 1 \pmod{p}$.

THEOREM 4. *The number of non-cyclic subgroups of order p^s of a non-cyclic group of order p^g that contain a particular cyclic subgroup of order p^γ is* $\equiv 1 \pmod{p}$ $(p>2, 0\leqq\gamma<s, 2\leqq s\leqq g)$.*

Let Γ be the subgroup of order p^γ. Let $A(\Gamma/X)=1$ if X is a non-cyclic group of order p^s that contains Γ, and 0 otherwise. Then

$$A'(\Gamma/X) = \sum_{\Gamma/D/X} A(\Gamma/D)$$

is the number of non-cyclic subgroups of order p^s that contain Γ. The theorem being trivial if $s=g$, we suppose $s<g$. We shall prove the theorem by induction on g, assuming that $A'(\Gamma/K)\equiv 1 \pmod{p}$ if K is a non-cyclic group of order p^k $(s<k<g)$, and proving that $A'(\Gamma/G)\equiv 1 \pmod{p}$, where G is a group of order p^g.

By the first inversion formula we have, with $X=G$,

$$A'(\Gamma/G) \equiv \sum A'(\Gamma/G_{g-1}) \pmod{p},$$

where G_{g-1} ranges over the maximal subgroups of G that contain Γ. If G_{g-1} is cyclic, $A'(\Gamma/G_{g-1})=0$. If G_{g-1} is non-cyclic, $A'(\Gamma/G_{g-1})\equiv 1 \pmod{p}$, by assumption. Hence, if exactly m maximal subgroups of G contain Γ, and of these n are cyclic,

$$A'(\Gamma/G) \equiv m - n \pmod{p}.$$

If $n\geqq 1$, G contains an element of order p^{g-1}. Now there are only two types of non-cyclic groups of order p^g $(p>2, g>2)$ containing an element of order p^{g-1}. For these groups the theorem may be verified directly. We therefore suppose that $n=0$. As $m\equiv 1 \pmod{p}$ by Theorem 3, we conclude that $A'(\Gamma/G)\equiv 1 \pmod{p}$.

* The special case $\gamma=0$ was first treated by G. A. Miller. See Miller, Blichfeldt and Dickson, *Finite Groups*, p. 128.

6. **Number of sets of generators.** When Γ is the identity group, the first inversion formula may be written*

$$A(X) = \sum_{r=0}^{x} (-1)^r p^{r(r-1)/2} \sum A'(X_{x-r}), \tag{12}$$

where X_i ranges over all invariant subgroups of order p^i of X such that $X \div X_i$ is an abelian group of type $(1, 1, 1, \cdots)$. Let X' be the cross-cut of all the maximal subgroups of X; and let $p^{\nu(X)}$ be the order of $X \div X'$. It is known that X' is characterized by the fact that it is the smallest invariant subgroup of X whose corresponding quotient is an abelian group of type $(1, 1, 1, \cdots)$. It is readily proved that X' is a subgroup of every invariant subgroup of X whose corresponding quotient group is an abelian group of type $(1, 1, 1, \cdots)$. It follows from (7) that *the number of terms of* $\sum A'(X_{x-r})$ *in* (12) *is*

$$\frac{(p^\nu - 1) \cdots (p^{\nu-r+1} - 1)}{(p - 1) \cdots (p^r - 1)} \qquad (r \geqq 1, \nu = \nu(X)). \tag{13}$$

These facts are useful in applying (12).

Let X be a subgroup of order p^x of a group G of order p^g, and let $f(X)$ be the number of ordered sets of k (not necessarily distinct) elements of X that generate X. As the number of ordered sets of k elements of X is p^{kx}, and each set generates some subgroup of X,

$$p^{kx} = \sum_{D/X} f(D).$$

Applying (12) and (13), observing that

$$A'(X_{x-r}) = p^{k(x-r)},$$

and taking $X = G$, we have

$$f(G) = p^{kg} + \sum_{r=1}^{\nu-1} (-1)^r p^{r(r-1)/2} p^{k(g-r)} \frac{(p^\nu - 1) \cdots (p^{\nu-r+1} - 1)}{(p - 1) \cdots (p^r - 1)} \qquad (\nu = \nu(G)).$$

This series is easily summed with the aid of Cauchy's identity (9).

THEOREM 5. *The number of ordered sets of k (not necessarily distinct) elements of a group G of order p^g that generate G is*

$$p^{(g-\nu)k} \prod_{r=0}^{\nu-1} (p^k - p^r) \qquad (\nu = \nu(G)):$$

* Compare with the enumeration principle of P. Hall, *A contribution to the theory of groups of prime-power order*, Proceedings of the London Mathematical Society, vol. 36 (1933), p. 39.

This number vanishes for $k<\nu$, confirming the known fact that G cannot be generated by $<\nu(G)$ of its elements.

The next theorem is proved in a similar manner.

THEOREM 6. *The number of sets of k distinct elements of a group G of order p^σ that generate G is*

$$\binom{p^\sigma}{k}+\sum_{r=1}^{\nu-1}(-1)^r p^{r(r-1)/2}\binom{p^{\sigma-r}}{k}\frac{(p^\nu-1)\cdots(p^{\nu-r+1}-1)}{(p-1)\cdots(p^r-1)} \qquad (\nu=\nu(G)).$$

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GENERAL RELATIONS BETWEEN BERNOULLI, EULER, AND ALLIED POLYNOMIALS*

BY
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1. **Introduction.** The derivation of a complete set of general relations (§§5–9, 10) between the polynomials of Bernoulli, Euler, Genocchi, and Lucas, is reduced by the symbolic method (§§3, 4) to elementary algebraical operations (addition, multiplication, resolution into partial fractions) on four rational functions of the form $N(t)/D(t)$, where N, D are polynomials of degree $\leqq 2$ in t. In §9 it is shown that the relations imply a complete set of relations between the Bernoulli and allied numbers. The second of the transformations in §10, by specifying h, gives relations between the polynomials (or associated numbers) when their ranks are in arithmetical progression with any positive common difference.

2. **Notation.** The even-suffix notation is used for the numbers B, E, G, R of Bernoulli, Euler, Genocchi, and Lucas:

$$B_0 = 1,\quad B_1 = -\frac{1}{2},\ B_{2s+1} = 0(s > 0),\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30}, B_6 = \frac{1}{42},\ \cdots;$$

$$E_0 = 1,\quad E_{2s+1} = 0(s \geqq 0),\quad E_2 = -1,\quad E_4 = 5,\quad E_6 = -61,\ \cdots;$$

$$G_0 = 0,\quad G_1 = 1,\quad G_{2s+1} = 0(s > 0),\quad G_2 = -1,\quad G_4 = 1,\quad G_6 = -3;$$

$$R_0 = \frac{1}{2},\quad R_{2s+1} = 0(s \geqq 0),\quad R_2 = -\frac{1}{6},\quad R_4 = \frac{7}{30},\quad R_6 = -\frac{31}{42};$$

$$G_s = 2(1-2^s)B_s,\quad R_s = (1-2^{s-1})B_s\ (s \geqq 0);$$

the E, G are integers. Let x denote a complex variable. The above numbers have the following symbolic generators, in which the expansions of the exponentials converge absolutely for some $|x| \neq 0$:

$$\frac{x}{e^x - 1} = e^{Bx},\quad \frac{2e^x}{e^{2x}+1} = e^{Ex},\quad \frac{2x}{e^x+1} = e^{Gx},\quad \frac{xe^x}{e^{2x}-1} = e^{Rx}.$$

Let u be a complex variable and α the umbra of the sequence α_n ($n = 0, 1, 2, \cdots$). The Appell polynomial of degree n in u with the base α is

$$(u+\alpha)^n \equiv \sum_{s=0}^{n} {}_nC_s \alpha_s u^{n-s} \quad ({}_0C_0 = 1);$$

* Presented to the Society, September 13, 1935; received by the editors December 4, 1934.

its generator is obtained by multiplying the generator $e^{z\alpha}$ of α by e^{zu}, thus, $e^{zu}e^{z\alpha}=e^{z(u+\alpha)}$.

The Appell polynomials in u with the respective bases B, E, G, R are (here) called the Bernoulli, Euler, Genocchi, and Lucas polynomials in u:

$$\beta_n(u) \equiv (u+B)^n,\quad \eta_n(u) \equiv (u+E)^n,\quad \gamma_n(u) \equiv (u+G)^n,\quad \rho_n(u) \equiv (u+R)^n.$$

These definitions of the polynomials, instead of any of the numerous slightly different definitions in the literature, are chosen on account of the symmetry and simplicity of all calculations with the polynomials consequent upon their use. Other definitions can be readily translated into terms of these if necessary.

From the generators of B, E, G, R we write down those of $\beta(u)$, $\eta(u)$, $\gamma(u)$, $\rho(u)$:

$$t \equiv e^x, \qquad w \equiv e^{xu};$$

$$\frac{xw}{t-1} = e^{x\beta(u)} \equiv P(t) \equiv P,$$

$$\frac{2wt}{t^2+1} = e^{x\eta(u)} \equiv Q(t) \equiv Q,$$

$$\frac{2xw}{t+1} = e^{x\gamma(u)} \equiv S(t) \equiv S,$$

$$\frac{xwt}{t^2-1} = e^{x\rho(u)} \equiv T(t) \equiv T.$$

The generators are absolutely convergent for $|x|$, $|u|$ properly restricted and $\neq 0$.

Let $f(h)$ denote either a polynomial in the complex variable h or, if absolutely convergent for some $h\neq 0$, a power series in h. The results of substituting z for h in the successive derivatives $f'(h), f''(h), \cdots$ of $f(h)$ with respect to h will be written $f'(z), f''(z), \cdots$. This applies in particular if z contains umbrae.

All of the foregoing notation will be used without further reference.

3. **Order of relations; reductions.** The umbrae α, σ are said to be distinct if and only if $\alpha_n\neq\sigma_n$ for some integer $n\geqq 0$. Let $h, a, \cdots, s$ be ordinaries (complex numbers) and $\alpha, \cdots, \sigma$ umbrae. Symbolically, $(h\alpha)^n$ means $h^n\alpha_n$. A linear expression of the form $h+a\alpha+\cdots+s\sigma$ is said to be of order p (in the umbrae) if precisely p umbrae $\alpha, \cdots, \sigma$ occur in it. The order of a relation involving functions of the form $f(h+a\alpha+\cdots+s\sigma)$ is by definition the highest order of any expression occurring as an argument of f in the relation.

To find relations of order p in $\beta(u)$, $\eta(u)$, $\gamma(u)$, $\rho(u)$ we multiply together precisely p of their generators (if $p>4$, at least one generator will occur to a power >1). The product of generators, considered as a function of t, is then separated into partial fractions, from which it is easy to exhibit the product as a linear homogeneous function of the generators in the product, with coefficients of the form $H(x, w, t)$, where H is a rational function. From the result we can write down a relation involving a finite number of functions f and their derivatives f', f'', $\cdots$ in which the argument of one function is of order p and the arguments of the rest are of order 1. One example of the simple process of reduction by which the results stated later were obtained will suffice.

To reduce PQ to a linear homogeneous function of P, Q we have

$$PQ = \frac{2xw^2t}{(t-1)(t^2+1)};$$

$$\frac{t}{(t-1)(t^2+1)} = \frac{1}{2(t-1)} - \frac{t-1}{2(t^2+1)};$$

$$PQ = w\frac{xw}{t-1} - \frac{(t-1)xw}{2t}\frac{2wt}{t^2+1},$$

$$= wP - \frac{(t-1)xw}{2t}Q;$$

$$2tPQ = 2wtP - xw(t-1)Q.$$

4. **Derivation of general relations.** Let $h, x, a, d, \cdots, s, a', d', \cdots, s'$ be ordinaries, $\alpha, \delta, \cdots, \sigma$ umbrae ($\alpha^n = \alpha_n$ for $n=0, 1, \cdots$, $\alpha^m = \alpha_m = 0$ for $m<0$, and similarly for $\delta, \cdots, \sigma$), and i, j, integers $\geqq 0$. Let

$$a'e^{x\alpha} = d'x^ie^{x\delta} + \cdots + s'x^je^{x\sigma}$$

be an identity in x. Multiply throughout by e^{xh}, and equate coefficients of x^n. Then, with $D_h{}^p \equiv d^p/dh^p$, we have, for $n=0, 1, 2, \cdots$,

$$a'(h+\alpha)^n = D_h{}^id'(h+\delta)^n + \cdots + D_h{}^js'(h+\sigma)^n;$$

and hence, if $f(x) = \sum_{n=0} c_nx^n$,

$$a'f(h+\alpha) = d'f^{(i)}(h+\delta) + \cdots + s'f^{(j)}(h+\sigma),$$

where $f^{(p)}$ denotes the pth derivative (as in §2). Obviously the final relation can be written down at once from the given identity.

For example, the identity at the end of §3 is

$$2e^xe^{x(\beta(u)+\eta(u))} = 2e^{xu}e^xe^{\beta(u)} - xe^{xu}(e^x-1)e^{\eta(u)};$$

that is,

$$2e^{x(1+\beta(u)+\eta(u))} = 2e^{x(u+1+\beta(u))} - x\left[e^{x(u+1+\eta(u))} - e^{x(u+\eta(u))}\right];$$

hence

$$2f(h + \beta(u) + \eta(u) + 1) = 2f(h + \beta(u) + u + 1) - f'(h + \eta(u) + u + 1) + f'(h + \eta(u) + u).$$

We consider first relations in which no umbra occurs twice in the argument of any function in the relation.

5. **Relations of order 1.** From the generators of β, η, γ, ρ in §2 we have

$$(t - 1)P = xw, \qquad (t^2 + 1)Q = 2wt,$$
$$(t + 1)S = 2xw, \qquad (t^2 - 1)T = xwt.$$

Hence (as in §3) we write down the general relations

(1) $$f(h + \beta(u) + 1) - f(h + \beta(u)) = f'(h + u);$$

(2) $$f(h + \eta(u) + 2) + f(h + \eta(u)) = 2f(h + u + 1);$$

(3) $$f(h + \gamma(u) + 1) + f(h + \gamma(u)) = 2f'(h + u);$$

(4) $$f(h + \rho(u) + 2) - f(h + \rho(u)) = f'(h + u + 1).$$

Again, directly from the generators,

$$4T(x) = 2P(x) + S(x), \qquad 2wT(x) = tP(2x), \qquad 2xwQ(x) = tS(2x);$$

(5) $4f(h + \rho(u)) = 2f(h + \beta(u)) + f(h + \gamma(u));$

(6) $2f(h + \rho(u) + u) = f(h + 2\beta(u) + 1);$

(7) $2f'(h + \eta(u) + u) = f(h + 2\gamma(u) + 1).$

The six possible pairs chosen from β, η, γ, ρ give relations of the first order from

$$P(x) = \frac{x(t^2 + 1)}{2t(t - 1)} Q(x) = \frac{t + 1}{2(t - 1)} S(x) = \frac{t + 1}{t} T(x);$$

(8) $2f(h + \beta(u) + 2) - 2f(h + \beta(u) + 1) = f'(h + \eta(u) + 2) + f'(h + \eta(u));$

(9) $2f(h + \beta(u) + 1) - 2f(h + \beta(u)) = f(h + \gamma(u) + 1) + f(h + \gamma(u));$

(10) $f(h + \beta(u) + 1) = f(h + \rho(u) + 1) + f(h + \rho(u));$

(11) $f'(h + \eta(u) + 2) + f'(h + \eta(u)) = f(h + \gamma(u) + 2) + f(h + \gamma(u) + 1);$

(12) $f'(h + \eta(u) + 2) + f'(h + \eta(u)) = 2f(h + \rho(u) + 2) - 2(f + \rho(u));$

(13) $f(h + \gamma(u) + 1) = 2f(h + \rho(u) + 1) - 2f(h + \rho(u)).$

These also follow easily from (1)–(4).

6. **Relations of order 2.** These are given by the reductions of PQ, PS, PT, QS, QT, ST. We have

$$
\begin{aligned}
2tPQ &= 2wtP - xw(t-1)Q;\\
2PS &= 2xwP - xwS,\\
tPS &= 2xwT;\\
2t(t-1)PT &= xwtP + xw(t-1)T;\\
tQS &= xw(t+1)Q - wtS;\\
2tQT &= xwQ + 2wT,\\
Q(x)T(x) &= T(2x);\\
2t(t+1)ST &= xwtS + 2xw(t+1)T.
\end{aligned}
$$

To these correspond respectively

$$
\begin{aligned}
(14)\quad & 2f(h+\beta(u)+\eta(u)+1) = f(h+u+\beta(u)+1)\\
& \qquad - f'(h+u+\eta(u)+1) + f'(h+u+\eta(u));\\
(15)\quad & 2f(h+\beta(u)+\gamma(u)) = 2f'(h+u+\beta(u)) - f'(h+u+\gamma(u)),\\
(16)\quad & f(h+\beta(u)+\gamma(u)+1) = 2f'(u+\rho(u));\\
(17)\quad & 2f(h+\beta(u)+\rho(u)+2) - 2f(h+\beta(u)+\rho(u)+1)\\
& \qquad = f'(h+\beta(u)+u+1) + f'(h+\rho(u)+u+1) - f'(h+u+\rho(u));\\
(18)\quad & f(h+\eta(u)+\gamma(u)+1) = f'(h+\eta(u)+u+1)\\
& \qquad + f'(h+\eta(u)+u) - f(h+\gamma(u)+u+1);\\
(19)\quad & 2f(h+\eta(u)+\rho(u)+1) = f'(h+\eta(u)+u) + 2f(h+\rho(u)+u),\\
(20)\quad & f(h+\eta(u)+\rho(u)) = f(h+2\rho(u));\\
(21)\quad & 2f(h+\gamma(u)+\rho(u)+2) + 2f(h+\gamma(u)+\rho(u)+1)\\
& \qquad = f'(h+\gamma(u)+u+1) + 2f'(h+\rho(u)+u+1)\\
& \qquad\quad + 2f'(h+\rho(u)+u).
\end{aligned}
$$

7. **Relations of orders 3, 4.** The reductions of QST, PST, PQT, PQS provide several alternatives; we choose the simplest.

$$
\begin{aligned}
& 2t(t+1)QST = -x^2w^2(t+1)^2Q + xw^2tS + 2xw^2t(t+1)T;\\
& (t^2-1)PST = 2x^2w^2T;\\
& 4t(t-1)PQT = 2xw^2tP - x^2w^2(t^2-1)Q + 2xw^2t(t-1)T;\\
& 2PQS = 2xw^2P - 2x^2w^2Q + xw^2S.
\end{aligned}
$$

The simplest relation of order 4 is

$$2t(t^2-1)PQST = 2x^2w^3(t^2+1)T - x^3w^3t(t^2-1)Q.$$

$$\begin{aligned}(22)\quad & 2f(h+\eta(u)+\gamma(u)+\rho(u)+2)+2f(h+\eta(u)+\gamma(u)+\rho(u)+1)\\ &= -f''(h+\eta(u)+2u+2)-2f''(h+\eta(u)+2u+1)\\ &\quad -f''(h+\eta(u)+2u)+f'(h+\gamma(u)+2u+1)\\ &\quad +2f'(h+\rho(u)+2u+2)+2f'(h+\rho(u)+2u+1);\end{aligned}$$

$$\begin{aligned}(23)\quad & f(h+\beta(u)+\gamma(u)+\rho(u)+2)-f(h+\beta(u)+\gamma(u)+\rho(u))\\ &= 2f''(h+\rho(u)+2u);\end{aligned}$$

$$\begin{aligned}(24)\quad & 4f(h+\beta(u)+\eta(u)+\rho(u)+2)-4f(h+\beta(u)+\eta(u)+\rho(u)+1)\\ &= 2f'(h+\beta(u)+2u+1)-f''(h+\eta(u)+2u+2)\\ &\quad +f''(h+\eta(u)+2u)+2f'(h+\rho(u)+2u+2)\\ &\quad -2f'(h+\rho(u)+2u+1);\end{aligned}$$

$$\begin{aligned}(25)\quad & f(h+\beta(u)+\eta(u)+\gamma(u)) = 2f'(h+\beta(u)+2u)-2f''(h+\eta(u)+2u)\\ &\quad +f'(h+\gamma(u)+2u);\end{aligned}$$

$$\begin{aligned}(26)\quad & 2f(h+\beta(u)+\eta(u)+\gamma(u)+\rho(u)+3)\\ &\quad -2f(h+\beta(u)+\eta(u)+\gamma(u)+\rho(u)+1)\\ &= 2f''(h+\rho(u)+3u+2)+2f''(h+\rho(u)+3u)\\ &\quad -f'''(h+\eta(u)+3u+3)+f'''(h+\eta(u)+3u+1).\end{aligned}$$

8. **Relations with repeated umbrae.** In an expression involving $\alpha+\alpha+\cdots+\alpha$, where the umbra α is repeated precisely n times, the α's are replaced by $\alpha', \alpha'', \cdots$ respectively until after all exponents have been lowered to suffixes, when all accents are dropped, thus $\alpha'_s=\alpha''_s=\cdots=\alpha_s$; the expression $\alpha+\alpha+\cdots+\alpha$ (n α's) will be written $\alpha^{(n)}$. As before, $f^{(n)}$ denotes the nth derivative of f (as in §2).

We have $x^nw^n=(t-1)^nP^n$; hence

$$(27)\qquad f^{(n)}(h+nu) = \sum_{s=0}^{n}(-1)^s{}_nC_sf(h+\beta^{(n)}(u)+n-s).$$

Similarly

$$(28)\qquad 2^nf(h+nu+n) = \sum_{s=0}^{n}{}_nC_sf(h+\eta^{(n)}(u)+2n-2s);$$

$$(29)\qquad 2^nf^{(n)}(h+nu) = \sum_{s=0}^{n}{}_nC_sf(h+\gamma^{(n)}(u)+n-s);$$

$$(30)\qquad f^{(n)}(h+nu+n) = \sum_{s=0}^{n}(-1)^s{}_nC_sf(h+\rho^{(n)}(u)+2n-2s).$$

For $n=1$, these become (1)–(4). The generators of $\beta^{(n)}(u)$, $\eta^{(n)}(u)$, $\gamma^{(n)}(u)$, $\rho^{(n)}(u)$ being P^n, Q^n, S^n, T^n respectively, we could proceed as before to find relations of given orders for the $\beta^{(n)}(u), \cdots (n=1, 2, \cdots)$.

9. **Relations for** B, E, G, R. From the generators in §2 it is clear that P, Q, S, T become the generators of B, E, G, R respectively when $w=1$. Hence in (1)–(30) we may replace u by 0 and $\beta(u)$, $\eta(u)$, $\gamma(u)$, $\rho(u)$ by B, E, G, R respectively.

10. **Transformations.** Considering $P\equiv P(x, u)$ as a function of two variables x, u, and similarly for Q, S, T, we easily find that all the rational transformations of x, u which leave P, Q, S, T invariant to within constant factors are the following:

$$P(x, u) = P(-x, 1-u), \qquad Q(x, u) = Q(-x, -u),$$
$$S(x, u) = -S(-x, 1-u), \qquad T(x, u) = T(-x, -u).$$

Hence

$$e^{x\beta(u)} = e^{-x\beta(1-u)}, \qquad e^{x\eta(u)} = e^{-x\eta(-u)},$$
$$e^{x\gamma(u)} = -e^{-x\gamma(1-u)}, \qquad e^{x\rho(u)} = e^{-x\rho(-u)}.$$

Multiplying each of these on both sides by $e^{x(h+ru+s)}$, where h, r, s are ordinaries, and equating coefficients of $x^n (n=0, 1, \cdots)$ in the results, we get

$$(h+\beta(u)+ru+s)^n = (h-\beta(1-u)+ru+s)^n,$$
$$(h+\eta(u)+ru+s)^n = (h-\eta(-u)+ru+s)^n,$$
$$(h+\gamma(u)+ru+s)^n = -(h-\gamma(1-u)+ru+s)^n,$$
$$(h+\rho(u)+ru+s)^n = (h-\rho(-u)+ru+s)^n;$$

whence, f being as before, we have

(31) $$f(h+\beta(u)+ru+s) = f(h-\beta(1-u)+ru+s);$$

(32) $$f(h+\eta(u)+ru+s) = f(h+\eta(-u)+ru+s);$$

(33) $$f(h+\gamma(u)+ru+s) = -f(h-\gamma(1-u)+ru+s);$$

(34) $$f(h+\rho(u)+ru+s) = f(h-\rho(-u)+ru+s),$$

and similarly for f', f'', $\cdots$. The arguments in (1)–(26) are of the form $h+\alpha(u)+ru+s$ $(\alpha=\beta, \eta, \gamma, \rho)$, for r, s properly chosen. Thus (31)–(34) enable us to write down from each of (1)–(26) at least one more general relation. The like, with obvious modifications, applies to (27)–(30).

Another type of transformation produces relations from which follow (as special cases) relations between functions $\beta_n(u), \cdots$ whose ranks n are in arithmetical progression. Let m be a positive integer, and let f, $f^{(m)}$ be as before. Write

$$F_{m,s}(k) \equiv \sum_{j=0} \frac{f^{(jm+s)}(k)}{(jm+s)!} \qquad (s = 0, 1, \cdots, m-1),$$

$F'_{m,s}(z) \equiv$ the result of substituting z for k in $D_k F_{m,s}(k)$, etc. Then, in any of the relations (1)–(26) we may replace $f(k)$ by $F_{m,s}(k)$, $f'(k)$ by $F'_{m,s}(k)$, $f''(k)$ by $F''_{m,s}(k)$, etc., where, as above, $k \equiv h+\alpha(u)+ru+s$, or the corresponding argument on the right of (31)–(34).

To see this, replace h by $h+\mu$ in the argument of $f(h+\alpha(u)+ru+s)$, where μ is a primitive mth root of unity, and expand (as a function of h) by MacLaurin's theorem,

$$f(\mu + k) = \sum_{s=0}^{m-1} \mu^s F_{m,s}(k), \qquad k \equiv h + \alpha(u) + ru + s.$$

Proceed similarly with each $f, f', \cdots$ in a given relation. Since $1, \mu, \mu^2, \cdots, \mu^{m-1}$ are linearly independent, the result follows.

Added, March 27, 1935. In a recent abstract* D. H. Lehmer reports recurrences for Bernoulli numbers with gaps. These, apparently, are of a different type from those given by the above for special values of m.

* Bulletin of the American Mathematical Society, vol. 40 (1934), p. 51.

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ALGEBRAIC CHARACTERIZATIONS IN COMPLEX DIFFERENTIAL GEOMETRY†

BY

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1. In the treatment of differential geometry from the modern invariantive standpoint it is usually unnecessary that the coordinates and the functions which define the structure of the space under consideration be real quantities. Adopting the more general hypothesis of complex coordinates and structure functions we arrive at the concept of generalized spaces of complex character. This procedure has the advantage that it serves to distinguish the purely formal aspects of the theory which are identical under the real or complex hypothesis from non-trivial questions of reality arising in the transition from the complex to the corresponding real space.

Let S be a complex generalized space and denote by F_1 and F_2 systems of polynomials in the structure functions of S and their derivatives to a certain order, the coefficients of these polynomials being definitely specified constants. We shall say that the conditions

$$F_1 = 0, \qquad F_2 \neq 0 \tag{1.1}$$

constitute an *algebraic characterization* of a property P of the space S provided that necessary and sufficient conditions for the existence of the property P are furnished by (1.1). Since the property P is independent of the coordinate system adopted, the conditions (1.1) must be invariant under coordinate transformations.

In particular the equations $F_1 = 0$ alone may suffice for the algebraic characterization. This type of characterization may be described precisely as an *algebraic characterization in terms of equations*. We shall here be concerned exclusively with such characterizations, which for the sake of brevity will be referred to as *simple algebraic characterizations*. The results obtained will be seen to yield as immediate consequences certain algebraic characterizations which are not simple.

In the following discussion the above polynomials F will be found directly as polynomials in the components of a complete set of tensor differential invariants of the space S. It will therefore be convenient to consider these components rather than the structure functions and their derivatives as the independent variables in the polynomials F.

† Presented to the Society, September 13, 1935; received by the editors November 20, 1934.

Thus if S is an affinely connected space, a projective space of paths, a metric space, or a conformal space, the vanishing of the corresponding curvature tensor gives a simple algebraic characterization of the flat space S. As is well known, the equations

$$B_{\alpha\beta\gamma\delta} = K(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}), \tag{1.2}$$

in which the B's are the components of the curvature tensor and the g's the components of the fundamental metric tensor, express the conditions for a metric space to be of constant curvature K. The above equations do not constitute a strict algebraic characterization of the space of constant curvature since the constant K is arbitrary. However, an algebraic characterization is obtained by elimination of K which gives

$$(g_{ad}g_{bc} - g_{ac}g_{bd})B_{\alpha\beta\gamma\delta} - (g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta})B_{abcd} = 0. \tag{1.3}$$

Since the determinant $|g_{\alpha\beta}|$ does not vanish by hypothesis, not all of the expressions in parentheses in (1.3) will vanish as these are the second-order minors of $|g_{\alpha\beta}|$. Hence we can pass from (1.3) to (1.2) in which the quantity K is at most a function of position. It then follows by Schur's theorem that K is a constant and hence the equations (1.3) give an algebraic characterization of the metric spaces of constant curvature.

In a recent paper by J. Levine and myself a proof of the existence of algebraic characterizations was given for a certain class of problems in differential geometry.† It was shown in particular that a simple algebraic characterization exists for the metric representations of an affinely connected space provided that the dimensionality of the representations is unspecified. We now give a more exhaustive treatment of this problem on the basis of the theory of algebraic manifolds and the Kronecker theory of algebraic elimination. Our methods are quite general and permit a wide range of application beyond the particular problem treated in this paper. We have shown that there exist n irreducible algebraic manifolds which are of significance for our characterization problem. One of these irreducible manifolds furnishes a simple algebraic characterization of the 1-dimensional metric representations. The others give necessary conditions for the existence of representations of dimensionality $r>1$ but fail to meet the sufficiency requirement, with the result that none of these latter representations admit a simple algebraic characterization.

2. Consider the system of equations

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = g_{\sigma\beta}\Gamma^\sigma_{\alpha\gamma} + g_{\alpha\sigma}\Gamma^\sigma_{\beta\gamma} \tag{2.1}$$

† T. Y. Thomas and J. Levine, *On a class of existence theorems in differential geometry*, Bulletin of the American Mathematical Society, vol. 40 (1934), p. 721.

in the set of symmetric unknowns $g_{\alpha\beta}$ and the given components $\Gamma^{\alpha}_{\beta\gamma}(x)$ defining the (symmetric) connection of a complex space S of $n(\geqq 2)$ dimensions. As integrability conditions of (2.1) we derive the following sequence:

$$
\begin{aligned}
g_{\sigma\alpha}B^{\sigma}_{\beta\gamma\delta} + g_{\beta\sigma}B^{\sigma}_{\alpha\gamma\delta} &= 0, \\
g_{\sigma\alpha}B^{\sigma}_{\beta\gamma\delta,\epsilon} + g_{\beta\sigma}B^{\sigma}_{\alpha\gamma\delta,\epsilon} &= 0, \\
g_{\sigma\alpha}B^{\sigma}_{\beta\gamma\delta,\epsilon,\zeta} + g_{\beta\sigma}B^{\sigma}_{\alpha\gamma\delta,\epsilon,\zeta} &= 0, \\
\cdots\cdots\cdots\cdots\cdots &\cdot,
\end{aligned}
\tag{2.2}
$$

where the $B^{\alpha}_{\beta\gamma\delta}$, $B^{\alpha}_{\beta\gamma\delta,\epsilon}$, $\cdots$ are the components of the curvature tensor and its successive covariant derivatives. It can be proved† that there exists an integer N such that the vanishing of the resultant system $R(B)$ of the first N sets of equations of the sequence (2.2) is necessary and sufficient for the existence of a solution of (2.1). If a solution $g_{\alpha\beta}(x)$ of (2.1) exists such that the rank of the matrix $\|g_{\alpha\beta}\|$ is n, then the $\Gamma^{\alpha}_{\beta\gamma}$ are Christoffel symbols with respect to the $g_{\alpha\beta}$ and the space S is said to reduce to a metric space or to admit an n-dimensional metric representation. If the rank of the solution matrix $\|g_{\alpha\beta}\|$ is r, where $1\leqq r\leqq n-1$, and if a metric is defined in the space S by the degenerate quadratic differential form $g_{\alpha\beta}dx^{\alpha}dx^{\beta}$, the space will be multiply isomorphic to an r-dimensional metric space S^*, the metric of S^* being defined by a form which is not degenerate; we then say that the space S admits an r-dimensional metric representation. The equations $R(B)=0$ therefore give an algebraic characterization of the metric representations of the space S under the hypothesis that the dimensionality of the representations is unspecified.

3. Now suppose that (2.1) admits a solution $g_{\alpha\beta}(x)$ for which the matrix $\|g_{\alpha\beta}\|$ is of rank n. We can then solve (2.1) for the $\Gamma^{\alpha}_{\beta\gamma}$ so as to express these quantities as Christoffel symbols

$$
\Gamma^{\alpha}_{\beta\gamma} = \tfrac{1}{2}g^{\alpha\sigma}\left(\frac{\partial g_{\sigma\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\sigma\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\sigma}}\right)
\tag{3.1}
$$

in terms of the solution $g_{\alpha\beta}(x)$. Substituting (3.1) into the expression defining any component B as a function of the Γ's and their derivatives it is seen that the component B is given by an expression of the form

$$
B = \frac{P(g_{\alpha\beta}, \partial g_{\alpha\beta}/\partial x^{\gamma}, \cdots)}{|\, g_{\alpha\beta}\,|^{m}},
\tag{3.2}
$$

where P denotes a definitely determined polynomial, with rational coefficients, in the $g_{\alpha\beta}$ and a finite number of their derivatives, and where m is a

† See T. Y. Thomas and J. Levine, loc. cit., p. 721.

suitable positive integer. If the B's occurring in the resultant system $R(B)$ have numerical values given by the parametric equations

$$B = \frac{P(q_{\alpha\beta}, q_{\alpha\beta\gamma}, \cdots)}{|q_{\alpha\beta}|^m}, \tag{3.3}$$

where the q's are arbitrary subject to the symmetry conditions on the corresponding quantities in the right members of (3.2) and such that the determinant $|q_{\alpha\beta}|$ does not vanish, then $R(B)=0$, and the first N sets of equations (2.2) admit a numerical solution; in fact $g_{\alpha\beta}=q_{\alpha\beta}$ is a solution of these equations. This follows from the fact that we can define a set of polynomial functions

$$g_{\alpha\beta}(x) = q_{\alpha\beta} + q_{\alpha\beta\gamma}x^\gamma + \tfrac{1}{2}q_{\alpha\beta\gamma\delta}x^\gamma x^\delta + \cdots$$

having the above quantities q in (3.3) as coefficients and these functions can be used to determine the $\Gamma^\alpha_{\beta\gamma}$ by means of (3.1). At $x^\alpha=0$ the resulting components B will have values given by (3.3) and since (2.1) is merely another form of (3.1) for the case under consideration the first N sets of equations (2.2), as integrability conditions of (2.1), will admit the above mentioned solution. Hence the number N_1 of algebraically independent components B in $R(B)$ which correspond to a space S admitting an n-dimensional metric representation is exactly determined by the parametric representation (3.3).

Let Δ denote a set of algebraically independent components B appearing in the resultant system $R(B)$ without the restriction that the space S admits an n-dimensional metric representation, the complete sets of identities of the B's in S being used for the determination of these independent components.† If there are N_2 independent components B in the set Δ, we can interpret these as the coordinates of a space E of N_2 dimensions. The inequality $N_2>N_1$ evidently holds.

Let M_n denote the least algebraic manifold in E defined by the parametric equations (3.3). By recourse to the theory of polynomial ideals it can be shown that M_n is irreducible.‡ The algebraic equations $F_n(B)=0$ which define M_n are necessarily satisfied by the components B of a space S admitting an n-dimensional metric representation. Under such conditions we shall say as a matter of terminology that a space S which admits an n-dimensional metric representation belongs to the manifold M_n. Since M_n is the least algebraic manifold satisfying the required conditions given by (3.3) the equations $F_n(B)=0$ must give a simple algebraic characterization of the n-dimensional metric representations of S if such a characterization exists.

† See T. Y. Thomas, *The Differential Invariants of Generalized Spaces*, Cambridge University Press, 1934, p. 134.

‡ B. L. van der Waerden, *Moderne Algebra*, II, Berlin, Springer, 1931, p. 58.

4. We shall now extend the above discussion to the case of the r-dimensional metric representations of S where it is assumed that $r<n$. For this purpose we suppose that (2.1) admits a solution $g_{\alpha\beta}(x)$ with matrix $\|g_{\alpha\beta}\|$ of rank r. It is then possible to make a non-singular coordinate transformation $x\rightarrow y$ such that $\|g_{\alpha\beta}\|$ assumes the form

$$\left\|\begin{array}{ccc|ccc} h_{11} & \cdots & h_{1r} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ h_{r1} & \cdots & h_{rr} & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right\|,$$

where the quantities $h_{\alpha\beta}$ depend on the variables $y^1, \cdots, y^r$ alone. If $C^\alpha_{\beta\gamma}$ denote the components of the connection with respect to the y coordinate system, then these components must vanish for $\alpha=1, \cdots, r$; $\beta=r+1, \cdots, n$; $\gamma=1, \cdots, n$. The components $C^\alpha_{\beta\gamma}$ for $\alpha>r$ and $\beta, \gamma=1, \cdots, n$ are arbitrary analytic functions of the coordinates $y^1, \cdots, y^n$. The remaining components $C^\alpha_{\beta\gamma}$ where $\alpha, \beta, \gamma=1, \cdots, r$ are Christoffel symbols with respect to the h's in the rth-order minor in the upper left hand corner of the above matrix,† i.e.,

$$C^\alpha_{\beta\gamma} = \tfrac{1}{2}h^{\alpha\sigma}\left(\frac{\partial h_{\sigma\beta}}{\partial y^\gamma} + \frac{\partial h_{\sigma\gamma}}{\partial y^\beta} - \frac{\partial h_{\beta\gamma}}{\partial y^\sigma}\right) \qquad (\text{indices} = 1, \cdots, r). \tag{4.1}$$

If we denote by D the components B when taken with respect to the y coordinates, we obtain a parametric representation of the D's corresponding to (3.3) on the basis of the conditions (4.1) and the fact that certain of the C's vanish while others are completely arbitrary; in this representation the non-vanishing determinant $|q_{\alpha\beta}|$ of the rth order will occur in place of the corresponding nth-order determinant in the denominators of (3.3). To obtain the parametric representation with respect to the arbitrary x coordinate system we have merely to transform the above expressions for the D's into the B's by the tensor transformations, i.e.,

$$B^\alpha_{\beta\gamma\delta} = D^\mu_{\nu\sigma\tau}u^\nu_\beta u^\sigma_\gamma u^\tau_\delta v^\alpha_\mu, \cdots, \tag{4.2}$$

---

† In particular if $r=1$, these equations assume the form

$$C^1_{11} = \frac{1}{2}\cdot\frac{1}{h_{11}}\cdot\frac{\partial h_{11}}{\partial y^1}.$$

See T. Y. Thomas and J. Levine, loc. cit., p. 721.

where the u^α_β are arbitrary parameters such that the determinant $|u^\alpha_\beta|$ is not equal to zero and the v^α_β are their normalized cofactors.

If we denote by M_r the least algebraic manifold in E defined by (4.2) this manifold is irreducible as was the case for the manifold M_n, and the algebraic equations $F_r(B)=0$ which define M_r constitute necessary conditions on the space S for this space to admit an r-dimensional metric representation. If a simple algebraic characterization exists it must evidently be given by these equations.

THEOREM. *There exist n irreducible algebraic manifolds $M_1, \cdots, M_n$ such that any space S which admits an r-dimensional metric representation belongs to the manifold M_r.*

5. We shall now prove a lemma which will have application in the following section.

LEMMA. *If r is any integer of the set $1, \cdots, n-1$ and if the solution matrix $\|g_{\alpha\beta}\|$ of (2.1) is of rank r at a point $x^\alpha = p^\alpha$ of its domain of definition, then there exists a neighborhood of p^α in which $\|g_{\alpha\beta}\|$ is of rank r.*

Take first the case $r=n-1$. For the derivative of the determinant $|g_{\alpha\beta}|$ of the matrix $\|g_{\alpha\beta}\|$ we have

$$\frac{\partial |g_{\mu\nu}|}{\partial x^\gamma} = \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} A^{\alpha\beta}, \tag{5.1}$$

where $A^{\alpha\beta}$ denotes the cofactor of the corresponding element $g_{\alpha\beta}$ of the matrix $\|g_{\alpha\beta}\|$. Substituting from (2.1), equations (5.1) become

$$\frac{\partial |g_{\mu\nu}|}{\partial x^\gamma} = (g_{\sigma\beta}\Gamma^\sigma_{\alpha\gamma} + g_{\alpha\sigma}\Gamma^\sigma_{\beta\gamma})A^{\alpha\beta} = 2|g_{\mu\nu}|\Gamma^\sigma_{\sigma\gamma}. \tag{5.2}$$

Since $|g_{\mu\nu}|=0$ at $x^\alpha = p^\alpha$ the left member of (5.2) vanishes at $x^\alpha = p^\alpha$ and consequently successive derivatives of the determinant $|g_{\mu\nu}|$ will likewise vanish at $x^\alpha = p^\alpha$. By hypothesis the rank of $\|g_{\alpha\beta}\|$ is $r=n-1$ at $x^\alpha=p^\alpha$ and hence the rank of this matrix is $n-1$ in a certain neighborhood of p^α.

If $r<n-1$ we consider the determinant $|g_{\mu\nu}|$ in the left member of (5.1) to be any determinant of order $r+1$ selected from the matrix $\|g_{\alpha\beta}\|$, the $A^{\alpha\beta}$ being the cofactors of corresponding elements of the determinant $|g_{\mu\nu}|$. Then (5.1) is satisfied, and making the substitution (2.1) we obtain the first set of equations (5.2), in which the indices α, β are now restricted by the particular selection of the $(r+1)$st-order determinant $|g_{\mu\nu}|$. The right members of these latter equations now expand into a linear and homogeneous expression in determinants of order $r+1$ of the matrix $\|g_{\alpha\beta}\|$. Hence the rank of $\|g_{\alpha\beta}\|$ is r in a neighborhood of $x^\alpha = p^\alpha$ and the above lemma is proved.

6. Suppose that S belongs to M_r. We shall then show that S admits a metric representation of dimensionality $m \leqq r$. To prove this result we make use of the theorem on resultant systems of homogeneous algebraic equations.†

Denote by $L_{n-1}(B)=0$ the resultant system of the first N sets of equations (2.2) combined with the determinant equation

$$\begin{vmatrix} g_{11} & \cdots & g_{1n} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ g_{n1} & \cdots & g_{nn} \end{vmatrix} = 0. \tag{6.1}$$

More generally denote by $L_r(B)=0$ for $r=1, \cdots, n-1$ the resultant system of the first N sets of equations (2.2) and the set of equations obtained by equating to zero all determinants of order $r+1$ in the left member of (6.1). Then $L_r(B)=0$ will define an algebraic manifold G_r in the space E. The manifold defined in E by the resultant system $R(B)=0$ of the first N sets of equations (2.2) will be denoted by G_n. It is evident from the equations of definition of the manifolds G_r for $r=1, \cdots, n$ that

$$G_n \supset G_{n-1} \supset \cdots \supset G_1. \tag{6.2}$$

Now $M_n \subset G_n$. Since the coordinates B of any point P of M_{n-1} having a parametric representation of the form (4.2) permit a solution of the first N sets of equations (2.2) such that (6.1) is satisfied it follows that P lies in G_{n-1}. If the point P of M_{n-1} does not have the parametric representation (4.2) used in the determination of M_{n-1} it follows from a theorem in the theory of algebraic manifolds that P is the limit of a sequence of points $P_1, P_2, \cdots$ of M_{n-1} having this representation‡ and hence must likewise belong to G_{n-1}, i.e., $M_{n-1} \subset G_{n-1}$. Continuing we obtain the set of relations

$$M_n \subset G_n,\ M_{n-1} \subset G_{n-1}, \cdots, M_1 \subset G_1. \tag{6.3}$$

Suppose that S belongs to the manifold G_r where r is an integer of the set $1, \cdots, n$. The equations (2.1) will therefore admit a solution and hence there exists an integer N^* such that the first N^* sets of equations (2.2), which we shall call the equations H for brevity, are algebraically consistent and all their solutions satisfy the (N^*+1)st set of these equations. Let $x^\alpha = p^\alpha$ denote a

† B. L. van der Waerden, loc. cit., p. 14.

‡ See J. F. Ritt, *Differential Equations from the Algebraic Standpoint*, Colloquium Publications of the American Mathematical Society, vol. 14, New York, 1932, p. 91. Also B. L. van der Waerden, *Zur algebraischen Geometrie*, III, Mathematische Annalen, vol. 108 (1933), p. 694.

point of S where the rank of the matrix of H has its greatest value. Denote by $(B)_p$ the values of the corresponding B's in H. These equations, in which the coefficients B have the values $(B)_p$, will possess a solution $g_{\alpha\beta}=(g_{\alpha\beta})_p$ such that the rank of the matrix $\|(g_{\alpha\beta})_p\|$ is $m\leqq r$ since the resultant system $L_r(B)=0$ is satisfied by the values $(B)_p$. Since the rank of the matrix of the equations H for B's in a certain neighborhood Δ^* of $x^\alpha=p^\alpha$ is the same as when the B's have the values $(B)_p$, it follows that the general solution $g_{\alpha\beta}$, valid in Δ^*, will yield the values $(g_{\alpha\beta})_p$ for $x^\alpha=p^\alpha$ and for the selection $(g_{\alpha\beta})_p$ of the values of those $g_{\alpha\beta}$ which enter as arbitrary quantities in the general solution. Now in accordance with the theory of equations of the type (2.1),† these arbitrary quantities $g_{\alpha\beta}$ are to be determined as the solutions of a completely integrable system of differential equations, the general solution of (2.1) then being obtainable as the algebraic solution of H. Hence there exists a solution $g_{\alpha\beta}$ of (2.1) such that $g_{\alpha\beta}=(g_{\alpha\beta})_p$ at $x^\alpha=p^\alpha$. By the lemma of §5 the matrix $\|g_{\alpha\beta}\|$ of this solution will be of rank $m\leqq r$ in the neighborhood Δ^* of $x^\alpha=p^\alpha$. *Hence if S belongs to G_r the space S will admit a metric representation of dimensionality $m\leqq r$.* Since $M_r\subset G_r$ by (6.3) we have the following result:

THEOREM. *If the space S belongs to the irreducible manifold M_r when r is any integer of the set $1,\cdots,n$, it will admit a metric representation of dimensionality $m\leqq r$.*

This theorem when combined with the theorem of §4 gives the following

COROLLARY. *A necessary and sufficient condition for the space S to admit a 1-dimensional metric representation is that it belong to the irreducible manifold M_1.*

It is evident from the above that the manifold M_1 can be replaced by the manifold G_1 in the statement of this corollary. Hence the resultant system $L_1(B)=0$ gives an algebraic characterization of the 1-dimensional metric representations of S.

7. It will now be shown that the equations $F_r(B)=0$ which define the manifold M_r where r has any value of the set $2,\cdots,n$ are not sufficient to insure the existence of an r-dimensional metric representation of S, and hence in accordance with the observation of §4 that the r-dimensional metric representations of S do not admit a simple algebraic characterization.

Consider a one-parameter family of n-dimensional metric spaces S_a the components of the fundamental metric tensors of which are given by the matrix

† See, for example, T. Y. Thomas, *Differential Invariants of Generalized Spaces*, p. 202.

$$
(7.1) \qquad \left\| \begin{matrix} \sin a\phi & 0 & \cdots & 0 \\ 0 & e^{a\phi} & \cdots & 0 \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ 0 & 0 & \cdots\ 0 & e^{a\phi} \end{matrix} \right\|,
$$

where a denotes the (real) parameter and ϕ is an analytic function of the coordinates $x^1, \cdots, x^n$ in the neighborhood of the values $x^\alpha = 0$ at which the function ϕ is assumed to be different from zero; all other components of the tensors except those appearing in the diagonal of the matrix vanish identically. The spaces S_a are thus defined for values of the parameter a different from zero. Calculation of the resulting Christoffel symbols give

$$
(7.2) \qquad
\begin{aligned}
&\left\{ \begin{matrix} J \\ II \end{matrix} \right\} = -\frac{a}{2}\frac{\partial\phi}{\partial x^J}, &&\left\{ \begin{matrix} J \\ 11 \end{matrix} \right\} = -\frac{a}{2}\left(\frac{\cos a\phi}{e^{a\phi}}\right)\frac{\partial\phi}{\partial x^J}, \\
&\left\{ \begin{matrix} 1 \\ II \end{matrix} \right\} = -\frac{a}{2}\left(\frac{e^{a\phi}}{\sin a\phi}\right)\frac{\partial\phi}{\partial x^1}, &&\left\{ \begin{matrix} 1 \\ 1J \end{matrix} \right\} = \frac{a}{2}\left(\frac{a}{\tan a\phi}\right)\frac{\partial\phi}{\partial x^J}, \\
&\left\{ \begin{matrix} I \\ 1I \end{matrix} \right\} = \frac{a}{2}\frac{\partial\phi}{\partial x^1}, &&\left\{ \begin{matrix} I \\ IJ \end{matrix} \right\} = \frac{a}{2}\frac{\partial\phi}{\partial x^J}, \\
&\left\{ \begin{matrix} I \\ II \end{matrix} \right\} = \frac{a}{2}\frac{\partial\phi}{\partial x^I}, &&\left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} = \frac{1}{2}\left(\frac{a}{\tan a\phi}\right)\frac{\partial\phi}{\partial x^1},
\end{aligned}
$$

where $I, J = 2, \cdots, n$ and the remaining symbols vanish identically. Now let $a \to 0$ and denote the limiting values of the Christoffel symbols $\left\{ \begin{smallmatrix} \alpha \\ \beta\gamma \end{smallmatrix} \right\}$ by the corresponding symbols $\Gamma^\alpha_{\beta\gamma}$. Then those symbols in (7.2) which do not approach zero become

$$
(7.3) \qquad \left.\begin{aligned} \Gamma^1_{II} &= -\tfrac{1}{2}\theta_1 \\ \Gamma^1_{1\alpha} &= \tfrac{1}{2}\theta_\alpha \end{aligned}\right\} \text{ where } \theta = \log\phi,\ \theta_\alpha = \frac{\partial\theta}{\partial x^\alpha}.
$$

Denote by S^* the affinely connected space for which the components of connection Γ^1_{II} and $\Gamma^1_{1\alpha}$ are given by (7.3), the remaining components being zero identically.

Now observe that the Christoffel symbols for the spaces S_a can be represented by expansions of the form

$$
\psi_0 + \psi_1 a + \psi_2 a^2 + \cdots,
$$

where the ψ's are analytic functions of the coordinates x^α in the neighborhood of $x^\alpha = 0$. The functions represented by ψ_0 in these expansions are the components of the connection of the space S^*. Hence the components B of the curvature tensor and its successive covariant derivatives for the spaces S_a

approach, as $a \to 0$, the corresponding components B for the space S^*. Since the spaces S_a are such that the equations $F_n(B)=0$ which define the manifold M_n are satisfied, it follows that these equations are likewise satisfied for the space S^*, i.e. the *space S^* belongs to the algebraic manifold M_n*.

Now consider the equations (2.1) in which the Γ's are the components of connection of the space S^*. The first set of integrability conditions (2.2) then reduces to

$$g_{\alpha 1} \overset{1}{B}_{\beta\gamma\delta} + g_{\beta 1} \overset{1}{B}_{\alpha\gamma\delta} = 0. \tag{7.4}$$

Take $\alpha=\beta=\gamma=I$ where $I=2, \cdots, n$ and $\delta=1$. Then (7.4) becomes

$$g_{I1} \overset{1}{B}_{II1} = 0, \quad \text{or} \quad g_{I1}\left(\frac{\partial \theta_I}{\partial x^I} + \frac{\partial \theta_1}{\partial x^1} + \frac{1}{2}\theta_I^2 + \frac{1}{2}\theta_1^2\right) = 0,$$

where it is of course understood that no summation is involved in these equations. Hence if we choose the function ϕ so that

$$\frac{\partial \theta_I}{\partial x^I} + \frac{\partial \theta_1}{\partial x^1} + \frac{1}{2}\theta_I^2 + \frac{1}{2}\theta_1^2 \neq 0 \qquad (I = 2, \cdots, n), \tag{7.5}$$

it follows that $g_{I1}=0$. Now take $\alpha=\delta=1$, and $\beta=\gamma=I$ in (7.4); then these equations become

$$g_{11}\left(\frac{\partial \theta_I}{\partial x^I} + \frac{\partial \theta_1}{\partial x^1} + \frac{1}{2}\theta_I^2 + \frac{1}{2}\theta_1^2\right) = 0,$$

and hence the above condition (7.5) likewise gives $g_{11}=0$. It now follows from the equations (2.1) that $\partial g_{IJ}/\partial x^\alpha=0$ for $\alpha=1, \cdots, n$ and $I, J=2, \cdots, n$, so that the most general possible solution of these equations under the condition (7.5) is that given by the matrix

$$\left\| \begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & e_{22} & \cdots & e_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & e_{n2} & \cdots & e_{nn} \end{array} \right\|,$$

where the e's are arbitrary constants. Conversely it is seen immediately that this matrix represents a solution of (2.1) for the case under consideration. Hence the space S^* does not admit an n-dimensional metric representation although it belongs to the manifold M_n. It follows that *the n-dimensional*

metric representations of an affinely connected space S do not admit a simple algebraic characterization.†

8. To extend this result to the case of the r-dimensional metric representations for $r=2, \cdots, n-1$ we can consider a set of symmetric quantities $h_{\alpha\beta}$, where $\alpha, \beta=1, \cdots, r$, defined by a matrix of the form (7.1) in which the function ϕ depends on the coordinates $y^1, \cdots, y^r$ alone. We then use the equations (4.1) to define the components $C^{\alpha}_{\beta\gamma}$ for $\alpha, \beta, \gamma=1, \cdots, r$ and take the remaining C's subject to the restrictions stated in §4. Allowing the parameter a to approach zero we obtain a set of functions Γ defining the connection of a space S^{**} which must belong to the manifold M_r, although a consideration of the equations (2.1) for S^{**} shows that this space , subject to restrictions corresponding to (7.4), can admit at most an $(r-1)$-dimensional metric representation. While this process is thus analogous to that carried out in §7 it is desirable nevertheless to give the details of the process since certain formal difficulties present themselves.‡

For definiteness in our discussion we shall employ only the following letters as indices with the ranges indicated:

$$I, J, K = 2, \cdots, r,$$

$$h, i, j, k, l = r+1, \cdots, n,$$

$$\alpha, \beta, \gamma, \sigma = 1, \cdots, r,$$

$$\Lambda, \Theta, \Phi, \Psi, \Omega = 1, \cdots, n.$$

Corresponding to (7.2) the limiting values are now given by

$$(8.1)\qquad \begin{cases} \left.\begin{array}{l} \Gamma^{1}_{II} = -\tfrac{1}{2}\theta_1, \quad \Gamma^{1}_{1\alpha} = \tfrac{1}{2}\theta_\alpha \\ \text{Other } \Gamma^{\alpha}_{\beta\gamma} = 0 \end{array}\right\} \text{ where } \theta = \log\phi, \theta_\alpha = \dfrac{\partial\theta}{\partial y^\alpha}; \\ \Gamma^{\alpha}_{\Phi i} = 0, \\ \Gamma^{i}_{\Phi\Psi} = \text{arbitrary analytic function of the variables } y^1, \cdots, y^n. \end{cases}$$

Using these values of the Γ's we find that

$$(8.2)\qquad B^{1}_{1\alpha I} = B^{I}_{\alpha\beta\gamma} = B^{I}_{\alpha\beta i} = B^{\alpha}_{i\beta\gamma} = B^{\alpha}_{i\beta j} = B^{\alpha}_{\Phi jk} = 0,$$

$$(8.3)\qquad B^{i}_{jkl} = \frac{\partial\Gamma^{i}_{jk}}{\partial y^l} - \frac{\partial\Gamma^{i}_{jl}}{\partial y^k} + \Gamma^{i}_{hl}\Gamma^{h}_{jk} - \Gamma^{i}_{hk}\Gamma^{h}_{jl}.$$

---

† The method employed to deduce this result shows that necessary and sufficient conditions for the existence of an n-dimensional metric representation of the space S can not be expressed by any system of equations of the form $F(B)=0$ the left members of which are continuous functions of the components B. As the algebraic characterization is however of primary interest we have limited the above statement to such characterizations.

‡ I am indebted to Dr. J. Levine for the details of the treatment given in this section.

Now consider the first set of integrability conditions (2.2) of the equations (2.1) determined by (8.1), namely

$$g_{\Phi\Omega}B^{\Omega}_{\Psi\Lambda\Theta} + g_{\Omega\Psi}B^{\Omega}_{\Psi\Lambda\Theta} = 0. \tag{8.4}$$

Taking $\Phi, \Psi, \Lambda, \Theta = i, j, k, l$ respectively and making use of (8.2), these equations become

$$g_{ih}B^{h}_{jkl} + g_{jh}B^{h}_{ikl} = 0. \tag{8.5}$$

In these latter equations put $i=j$ so as to obtain

$$g_{ih}B^{h}_{ikl} = 0 \qquad (i \text{ not summed}). \tag{8.6}$$

Put $W=n-r$. Then in (8.6) for each value of i we have $W(W-1)/2$ equations in the W unknowns g_{ih}.

Case I. $W \geqq 3$. Since $W \leqq W(W-1)/2$ we can form from the matrix of the coefficients g_{ih} in (8.6) the following determinant of order W:

$$B_i \equiv \begin{vmatrix} B^{r+1}_{i\ r+1\ r+2} & B^{r+2}_{i\ r+1\ r+2} & \cdots & B^{n}_{i\ r+1\ r+2} \\ B^{r+1}_{i\ r+1\ r+3} & B^{r+2}_{i\ r+1\ r+3} & \cdots & B^{n}_{i\ r+1\ r+3} \\ \cdot & \cdot & \cdot & \cdot \\ B^{r+1}_{i\ r+1\ n} & B^{r+2}_{i\ r+1\ n} & \cdots & B^{n}_{i\ r+1\ n} \\ B^{r+1}_{i\ r+2\ r+3} & B^{r+2}_{i\ r+2\ r+3} & \cdots & B^{n}_{i\ r+2\ r+3} \end{vmatrix}.$$

None of the determinants B_i is identically zero since the elements B in any determinant B_i are algebraically independent quantities; this follows by recourse to the complete set of identities satisfied by the B's as defined by (8.3) in terms of the arbitrary functions Γ^i_{jk} in (8.1). Hence we can choose the Γ^i_{jk} so that $B_i \neq 0$ for $i=r+1, \cdots, n$. Hence $g_{ih}=0$ in consequence of (8.6).

From (8.4) we now obtain

$$g_{ah}B^{h}_{ikl} = 0, \tag{8.7}$$

use being made of the fact that the above quantities g_{ih} are equal to zero. Hence $g_{ah}=0$ since a determinant B_i is contained in the matrix of the coefficients of (8.7).

Case II. $W=2$. From (8.4) select the following three equations:

$$\begin{aligned} g_{mm}B^{m}_{mmn} + g_{mn}B^{n}_{mmn} &= 0, \\ g_{nm}B^{m}_{nmn} + g_{nn}B^{n}_{nmn} &= 0, \\ g_{nn}B^{n}_{mmn} + g_{mn}(B^{m}_{mmn} + B^{n}_{nmn}) + g_{mm}B^{m}_{nmn} &= 0, \end{aligned}$$

where m is used to denote $n-1$. Hence if the arbitrary Γ's in (8.1) are chosen so that

$$
(8.8)\qquad \begin{vmatrix} B^{m}_{mmn} & B^{n}_{mmn} & 0 \\ 0 & B^{m}_{nmn} & B^{n}_{nmn} \\ B^{m}_{nmn} & B^{m}_{mmn} + B^{n}_{nmn} & B^{n}_{mmn} \end{vmatrix}
$$

$$
= (B^{m}_{mmn} + B^{n}_{nmn})(B^{m}_{nmn}B^{n}_{mmn} - B^{m}_{mmn}B^{n}_{nmn}) \neq 0,
$$

it will follow that $g_{mm}=g_{mn}=g_{nn}=0$. Now from (8.7) we obtain

$$
g_{\alpha m}B^{m}_{mmn} + g_{\alpha n}B^{n}_{mmn} = 0,
$$
$$
g_{\alpha m}B^{m}_{nmn} + g_{\alpha n}B^{n}_{nmn} = 0,
$$

the determinant of which is a factor of the above determinant (8.8). Hence $g_{\alpha m}=g_{\alpha n}=0$.

Case III. $W=1$. From (8.4) we obtain

$$
g_{nn}B^{n}_{n\alpha\beta} = 0.
$$

Choosing the arbitrary Γ's in (8.1) so that not all the coefficients $B^{n}_{n\alpha\beta}$ are equal to zero, we have $g_{nn}=0$. Hence from (8.4) we have

$$
g_{\alpha n}B^{n}_{n\beta n} = 0,
$$

from which $g_{\alpha n}=0$ can be obtained.

We have now shown that the arbitrary Γ's in (8.1) can be chosen so that the solution matrix of (2.1) will have the form

$$
\left\| \begin{array}{c|c} \begin{matrix} g_{11} & \cdots & g_{1r} \\ \cdot & \cdots & \cdot \\ g_{r1} & \cdots & g_{rr} \end{matrix} & 0 \\ \hline 0 & 0 \end{array} \right\|,
$$

where r has any value in the set $2, \cdots, n-1$. To make the rank of this matrix be $<r$ we now make $g_{1\alpha}=0$. By a suitable selection of the indices in (8.4) these equations give

$$
g_{11}\left(\frac{\partial\theta_I}{\partial y^J} + \frac{1}{2}\,\theta_I\theta_J\right) = 0, \qquad g_{k1}\left(\frac{\partial\theta_1}{\partial y^J} + \frac{1}{2}\,\theta_1\theta_J\right) = 0.
$$

Hence to have $g_{1\alpha}=0$ we have merely to choose ϕ so that one of the coefficients in each of these two sets of equations is different from zero. As in §7 the most general solution of the equations (2.1) is thus seen to be represented by the matrix

$$\left\| \begin{array}{c|ccc|c} 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & e_{22} & \cdots & e_{2r} & \\ \cdot & \cdot & \cdots & \cdot & 0 \\ 0 & e_{r2} & \cdots & e_{rr} & \\ \hline 0 & & 0 & & 0 \end{array} \right\|,$$

where the e's are arbitrary constants. This gives the following theorem which includes the result of §7.

THEOREM. *The r-dimensional metric representations of an affinely connected space S, where r is an integer of the set* $2, \cdots, n$, *do not admit a simple algebraic characterization.*

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MAXIMAL ORDERS IN RATIONAL CYCLIC ALGEBRAS OF ODD PRIME DEGREE†

BY

RALPH HULL‡

1. Introduction. Throughout this paper R denotes the field of rational numbers and n is a fixed odd prime. We consider cyclic algebras of degree n, order n^2, over R, that is, algebras A of the following type:§

A has an R-basis of the form

$$u^i z_k \qquad (i = 0, \cdots, n-1;\ k = 1, \cdots, n), \tag{1}$$

where the z_k form an R-basis of a cyclic sub-corps Z of A of degree n over R and $1, u, \cdots, u^{n-1}$ form a Z-basis of A, with the relations

$$zu = uz^S \quad \text{for every } z \text{ of } Z, \tag{2}$$

where S is a generating element of the Galois group of Z over R and z^S is the element of Z corresponding to z under the automorphism S, and

$$u^n = \alpha \neq 0 \text{ in } R. \tag{3}$$

This is called a *cyclic generation* of A and is denoted by

$$A = (\alpha, Z, S). \tag{4}$$

Artin|| has defined an order in a rational semi-simple algebra B as a subset I of elements of B with the following properties:

(a) The sum, difference, and product of any two elements of I are also in I.

(b) If b is any element of B there exists a rational integer μ such that μb is in I.

(c) The set I is of finite order, that is, there is a finite set of elements $a_1, a_2, \cdots, a_r$ of I such that every element a of I can be expressed in the form

$$a = \eta_1 a_1 + \cdots + \eta_r a_r$$

with rational integers $\eta_1, \cdots, \eta_r$.

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§ The description given here is that of Hasse: *Theory of cyclic algebras over an algebraic number field*, these Transactions, vol. 34 (1932), pp. 171–214. We shall refer to this paper hereafter as H. For complete bibliographies see H and the later paper by Hasse: *Die Struktur der R. Brauerschen Algebrenklassengruppe über einem algebraischen Zahlkörper*, Mathematische Annalen, vol. 107 (1933), p. 731.

|| Artin, *Zur Arithmetik hyperkomplexer Zahlen*, Abhandlungen des Mathematischen Seminars, Hamburg Universität, vol. 5 (1927).

An order is called maximal if it cannot be imbedded properly in any other order. Suppose I is maximal in B. Then it can be shown that every element of I satisfies an equation with rational integral coefficients and highest coefficient 1, and also that I contains the modulus of B. Thus a maximal order is an integral set according to the definition of Dickson.†

The construction of an order in an algebra A is trivial. If, for example, in (3), α is a rational integer and the z_k in (1) are any basis of Z which is also a basis for the integers of Z, i.e., the unique maximal order in Z, then the totality of linear combinations of the basal units (1) with rational integral coefficients is an order in A which is independent of the particular basis $z_1, \cdots, z_n$ of the maximal order of Z and may be called the order in A associated with the cyclic generation (4) of A. An order thus associated with a particular cyclic generation of an algebra A is not in general a maximal order. The importance of constructing maximal orders in an algebra arises from the simplicity of the arithmetic in a maximal order as compared with that in a non-maximal order.

Albert‡ has determined explicitly maximal orders for every rational cyclic algebra of degree 2, that is, every rational generalized quaternion algebra Q. Following a similar plan but using more general methods we shall obtain maximal orders for every algebra A. First, by means of Hasse's theory of invariants of cyclic algebras we shall obtain for each A cyclic generations of an especially simple form. These will be called canonical generations. We shall then exhibit n distinct maximal orders in A containing the order in A associated with a given canonical generation in the manner described above. Finally, by a consideration of π-adic components at all rational prime spots π of R, we shall show that these are the only maximal orders in A which contain the order associated with a given canonical generation.

2. **Canonical generations of the algebras A.** An algebra A defined by a cyclic generation (4) clearly depends upon α, Z, and S but these are by no means uniquely determined by A, and a given A has infinitely many cyclic generations involving distinct sub-corps Z and, for a given Z, distinct α and S. All cyclic generations are determined by means of Hasse's theory of the invariants of a cyclic algebra (see H). A complete set of invariants of A consists of the degree n and the totality of the integers ν_π modulo n, where π ranges over all prime spots of R, defined in terms of the given generation (4) by

$$((\alpha, Z)/\pi) = S^{\nu_\pi},$$

where $((\alpha, Z)/\pi)$ is the norm residue symbol. A necessary and sufficient con-

† Dickson, *Algebren und ihre Zahlentheorie*, p. 198.

‡ Albert, *Integral domains in rational generalized quaternion algebras*, Bulletin of the American Mathematical Society, 1934, p. 164.

dition that A be a total matric algebra is that $\nu_\pi \equiv 0 \pmod{n}$ for every π. An algebra A is either a total matric algebra or a division algebra since n is a prime. We assume in this section that A is a division algebra. Then there are a finite number, s, of distinct primes $q_1, \cdots, q_s$ such that $\pi_{q_j} \not\equiv 0 \pmod{n}$ $(j=1, \cdots, s)$ and

$$\sum_{j=1}^{s} q_j \equiv 0 \pmod{n}, \tag{5}$$

whereas $\nu_\pi \equiv 0 \pmod{n}$ for every π distinct from $q_1, \cdots, q_s$. In particular (5) shows that $s \geqq 2$. The primes $q_1, \cdots, q_s$ are characterized by the property that the q_j-adic fields R_{q_j} do not split A whereas R_π splits A for every other π. In the present case, n an odd prime, the single infinite prime spot π_∞ of R does not occur among the q_j since Z is real and hence R_{π_∞}, which is by definition the field of all real numbers, contains a subfield isomorphic to Z and so necessarily splits A. Thus $\sigma = q_1 \cdots q_s$ is a rational integer. For the purposes of this paper we now prove the existence of cyclic generations of A of the type, which we shall call canonical, described in

THEOREM 1. *Let A be a cyclic division algebra of odd prime degree n over R, and let $q_1, \cdots, q_s$ $(s \geqq 2)$ be the distinct rational primes at which A does not split. Let $\sigma = q_1 \cdots q_s$. Then there exist infinitely many rational primes p with the following properties: $p \equiv 1 \pmod{n}$ and is prime to σ; $q_1, \cdots, q_s$ are n-ic non-residues modulo p and σ is an n-ic residue modulo p; the unique cyclic field Z of degree n over R, of conductor (Führer) p, discriminant p^{n-1}, has an automorphism S such that (σ, Z, S) is a cyclic generation of A.*

We first prove the following

LEMMA.† *Let n be a fixed odd prime. If $q_1, \cdots, q_s$ $(s \geqq 2)$ are distinct rational primes and $\beta_2, \cdots, \beta_s$ are rational integers prime to n, then there exist infinitely many rational primes p with the following properties: $p \equiv 1 \pmod{n}$ and prime to σ; q_1 is an n-ic non-residue modulo p; there exist rational integers $y_2, \cdots, y_s$ such that*

$$q_1^{\beta_j} q_j \equiv y_j^n \pmod{p}. \tag{6}$$

To prove the lemma let ζ be a primitive nth root of unity and $K = R(\zeta)$. Let

$$\alpha_1 = q_1, \qquad \alpha_j = q_1^{\beta_j} q_j \qquad (j = 2, \cdots, s).$$

Suppose the quantity

† This lemma is similar to a lemma used by Artin in his proof of the general law of reciprocity. See Hasse's *Bericht*, II, Jahresbericht der Deutschen Mathematiker-Vereinigung, Ergänzungsband 6, p. 18.

$$P = \alpha_1^{x_1} \cdots \alpha_s^{x_s},$$

where $x_1, \cdots, x_s$ are rational integers, is the nth power of a quantity a of K. Then

$$P = a^n, \; a \text{ in } K,$$
$$P^{n-1} = N_{KR}(a^n) = (N_{KR}(a))^n.$$

Thus p^{n-1} is the nth power of the rational quantity $N_{KR}(a)$. It follows that P is itself the nth power of a rational quantity since $n-1$ is prime to n, and hence that $x_j \equiv 0 \pmod{n}$ $(j=1, \cdots, s)$. From this we can conclude that the fields

$$K(\alpha_1^{1/n}), \cdots, K(\alpha_s^{1/n})$$

are independent† and hence that, if

$$K_1 = K(\alpha_2^{1/n}, \cdots, \alpha_s^{1/n}), \; K_2 = K_1(\alpha_1^{1/n}),$$

K_2 is cyclic of degree n over K_1. It is known that K_2 is therefore a class field over K_1 for a certain cyclic ideal class group H of order n of the ideals of K_1. It is also known that in any generating class of H there are infinitely many prime ideals of the first degree relative to R and prime to $n\sigma$. Let $\mathfrak{p}$ be such a prime ideal of K_1 and $N_{KR}(\mathfrak{p}) = p$, a rational prime. Then p satisfies the conditions of the lemma as we shall now show.

Since $\mathfrak{p}$ is of the first degree relative to R we must have in K_1

$$(7) \qquad \alpha_j^{1/n} \equiv y_j \pmod{\mathfrak{p}} \qquad (j = 2, \cdots, s),$$

where $y_2, \cdots, y_s$ are rational integers which are prime to p since $\mathfrak{p}$ was chosen prime to $n\sigma$ and hence does not divide the quantities on the left of (7). From (7) we get

$$q_1^{\beta_j} q_j \equiv y_j^n \qquad \pmod{\mathfrak{p}},$$

whence we get (6) since the quantities in the last congruence are rational. Suppose now we have $q_1 \equiv \gamma^n \pmod{p}$ with a rational integer γ. Then $q_1 \equiv \gamma^n \pmod{\mathfrak{p}}$. But by a known theorem‡ applied to the Kummer field $K_2 = K_1(\alpha_1^{1/n}) = K_1(q_1^{1/n})$, the power residue symbol $(q_1/\mathfrak{p})$ is 1 if and only if $\mathfrak{p}$ is in the identity class of the group H, whereas $\mathfrak{p}$ was chosen in a generating class of this group. This contradiction shows that q_1 is an n-ic non-residue modulo p, and hence also that $p \equiv 1 \pmod{n}$. This completes the proof of the lemma.

We turn now to the proof of Theorem 1. Let the invariants of A corre-

† *Bericht*, II, loc. cit., p. 43.
‡ *Bericht*, II, loc. cit., p. 51.

sponding to $q_1, \cdots, q_s$ be $\nu_1, \cdots, \nu_s$, respectively, so that we have $\nu_j \not\equiv 0$ (mod n) $(j=1, \cdots, s)$, and

$$\sum_{j=1}^{s} \nu_j \equiv 0 \qquad (\text{mod } n). \tag{8}$$

We shall prove that a rational prime p satisfying the conditions of the lemma with rational integers $\beta_2, \cdots, \beta_s$, chosen such that

$$\nu_1\beta_j \equiv -\nu_j \ (\text{mod } n) \qquad (j = 2, \cdots, s), \tag{9}$$

satisfies the conditions of Theorem 1.

There is a unique cyclic field Z of degree n over R with the conductor p, discriminant p^{n-1}, namely, the class field over R for the ideal class group in R whose identity class consists of the ideals in R which are generated by n-ic residues modulo p. Since q_1 is an n-ic non-residue modulo p, q_1 is not in this identity class and hence the Frobenius-Artin symbol (Z/q_1) is not the identity automorphism E of Z over R. Since ν_1 is prime to n the automorphism S of Z defined by

$$S^{-\nu_1} = (Z/q_1)$$

is different from E. Thus $A' = (\sigma, Z, S)$ is a cyclic algebra of degree n over R. Denote its invariants by ν_r'. The ν_r' are easily calculated from the properties of the norm residue symbol (H, p. 175). Thus, for a prime π not contained in $p\sigma$, we have $((\sigma, Z)/\pi) = E$, and hence $\nu_r' \equiv 0$ (mod n). To determine ν_{q_1}' we have

$$((\sigma, Z)/q_1) = (Z/q_1)^{-1} = S^{\nu_1}$$

by the definition of S, since q_1 is prime to the conductor p of Z and occurs exactly to the first power in σ. Hence $\nu_{q_1}' \equiv \nu_1$ (mod n). Next let $j \geqq 2$. We have, as for q_1,

$$((\sigma, Z)/q_j) = (Z/q_j)^{-1}.$$

By the general law of reciprocity,† condition (3) of the lemma implies

$$(Z/q_j) = (Z/q_1)^{-\beta_j} \qquad (j = 2, \cdots, s).$$

Combining these results with the definition of S and (9) we get

$$((\sigma, Z)/q_j) = S^{\nu_j},$$

and hence $\nu_{q_j}' \equiv \nu_j$ (mod n) $(j=2, \cdots, s)$. For the algebra A' the corresponding condition to (8) is

† *Bericht*, II, loc. cit., p. 11.

$$\sum_{j=1}^{s} \nu'_{q_j} + \nu'_p \equiv 0 \qquad (\text{mod } n),$$

which becomes, from what we have just shown,

$$\sum_{j=1}^{s} \nu_j + \nu'_p \equiv 0 \qquad (\text{mod } n).$$

Then (8) implies $\nu'_p \equiv 0$ (mod n).

We have now shown that the algebra A' has the same invariants as A. It is therefore isomorphic to A by Hasse's Theorem A (H, p. 176). In other words A has the cyclic generation (σ, Z, S). To complete the proof of the theorem we have only to point out that $q_1, \cdots, q_s$ are n-ic non-residues modulo p by the lemma, and that condition (8) is equivalent to the statement that σ is an n-ic residue modulo p.

3. **Maximal orders in the algebras A.** We consider in this section a fixed cyclic algebra A of degree n over R and propose to determine maximal orders in A. For convenience we use the term *basis*, as distinguished from *R-basis* and *Z-basis*, to refer to a basis with respect to rational integers of an order in A or in one of its sub-corps. As previously stated, A is either a total matric or a division algebra over R. Suppose that A is a total matric algebra. Then it is well known that any complete set of matrix units of A is also a basis of a maximal order in A. We assume henceforth that A is a division algebra.

By Theorem 1, A has infinitely many canonical generations. Let

$$A = (\sigma, Z, S) \tag{10}$$

be a fixed canonical generation of A so that Z is the unique cyclic field of degree n over R with a certain fixed prime conductor $p \equiv 1$ (mod n), discriminant p^{n-1}, and $\sigma = q_1 \cdots q_s$, where the q_j and σ have the properties in Theorem 1. Let I denote the order in A associated with the generation (10). Then I has the basis

$$u^i z_k \qquad (i = 0, \cdots, n-1;\ k = 1, \cdots, n), \tag{11}$$

where the z_k form a basis for the integers of Z and the usual relations hold:

$$u^n = \sigma,\ zu = uz^S \text{ for every } z \text{ in } Z. \tag{12}$$

We shall exhibit n distinct maximal orders in A which contain I. We need certain properties of the field Z, and a well known representation of A as an algebra of matrices with elements in Z. These will now be described.

To describe the properties of Z let ξ be a primitive pth root of unity and let g be a primitive root modulo p. Then Z is the field $R(\eta)$, where η is the Gaussian period

$$\eta = \sum_{r=0}^{h-1} \xi^{g^{nh}}, \qquad p = hn + 1.$$

It is known that the n conjugates

$$\eta, \eta^{S}, \cdots, \eta^{S^{n-1}} \tag{13}$$

form a so-called *normal basis for the integers of* Z. We shall assume that $z_1, \cdots, z_n$ in (11) are the quantities (13) in that order. Then (12) implies

$$u^n = \sigma, \qquad z_k u = u z_{k+1} \qquad (k = 1, \cdots, n), \tag{14}$$

where we agree that $z_r = z_s$ if $r \equiv s \pmod{n}$. In Z a rational prime π factors as follows. If π is distinct from p, π is the product of n distinct prime ideal factors in Z or is itself a prime in Z according as it is an n-ic residue or an n-ic non-residue modulo p. The rational prime p is the power

$$p = \mathfrak{p}^n, \tag{15}$$

of a prime ideal $\mathfrak{p}$ which is a principal ideal, being generated, for example, by the quantity

$$\beta = \prod_{r=0}^{h-1} (1 - \xi^{g^{hn}}), \tag{16}$$

which is in Z and satisfies the relations

$$N_{ZR}(\beta) = p, \qquad \mathfrak{p} = (\beta). \tag{17}$$

We summarize these properties of Z in

THEOREM 2. *The integers of Z have a normal basis $z_1, \cdots, z_n$ such that in A the relations* (14) *hold. In Z, the rational prime p is the power $\mathfrak{p}^n$ of a prime ideal $\mathfrak{p}$ which is a principal ideal, and there exists a quantity β of Z such that* (17) *holds. A rational prime π distinct from p is the product of n distinct prime ideal factors or is itself a prime in Z according as it is an n-ic residue or an n-ic non-residue modulo p.*

The basis (11) of I is also an R-basis of A and by means of it one obtains in a well known manner an algebra $\bar{A}$ of matrices with elements in Z which is isomorphic to A. Thus, consider the vector $(1, u, \cdots, u^{n-1})$ and let a be any element of A. We get, using (14),

$$\begin{aligned} a(1, u, \cdots, u^{n-1}) &= (a, au, \cdots, au^{n-1}) \\ &= (1, u, \cdots, u^{n-1})\bar{a}, \end{aligned}$$

where $\bar{a}$ is an n-rowed square matrix with elements in Z. Let

$$a = \sum_{i,k} \alpha_{ik} u^i z_k, \qquad \alpha_{ik} \text{ rational}. \tag{18}$$

This may be written

$$a = \sum_i u^i x_i, \qquad x_i = \sum_k \alpha_{ik} z_k, \tag{19}$$

where the x_i are elements of Z. Denote the conjugates of x_i by

$$x_i^{(0)} = x_i,\ x_i^{(1)},\ \cdots,\ x_i^{(n-1)},$$

where $x_i^{(1)} = x_i^S$ and so on. Then we have

$$a \to \bar{a} = \left\| \begin{matrix} x_0, & \sigma x_{n-1}^{(1)}, & \cdots, & \sigma x_1^{(n-1)} \\ x_1, & x_0^{(1)}, & \cdots, & \sigma x_2^{(n-1)} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ x_{n-1}, & x_{n-2}^{(1)}, & \cdots, & x_0^{(n-1)} \end{matrix} \right\| \tag{20}$$

in which σ appears as a factor of every element above the main diagonal. The characteristic equation of $\bar{a}$ is the so-called principal equation of a, and the trace and determinant of $\bar{a}$ are the reduced trace and norm, respectively, of a. We shall denote them by $T(a)$ and $N(a)$. It is to be noted that $T(z)$ and $N(z)$, for a z in Z, are identical, respectively, with the trace and norm of z as an element of Z.

We now define the reduced discriminant of the order I. Let the basis (11) be denoted by the vector $v = (v_1, \cdots, v_{n^2})$ whose components are numbered as indicated by

$$v_1 = z_1, \cdots, v_n = z_n,\ v_{n+1} = uz_n, \cdots, v_{n^2} = u^{n-1}z_n. \tag{21}$$

Then the determinant

$$\Delta = \Delta(v) = | T(v_r v_t) | \qquad (r, t = 1, \cdots, n^2)$$

is called the reduced discriminant of the basis (11). Let P be any n^2-rowed non-singular rational matrix and define a vector w of elements of A by

$$w = Pv. \tag{22}$$

Then it can be shown that

$$\Delta(w) = | P |^2 \Delta(v). \tag{23}$$

Since a necessary and sufficient condition that w be a basis of I is that $|P| = 1$, the quantity Δ depends only upon I and is called the reduced discriminant of I. From (22) and (23) we are also led at once to the correspondence between

maximal orders and minimal discriminants here as in the case of algebraic number fields.†

The value of $\Delta(v)=\Delta(I)$ with the numbering (21) is easily calculated by means of the isomorphism $A\cong\bar{A}$, and the fact that the n-rowed determinant

$$| T(z_k z_j) | \qquad (k, j = 1, \cdots, n)$$

has the value p^{n-1} since it is the discriminant of Z. We find as the result of the calculations

$$\Delta = \Delta(I) = \sigma^{n(n-1)}p^{n(n-1)}. \tag{24}$$

This is the first part of

THEOREM 3. *The discriminant of I is given by* (24). *The discriminant of any maximal order in A which contains I is divisible by $\sigma^{n(n-1)}$.*

To prove the second part of Theorem 3 suppose the quantity a in (18) and (19) is in a maximal order I' which contains I. By closure the n^2 quantities av_t must also be in I'. We get, by combining (18) and (21) and multiplying by v_t ($t=1, \cdots, n^2$), n^2 equations of the form

$$av_t = \sum_{r=1}^{n^2} \alpha_r v_r v_t,$$

where the α_r are the α_{ik} renumbered. Taking traces and solving for the α_{ik} we see that the denominators of these rational coefficients are divisors of Δ, since the av_t must have integral traces. With a change of notation we can therefore write

$$\Delta\bar{a} = \left\| \begin{matrix} x_0, & \sigma \overset{(1)}{x}_{n-1}, & \cdots, & \sigma \overset{(n-1)}{x_1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ x_{n-1}, & \overset{(1)}{x}_{n-2}, & \cdots, & \overset{(n-1)}{x_0} \end{matrix} \right\|, \tag{25}$$

where now the x_i are given by (19) with rational integers α_{ik} and are integers of Z.

Since a was assumed to be in a maximal order, its principal equation, i.e., the characteristic equation of $\bar{a}$, must have rational integral coefficients and highest coefficient 1. From this it follows in particular that $|\bar{a}|$ must be a rational integer. From (25) we get

$$\Delta^n | \bar{a} | = N(x_0) + \sigma Q,$$

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† For details see Artin's paper cited in the Introduction.

where Q is a rational integer. Hence we must have

$$N(x_0) \equiv 0 \pmod{\sigma}. \tag{26}$$

By Theorem 1 each prime factor q of σ is an n-ic non-residue modulo p and is therefore a prime in Z by Theorem 2. Hence $N(x_0) \equiv 0 \pmod{q}$ implies $x_0 \equiv 0 \pmod{q}$, and (26) implies $x_0 \equiv 0 \pmod{\sigma}$ since the prime factors of σ are distinct. We use this in (25) which then implies that

$$\Delta^n \mid \bar{a} \mid = \sigma N(x_1) + \sigma^2 Q_1,$$

where Q_1 is a rational integer. This implies similarly that $x_1 \equiv 0 \pmod{\sigma}$. It is evident that the same argument then yields $x_2 \equiv 0 \pmod{\sigma}$ and so on, and we can cancel a factor σ in (25). Then we can repeat the whole argument for the resulting equation and so on. We are led ultimately to the condition

$$x_0 \equiv x_1 \equiv \cdots \equiv x_{n-1} \equiv 0 \pmod{\sigma^{n(n-1)}},$$

which implies that each α_{ik} in (19) is divisible by $\sigma^{n(n-1)}$.

We have now shown that *a necessary condition that a quantity of A be in a maximal order, say I', containing I, is that its coefficients, when it is expressed by means of the R-basis* (11) *of A, have denominators which are at most powers of p*. This condition is equivalent to the second part of Theorem 3. For it is easy to show by the usual argument that I' has a basis. Let its basis be w. Then w will be related to v as in (22) for some P, and (23) implies the equivalence just claimed.

We now construct a maximal order in A containing I. The congruence

$$\lambda^n \equiv \sigma \pmod{p} \tag{27}$$

has a rational integral solution by Theorem 1. Let λ be a fixed solution of (27), let $\alpha = \beta^{n-1}$, where β is the quantity in (16) and (17), and define a quantity y by

$$py = (\lambda - u)\alpha. \tag{28}$$

With these definitions we are ready to prove

THEOREM 4. *For a fixed rational integral solution λ of* (27), *the set $I(\lambda)$ of linear combinations with rational integral coefficients of the quantities*

$$y^i z_k \qquad (i = 0, \cdots, n-1;\ j = 1, \cdots, n), \tag{29}$$

where y is defined by (28) *with $\alpha = \beta^{n-1}$ and the z_k form the normal basis of the integers of Z described in Theorem* 2, *is a maximal order*† *in A.*

† Maximal orders similar to this in certain algebras A of degree 3 were found by F. S. Nowlan, *Arithmetics of rational division algebras of order nine*, Transactions of the Royal Society of Canada, (3), vol. 21 (1927).

To prove the theorem, we first verify that $I(\lambda)$ satisfies the order postulates (a), (b), and (c) stated in the Introduction, and then calculate the discriminant of its basis (29). From the value found for this discriminant the maximality of $I(\lambda)$ follows from Theorem 3.

The definition of $I(\lambda)$ is such that postulates (b) and (c) are automatically satisfied. It is also obvious that $I(\lambda)$ is closed under addition and subtraction. Of the order postulates there remains only to show that it is closed under multiplication. To do this it is clearly necessary and sufficient to show that y satisfies an equation of degree n with rational integral coefficients and highest coefficient 1, and that the n quantities $z_k y$ $(k=1, \cdots, n)$ are in $I(\lambda)$.

The quantity y satisfies the characteristic equation of the matrix $\bar{y}$ to which y corresponds in the isomorphism $A \cong \bar{A}$. Let the characteristic equation of $p\bar{y}$ be $f(t)=0$.

We have

$$(30) \qquad py \to p\bar{y} = \begin{Vmatrix} \lambda\alpha, & 0, & \cdots, & 0, & -\sigma\alpha^{(n-1)} \\ -\alpha, & \lambda\alpha^{(1)}, & \cdots & 0, & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0, & 0, & \cdots, & -\alpha^{(n-2)}, & \lambda\alpha^{(n-1)} \end{Vmatrix}$$

whence

$$f(t) = t^n - \lambda\gamma_1 t^{n-1} + \lambda^2\gamma_2 t^{n-2} - \cdots + \lambda^{n-1}\gamma_{n-1} t - \gamma_n,$$

where $\gamma_1, \cdots, \gamma_n$ are rational integers. Consider first a γ_k with $1 \leqq k \leqq n-1$. Then γ_k is the kth elementary symmetric function of α and its conjugates in Z and thus is of degree k in α and its conjugates. Since $\alpha = \beta^{n-1}$, by (17) we have $\alpha \equiv 0 \pmod{\mathfrak{p}^{n-1}}$, and since $\mathfrak{p}$ is unaltered by each of the automorphisms of Z, this implies $\alpha^{(j)} \equiv 0 \pmod{\mathfrak{p}^{n-1}}$ for $j=0, \cdots, n-1$. Hence

$$\gamma_k \equiv 0 \qquad (\mathrm{mod}\ \mathfrak{p}^{k(n-1)}),$$
$$\gamma_k \equiv 0 \qquad (\mathrm{mod}\ \mathfrak{p}^{(k-1)n+n-k}).$$

This shows that γ_k/p^{k-1} is a rational integer such that

$$\gamma_k/p^{k-1} \equiv 0 \qquad (\mathrm{mod}\ \mathfrak{p}^{n-k}),$$

whence

$$\gamma_k/p^{k-1} \equiv 0 \qquad (\mathrm{mod}\ p),$$

since $n-k>0$. Hence $\gamma_k \equiv 0 \pmod{p^k}$. From (30) we see that $\gamma_n = N(\alpha)(\lambda^n - \sigma)$. Hence $\gamma_n \equiv 0 \pmod{p^n}$ since $N(\alpha) = p^{n-1}$ and (27) holds. These results show

that the characteristic equation of $\bar{y}$ which is satisfied by y is of the type required.

Now consider a product $z_k y$. We have

$$\begin{aligned} pz_k y &= z_k(\lambda - u)\alpha = z_k\lambda\alpha - z_k u\alpha \\ &= z_k\lambda\alpha - z_{k+1}\lambda\alpha + z_{k+1}\lambda\alpha - uz_{k+1}\alpha \\ &= \lambda\alpha(z_k - z_{k+1}) + pyz_{k+1} \\ &= b_k + pyz_{k+1}, \end{aligned}$$

where b_k is an integer of Z, which we shall show is divisible by p. We have $\alpha \equiv 0 \pmod{\mathfrak{p}^{n-1}}$. But $z_k - z_{k+1} \equiv 0 \pmod{\mathfrak{p}}$ for each $k = 1, \cdots, n$, since the so-called group of inertia (*Trägheitsgruppe*) of $\mathfrak{p}$ is the whole Galois group of Z and $z_{k+1} = z_k^S$. Thus $b_k \equiv 0 \pmod{\mathfrak{p}^n}$, $b_k \equiv 0 \pmod{p}$. Thus we can write $z_k y = b_k/p - yz_{k+1}$, where b_k/p is an integer of Z, which shows that $z_k y$ is in $I(\lambda)$. *This completes the proof that $I(\lambda)$ is an order.* That it contains I is trivial since it obviously contains $z_1, \cdots, z_k$ and also $u = \lambda + yp/\alpha$, where $p/\alpha = p/\beta^{n-1}$ is an integer of Z.

We now evaluate the discriminant of $I(\lambda)$. Let the elements of the basis (29) of $I(\lambda)$ be denoted by $w_1, \cdots, w_{n^2}$ with similar numbering to that in (21). We then have a relation (22) with a rational matrix P which is easily seen to be of the form

$$P = \begin{Vmatrix} P_1 & 0 & 0 & \cdots & 0 \\ \cdot & P_2 & \cdot & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & P_n \end{Vmatrix}$$

where the P_k are n-rowed square matrices, and P has zeros everywhere above these. The determinants $|P_k|$ are easily calculated. It is clear that P_1 is the n-rowed identity matrix and its determinant is 1. To find P_2 we have

$$pw_{n+1} = (\lambda - u)\alpha z_1, \cdots, pw_{2n} = (\lambda - u)\alpha z_n.$$

Since α is an integer of Z we have

$$\alpha z_k = \sum_{j=1}^{n} \alpha_{jk} z_k \qquad (k = 1, \cdots, n),$$

where the α_{jk} are rational integers. It is clear that $pP_2 = \|\alpha_{jk}\|$. But it is well known that $|\alpha_{jk}| = N(\alpha) = p^{n-1}$ and hence $|P_2| = 1/p$. The coefficients of u^2 in $p^2 w_{2n+1}, \cdots, p^2 w_{3n}$, respectively, are $\alpha\alpha^{(1)} z_1, \cdots, \alpha\alpha^{(1)} z_n$, and we see that $p^2 P_3$ is a rational matrix whose determinant is $N(\alpha\alpha^{(1)}) = p^{2(n-1)}$ so that $|P_3| = 1/p^2$. Similarly in general we find that $|P_k| = 1/p^k$. Thus we get

$$|P|^2 = |P_1^2 \cdots P_n^2| = 1/p^{n(n-1)},$$

and $\Delta(w) = \sigma^{n(n-1)}$ by (23) and (24).

That $I(\lambda)$ is a maximal order in A now follows from the second part of Theorem 3 and the fact that an order with minimum discriminant is necessarily maximal. This completes the proof of Theorem 4.

In Theorem 4 a particular solution of (27) was chosen. The question now raises itself as to the effect of choosing a different solution of (27) or of replacing α in (28) by another integral quantity of Z whose norm is p^{n-1}. The latter would clearly have the effect only of yielding a different basis of the same order since any integral quantity of Z whose norm is p^{n-1} is the product of α and a unit of Z. The effect of the former is given in

Theorem 5. *There are n distinct maximal orders in A which contain I, corresponding to the n distinct solutions modulo p of* (27). *These n maximal orders are such that each can be obtained from any of the others by transformation with a suitable power of* β.

Let λ_1 and λ_2 be distinct solutions of (27), and let y_1 and y_2 be the quantities defined by (28) for $\lambda = \lambda_1$ and λ_2, respectively. Then the matrix $\bar{y}_1 - \bar{y}_2$ to which $y_1 - y_2$ corresponds in the isomorphism $A \cong \bar{A}$ does not have an integral determinant as its form readily shows. Thus $I(\lambda_1)$ and $I(\lambda_2)$ must necessarily be distinct. This proves the first part of Theorem 5.

To prove the second part of the theorem we consider the effect of transforming $I(\lambda)$, for a given λ, by β. The set $I' = \beta^{-1} I(\lambda)\beta$ is clearly a maximal order in A with the basis $y'^i z_k$, where $y' = \beta^{-1} y \beta$. We shall show that I' is identical with a maximal order $I(\lambda_1)$ for a solution λ_1 of (27) such that $\lambda_1 \not\equiv \lambda$ (mod p). We have

$$py' = p\beta^{-1}y\beta = \lambda\alpha - u\alpha(\beta/\beta^{(1)}),$$

$$\alpha\lambda\beta^{(1)}/\beta - u\alpha = py'\beta^{(1)}/\beta. \tag{31}$$

The quantity $\beta^{(1)}/\beta$ is a unit of Z, and since $\mathfrak{p}$ is of the first degree, we must have

$$\beta^{(1)}/\beta \equiv \gamma \pmod{\mathfrak{p}}, \qquad \beta^{(1)} \equiv \gamma\beta \pmod{\mathfrak{p}^2}, \tag{32}$$

where γ is a rational integer. To the first congruence in (32), we apply in succession the automorphisms $E = S^0, S, \cdots, S^{n-1}$, under which $\mathfrak{p}$ is invariant, and multiply the resulting congruences. We get $N(\gamma) \equiv N(\beta^{(1)}/\beta)$ (mod $\mathfrak{p}$) whence $\gamma^n \equiv 1$ (mod p). Moreover, $\gamma \not\equiv 1$ (mod p), since otherwise we would have from the second congruence (32)

$$\beta \equiv \beta^{(1)} \cdots \equiv \beta^{(n-1)} \pmod{\mathfrak{p}^2},$$

which with $T(\beta)\equiv 0 \pmod{p}$ would lead to $n\beta\equiv 0 \pmod{\mathfrak{p}^2}$, $\beta\equiv 0 \pmod{\mathfrak{p}^2}$, a contradiction.

We now set $\lambda_1=\gamma\lambda$. Then $\lambda_1^n\equiv\lambda^n\equiv\sigma$, $\lambda_1\not\equiv\lambda \pmod{p}$ and the results of the last paragraph show that we have

$$\alpha\lambda\beta^{(1)}/\beta \equiv \lambda_1\alpha \pmod{p}, \qquad \alpha\lambda\beta^{(1)}/\beta = \lambda_1\alpha + zp,$$

where z is an integer of Z, since $\alpha\equiv 0 \pmod{\mathfrak{p}^{n-1}}$ and $\beta^{(1)}/\beta\equiv\gamma \pmod{\mathfrak{p}}$. Substituting this in (31) we get

$$(\lambda_1 - u)\alpha = -pz + py'\beta^{(1)}/\beta,$$

which shows that $I(\lambda_1)\subseteqq I'$ and hence that $I(\lambda_1)=I'$ since it is maximal.

By a continuation of this discussion it is easy to show that the transformation of a given $I(\lambda)$ by the powers $\beta, \beta^2, \cdots, \beta^{n-1}$ yields maximal orders in A corresponding, respectively, to the distinct solutions $\lambda\gamma, \cdots, \lambda\gamma^{n-1}$ of (27), where γ is the rational integer defined in the last paragraph. An obvious argument now yields the second part of Theorem 5.

We shall show in the next section by less direct methods that the n maximal orders in A corresponding to the distinct solutions modulo p of (27) are the only maximal orders in A which contain I.

4. **The number of maximal orders containing an order A.** In this section we consider the order I in A associated with a fixed canonical generation (10) of A and show that there are exactly n distinct maximal orders in A containing I. We have already seen by Theorems 4 and 5 that there are n distinct such maximal orders and we now show that there are not more than n. To do this we consider the π-components of any maximal order which contains I, where π ranges over all prime spots of R. By the π-component of an order, for a fixed π, we mean the π-adic limit set of the order,† which is easily shown to be an order in the algebra A_π obtained from A by extending the centrum to be R_π. Thus A_π is the π-adic limit set of A. Conversely, we have the following fundamental theorem due to Hasse:

THEOREM 6. *A maximal order in A is the intersection of the totality of its π-components and A.*

With a view to the application of this theorem to our problem we now determine the number of maximal orders in the algebra A_π, for a fixed π, which contain the π-component I_π of I.

First suppose π is distinct from p. Let a be any quantity of A which is in a maximal order containing I. It was proved in §3 that, if a is expressed in terms

† This definition, and the proof of Theorem 6 and other properties used here, are given by Hasse, *Über p-adische Schiefkörper und ihre Bedeutung für die Arithmetik hyperkomplexer Zahlsysteme*, Mathematische Annalen, vol. 104 (1931), p. 495.

of the basis (11) of I, which is an R-basis of A, the rational coefficients have denominators which are at most powers of p. These denominators are units of R_π since $\pi \neq p$, and hence the π-adic limit set of any maximal order in A which contains I is I_π itself. In other words, if $\pi \neq p$, there is a single maximal order in A_π which contains I_π.

Now consider the case $\pi = p$. The algebra A_p is a total matric algebra by Theorem 2, since p is not one of the prime factors q of σ. Evidently A_p over R_p has the cyclic generation

$$A_p = (\sigma, Z_p, S_p),$$

where Z_p is the p-adic limit set of Z which is easily shown to be a cyclic field of degree n over R_p, and S_p is the automorphism of Z_p corresponding to the automorphism S of Z. We may regard A_p as the crossed product of Z_p and its Galois group with the operators $1, u, \cdots, u^{n-1}$ corresponding to the automorphisms $E, S_p, \cdots, S_p^{n-1}$, respectively, and the factor system consisting of 1's and σ's. The equation

$$x^n - \sigma = 0 \tag{33}$$

has a solution in R_p by a well known theorem on p-adic fields, since $x^n - \sigma$ factors modulo p into n distinct linear factors. Let ξ be a fixed solution in R_p of (33). We replace the operator u by v, where $u = \xi v$, which obviously has the effect of yielding a new factor system, equivalent to the former, consisting of 1's only.

With this operator v and factor system we can give explicitly all maximal orders in A_p. Let

$$V = 1 + v + \cdots + v^{n-1}.$$

Then every maximal order in A_p is of the form†

$$I(\mathfrak{m}) = \mathfrak{m}^* V \mathfrak{m},$$

where $\mathfrak{m}$ and $\mathfrak{m}^*$ are complementary moduls in Z_p. A necessary and sufficient condition that $I(\mathfrak{m})$ contain the maximal order of Z_p is that $\mathfrak{m}$ be an ideal of Z_p. The only ideals of Z_p are the prime ideal generated by the prime ideal $\mathfrak{p}$ of Z and its powers. We shall denote the prime ideal of Z_p also by $\mathfrak{p}$. Then the only maximal orders in A_p which contain the maximal order of Z_p are those of the form

† This explicit form for the maximal orders in a crossed product which is a total matric algebra over an algebraic number field was given by Emmy Noether: *Zerfallende verschränkte Produkte und ihre Maximalordnungen*. Actualités Scientifiques et Industrielles, No. 148 (Herbrand Memorial). A brief examination of Noether's proof of this and further consequences of it, in the case of an algebraic coefficient field, will show that the corresponding theorems hold almost trivially in the present case, namely, with the coefficient field R_p.

$$I(\mathfrak{p}^r) = \mathfrak{p}^{r*}V\mathfrak{p}^r, \; r \text{ a rational integer,} \tag{34}$$

where by $\mathfrak{p}^0$ will be meant the maximal order of Z_p. Since the different of Z_p is is $\mathfrak{p}^{n-1}$ we have

$$\mathfrak{p}^{r*} = \mathfrak{p}^{-r}\mathfrak{p}^{-(n-1)}.$$

The ideal $\mathfrak{p}^n$ of Z_p is the ideal generated by the rational prime p. Combining these we see that

$$I(\mathfrak{p}^{n+r}) = (\mathfrak{p}^{n+r})^*V\mathfrak{p}^{n+r} = (p\mathfrak{p}^r)^*Vp\mathfrak{p}^r = \mathfrak{p}^{r*}V\mathfrak{p}^r = I(\mathfrak{p}^r),$$

since $(p\mathfrak{p}^r)^* = \mathfrak{p}^{r*}/p$ and p is commutative with V. This shows that in the set (34) ($r=0, \pm 1, \pm 2, \cdots$) there are at most n distinct maximal orders. Hence there are at most n distinct maximal orders in A_p which contain the maximal order of Z_p and a fortiori, at most n which contain I_p.

The results obtained for the two cases $\pi \neq p$ and $\pi = p$, combined with Theorem 6, show that there are at most n distinct maximal orders in A which contain I. For, by Theorem 6, two such maximal orders in A must have distinct π-components for at least one π. But we have shown that their π-components can differ only for $\pi = p$ and that here there are at most n distinct possibilities. We combine this with the earlier theorems of §§2 and 3 and summarize the results of this paper in

THEOREM 7. *A cyclic division algebra A of odd prime degree n over R has infinitely many distinct canonical generations of the type described in Theorem* 1. *The order in A associated with a fixed such generation in the manner described in the Introduction, can be imbedded in exactly n distinct maximal orders in A and these maximal orders are of the type given in Theorems* 4 *and* 5.

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CARATHÉODORY MEASURE AND A GENERALIZATION OF THE GAUSS-GREEN LEMMA†

BY

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Introduction

1. The Gauss-Green lemma for the plane connects the double integral of a partial derivative of a function over a region R with the line integral of the function around the curve C bounding R. This connection is given by the formula

$$\int\int_R \frac{\partial f(x, y)}{\partial y}\,dxdy = \int_C f(x, y)\cos\beta\,ds$$

where s is the arc length of C and β is the angle made by the external normal to C at a variable point of C, with the positive y-axis. Thus for the line integral to have a meaning it is necessary that C have length and consequently be defined by two functions $x=\phi(t)$, $y=\psi(t)$ of bounded variation in t.

2. The applications of this lemma, through Green's theorem, are numerous. Consequently many investigations have been concerned with the types of regions and boundaries for which the lemma is valid. In most cases it has been deemed inherent in such a relation that the boundary be a curve, i.e., that an order relation be known among the points of the boundary. However, attacking the problem by radically different methods, J. P. Schauder‡ obtained results for a class of boundaries with no order relation prescribed. True, the only boundaries shown by Schauder to be admissible are those represented by two functions each satisfying the Lipschitz conditions, thus indirectly again introducing order and also a condition more restricting than that of bounded variation.

Also Schauder assumed that all points of the boundary of the second class§ project on the x-axis in a set of Lebesgue measure zero. Thus so simple a region as all points of the unit circle except the points $0\leqq x<1$ of the x-axis does not satisfy his conditions.

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† Presented to the Society, October 27, 1934; received by the editors January 8, 1935.

‡ Fundamenta Mathematicae, vol. 8 (1926), p. 1. Schauder states his results in terms of an integral over a volume and an integral over the boundary of the volume, but, as he points out, analogous results hold connecting n- and $(n-1)$-dimensional integrals, $n=2, 3, \cdots$. We discuss his results for $n=2$.

§ This subset of the boundary is defined in §16 below.

3. The present paper contains a proof of the Gauss-Green lemma under what seems extreme simplification of the conditions on the boundary. For a simply connected region there is no restriction except that the boundary have Carathéodory linear measure finite. Then by methods which have the effect of the usual crosscut scheme, applicable regions are extended to a wide class not simply connected. Furthermore, simplicity of restrictions on boundaries is gained by more careful analysis of properties of boundaries, and not by specialization of the function $f(x, y)$.

From the proofs given, analogous results are seen to hold connecting n- and $(n-1)$-dimensional integrals.

I. Carathéodory measure of a set and its closed subsets

4. Carathéodory† developed on five axioms a general theory of measure in which most of the theorems of the usual Lebesgue theory have analogues. The theorem that the inner measure of a set is the upper limit of the measures of closed subsets of the set, which plays such a central role in the Lebesgue theory, has not, however, been shown to follow from Carathéodory's five axioms. This closed subset theorem does follow, as proved by Hahn,‡ if in place of Carathéodory's fifth axiom the following modification is used:

Axiom V′. *To each point set A there is a sequence of open sets whose intersection contains A and has the same measure as the outer measure of A.*

After developing his general theory of measure, by merely postulating the existence of a number associated with each set, Carathéodory gave the following specific method of attaching a number to a set.

Let A be an arbitrary set in a euclidean space R_q of q dimensions. With ρ a positive number let $U_1, U_2, \cdots$ be a sequence of convex sets open in the space R_q,§ each with diameter less than ρ, whose union contains A. With d_k the diameter of U_k, consider the sums

$$d_1 + d_2 + \cdots$$

for all such sequences of point sets. Designate the least upper bound, which may be $+\infty$, of such sums by $L_\rho A$. Then $L_\rho A$ does not decrease as ρ decreases. Thus as $\rho\to 0$, $L_\rho A$ approaches a limit, finite or infinite, which in either case is called the exterior linear measure of A and is represented by L^*A.

The exterior two-dimensional measure of A is also defined by means of sets $U_1, U_2, \cdots$, each with diameter less than ρ, except that d_k is replaced

† Göttinger Nachrichten, 1914, p. 404.

‡ Hahn, *Theorie der reellen Funktionen*, vol. 1, 1921, Theorem III, p. 445.

§ Carathéodory did not assume the sets U_k to be open, but proved that the same number would be obtained if open convex sets were used.

by the two-dimensional diameter of U_k. The two-dimensional diameter of an open convex set is the least upper bound of the Lebesgue plane measures of the projections of the set on planes of all possible orientations.

Exterior linear measure is shown by Carathéodory to satisfy his five measure axioms. In proving that exterior linear measure satisfies his fifth axiom, Carathéodory did not use the fact that each U_k is open, but merely that it is linearly measurable. Upon following this proof, but using in addition the openness of U_k, one will see that Hahn's axiom V′ is also satisfied. It thus follows from the above reference to Hahn, that *if a set A has inner linear measure L_*A finite, there is a sequence of closed subsets of A whose union has the same linear measure as the inner linear measure of A.* This closed subset theorem plays a fundamental role in the work that follows.

5. One easily sees the following projection relation:

If A is a plane set and l a line of the plane, the projection A_l of A on l has Lebesgue exterior measure m^*A_l less than or equal to L^*A.

With this projection relation established, Gross† proved that if each point of A_l is the image of at least N points of A then $m^*A_l \leqq L^*A/N$. Specifically, *if L^*A is finite and each point of A_l is the image of an infinite number of points of A then $m^*A_l = 0$.*

It follows also from the above projection relation and the closed subset theorem that, *if A is linearly measurable with LA finite, then A_l is Lebesgue measurable.*

II. Measurable subsets of the union of a sequence of closed sets

6. A line perpendicular to the x-axis through a point of a plane set B may or may not contain a lowest point of B on it. We designate the set of all such lowest points, when they exist, by B^1. In general we designate by B^m the set of all points p of B such that exactly $m-1$ points of B lie below p on the same perpendicular to the x-axis.

Again a line perpendicular to the x-axis through a point of B may contain a finite or an infinite number of points of B. We designate by $\mathfrak{B}^m$ (or $\mathfrak{B}^\infty$) the collection of all points of B on all lines perpendicular to the x-axis that contain exactly m (or an infinite number of) points of B.

Arguments similar to those of Schauder show that if B is a Souslin set, each subset of B defined above is Carathéodory linearly measurable. Corresponding theorems of course hold if B is the union of a sequence of closed

† Monatshefte für Mathematik und Physik, vol. 29 (1918), pp. 174–176.

sets. However, simpler proofs can be made for the more restricted sets and at the same time furnish all that is necessary for our purpose.†

7. Suppose then B is the union of a sequence of closed sets $K_1, K_2, \cdots$.

Toward establishing the measurability of the subsets of B mentioned above we notice that the set $P(B)$, of all points on all lines perpendicular to the x-axis through points of B, is linearly measurable. For $P(B)=P(K_1)+P(K_2)+\cdots$ and each $P(K_i)$ is linearly measurable, since the part of it in or on any square is closed.

With k a positive integer let W_k^h be the collection of all points of the plane whose y-coordinates satisfy the relation $h/2^k \leqq y < (h+1)/2^k$, $h=1, 2, \cdots$. Since each set W_k^h, as well as B, is the union of a sequence of closed sets, the same is true of their intersection. Consequently, from the above proof, *the set* $P(BW_k^h)$ *is linearly measurable.*

Let $h_1 < h_2 < \cdots < h_m$ be m integers. We define

$$P^{h_1 h_2 \cdots h_m} \tag{1}$$

as all points of all lines perpendicular to the x-axis that contain points of B in each of the strips

$$W_k^{h_1}, W_k^{h_2}, \cdots, W_k^{h_m}, \tag{2}$$

but no point of B in any strip below $y=(h_m+1)/2^k$, other than these.

The point set (1) *is linearly measurable.* For the set of lines perpendicular to the x-axis with points of B in all the strips (2) may be written as

$$G_1 = \prod_{i=1}^{m} P(BW_k^{h_i}).$$

Then to obtain the set (1) we must remove from G_1 any line perpendicular to the x-axis that contains a point of B outside the strips (2), but still below $y=(h_m+1)/2^k$, i.e.,

$$G_2 = \sum_{j=-\infty}^{h_1-1} P(BW_k^j) + \sum_{j=h_1+1}^{h_2-1} P(BW_k^j) + \cdots + \sum_{j=h_{m-1}+1}^{h_m-1} P(BW_k^j).$$

Since each point set $P(BW_k^j)$ is linearly measurable, the point sets G_1 and G_2

† Schauder used a general measure Φ satisfying Carathéodory's axioms I–IV and a modification (different from Hahn's) of axiom V. He did not however prove the closed subset theorem for his general measure. To obtain this theorem he introduced a specific measure Φ_0, a modification of one defined by Gross (loc. cit.), which satisfies his modified axioms. In his proof he used some measure properties of B^m which in turn he obtained from measure properties of Souslin sets. For Carathéodory linear measure, indeed for the general measure satisfying Hahn's modification of the five axioms, the closed subset theorem followed without the use of Souslin sets and we now establish the requisite measure properties of B^m.

are linearly measurable. Consequently, $P^{h_1h_2\cdots h_m}=G_1-G_1G_2$ is also linearly measurable.

We now define the set

$$B_k^{h_1h_2\cdots h_m}$$

as all points p of B in the strip $W_k^{h_m}$ such that a line through p perpendicular to the x-axis contains points of B in all of the strips (2), but no point of B in any strip below $y=(h_m+1)/2^k$ other than these. The formulation for this set, $B_k^{h_1\cdots h_m}=(BW_k^{h_m})P^{h_1\cdots h_m}$, shows it to be linearly measurable, a fact we use in proving

THEOREM 1. *The subset B^m of B is linearly measurable.*

First consider the set

$$E_k=\sum B_k^{h_1\cdots h_m}+\sum B_{k+1}^{h_1\cdots h_m}+\cdots+\sum B_{k+i}^{h_1\cdots h_m}+\cdots,$$

where each summation sign means the union of the sets indicated as the m distinct integers $h_1<h_2<\cdots<h_m$ take on all possible values. The set E_k is then the union of a countable number of linearly measurable sets, so is itself linearly measurable.

We next assert that $B^m\subset E_k$. For if p is a point of B^m there are exactly $m-1$ other points of B below p on the same perpendicular to the x-axis. Then with k fixed, there is a number j so large that the distance between any two of these m points is greater than $1/2^{k+j}$. These m points then belong to some set of m strips $W_{k+j}^{h_1},\cdots,W_{k+j}^{h_m}$, and there are no points of B on their common line in any strip below $y=(h_m+1)/2^{k+j}$ other than these. Thus p belongs to some set $B_{k+j}^{h_1\cdots h_m}$ and so to E_k.

Thus *the set $E=E_1\cdot E_2\cdot\cdots$ is linearly measurable and also $B^m\subset E$.*

We assert, conversely, that $B^m\supset E$. For if p were a point of E and there were less than $m-1$ points of B below p on the same perpendicular to the x-axis, p and these points would not lie in any m distinct strips, so p would not belong to any set $B_k^{h_1\cdots h_m}$, so not to E. However, if there were more than $m-1$ points of B below p, for every k large enough, say $k>K$, there would be more than m strips including p and the points of B below p. Thus p would not belong to any set $B_k^{h_1\cdots h_m}$, $k>K$, so not to E_k, $k>K$, and finally not to E.

Thus B^m is the linearly measurable set E.

8. Toward proving $\mathcal{B}^m$ linearly measurable we designate by

$$\mathcal{B}_k^{h_1h_2\cdots h_m}$$

the totality of all points p of B such that the line through p perpendicular

to the x-axis contains points of B in each of the strips $W_k^{h_1}, \cdots, W_k^{h_m}$, but no others. This set is then the part of B on those lines $P^{h_1\cdots h_m}$ that do not contain points of B above $y=(h_m+1)/2^k$; that is,

$$\mathcal{B}_k^{h_1\cdots h_m} = B\left[P^{h_1\cdots h_m} - P^{h_1\cdots h_m}\sum_{j=h_m+1}^{\infty} P(BW_k^j)\right].$$

This formulation shows the set to be linearly measurable.

We now state

THEOREM 2. *The subset $\mathcal{B}^m$ (and $\mathcal{B}^\infty$) of B is linearly measurable.*

One will see that the linear measurability of $\mathcal{B}^m$ follows from that of $\mathcal{B}_k^{h_1\cdots h_m}$ in the same way the linear measurability of B^m followed from that of $B_k^{h_1\cdots h_m}$. Then $\mathcal{B}^\infty = B - \sum_{m=1}^{\infty}\mathcal{B}^m$ is also linearly measurable.

III. Normal sets and simply connected regions

9. Following Schauder, a family F_A of circles will be said to *cover* a plane set A if every point of A is the center of a sequence of circles of F_A with radii approaching zero. The set A is said to be *normal* with respect to a measure if in every F_A covering A there exists a mutually exclusive sequence of circles whose union contains almost all of A. Here a circle includes its circumference.

From the Vitali covering theorem† it follows that every bounded set A is normal with respect to Lebesgue plane measure. It is not known whether under any of the definitions of linear measure, the set A is normal with respect to linear measure even if A is linearly measurable with linear measure finite or, in fact, even if A is closed. The peculiar adaptability of Carathéodory linear measure to our problem is shown by the fact that the boundary B of of every simply connected region is normal with respect to Carathéodory linear measure if LB is finite. It is necessary, however, before proving this fact to obtain several auxiliary results.

10. With $c(p, r)$ a circle with center p and radius r, we shall call the limit superior and limit inferior as r approaches zero of the ratio

$$\frac{L^*Ac(p, r)}{2r},$$

the upper and lower exterior density of A at p and represent them by $\overline{D}^*(A, p)$ and $\underline{D}^*(A, p)$ respectively. If A is linearly measurable the asterisk is not used and the word "exterior" is dropped.

W. Sierpiński‡ has shown that in every F_A covering A there exists a se-

† Carathéodory, *Vorlesungen über Reelle Funktionen*, p. 229.

‡ Fundamenta Mathematicae, vol. 9, p. 172.

quence of mutually exclusive circles $c_1, c_2, \cdots$ such that if C_n is a circle (not necessarily belonging to F_A) with the same center, but diameter three times that of c_n, then the union $S = C_1 + C_2 + \cdots$ contains A. He also shows that if L^*A is finite, $\overline{D}^*(A, p) \leqq 1$ at almost all points of A. We use these two facts in proving

LEMMA 1. *If L^*A is finite and the lower exterior density of A is bounded from zero by a positive constant k for almost all of A, then A is normal.*†

From the conditions of the lemma and Sierpiński's density theorem, mentioned above, the subset A' of A where simultaneously $k \leqq \underline{D}^*(A, p)$ and $\overline{D}^*(A, p) \leqq 1$ has $L^*A' = L^*A$.

Let F_A be a family of circles covering A. Then the collection of all circles $c(p, r)$ of F_A such that, with p a point of A',

$$\frac{k}{2} < \frac{L^*Ac(p, r)}{2r}, \tag{1}$$

while the concentric circle $c(p, 3r)$, which may not belong to F_A, is such that

$$\frac{L^*Ac(p, 3r)}{6r} < 2, \tag{2}$$

is a family $F'_{A'}$ of circles covering A' in the sense defined above.

Then from Sierpiński's covering theorem, there exists in $F'_{A'}$ a sequence of mutually exclusive circles $c(p_1, r_1), c(p_2, r_2), \cdots$ such that the union of the larger circles $c(p_1, 3r_1), c(p_2, 3r_2), \cdots$ contains A'. Since (1) is true for each small circle and these circles are mutually exclusive,

$$k\sum_1^n r_i < \sum_1^n L^*Ac(p_i, r_i) = L^*\left[A\sum_1^n c(p_i, r_i)\right] \leqq L^*\left[A\sum_1^\infty c(p_i, r_i)\right] \leqq L^*A.$$

Thus the series of radii converge and

$$k\sum_1^\infty r_i \leqq L^*\left[A\sum_1^\infty c(p_i, r_i)\right]. \tag{3}$$

But the union of the larger circles contains A', almost all of A, and (2) is true for each of these circles, so

$$L^*A \leqq \sum_1^\infty L^*Ac(p_i, 3r_i) < 12\sum_1^\infty r_i. \tag{4}$$

From (3) and (4),

† Henceforth we use "normal" for "normal with respect to Carathéodory linear measure."

$$kL^*A < 12L^*\left[A\sum_1^\infty c(p_i, r_i)\right].$$

There is then a number n such that, for $s_1=\sum_1^n c(p_i, r_i)$, $kL^*A<12L^*As_1$. Thus $(12-k)L^*A>12(L^*A-L^*As_1)$, so, since s_1 is linearly measurable (closed),

$$L^*(A - As_1) < \left(\frac{12-k}{12}\right)L^*A. \tag{5}$$

Next let $F''_{A'}$ denote the collection of all circles of $F'_{A'}$ that have no point in common with s_1. Since s_1 is closed this family covers $A'-A's_1$. We now proceed as above, but use the set $A'-A's_1$ instead of A' and the family of circles $F''_{A'}$ instead of $F'_{A'}$, and obtain a set s_2 (the union of a finite number of mutually exclusive circles of $F''_{A'}$) and a relation similar to (5), namely,

$$L^*[(A - As_1) - (A - As_1)s_2] < \left(\frac{12-k}{12}\right)L^*(A - As_1).$$

But s_2 has no point in common with s_1, so, with the aid of (5),

$$L^*[A - A(s_1 + s_2)] < \left(\frac{12-k}{12}\right)L^*(A - As_1) < \left(\frac{12-k}{12}\right)^2 L^*A.$$

We thus see the existence of a sequence of mutually exclusive sets $s_1, s_2, \cdots$ each consisting of a finite number of mutually exclusive circles of F_A, such that for each n,

$$L^*[A - A(s_1 + s_2 + \cdots)] \leqq L^*[A - A(s_1 + \cdots + s_n)] < \left(\frac{12-k}{12}\right)^n L^*A.$$

But k is positive and L^*A is finite so A is normal.

11. By a *region* R we shall mean a connected portion of the plane, and by the *boundary* B of R, all points b such that every circle $c(b, r)$ contains both points of R and points of the complement of R. The boundary is a closed point set and is hence linearly measurable. A region is *simply connected* if every simple closed curve of the region contains on its interior only points of the region.

We now prove the proposition mentioned earlier.

THEOREM 3. *The boundary B of a simply connected region R (not the whole plane) is normal if LB is finite.*

This theorem follows from the above lemma upon showing that $\underline{D}(B, b) \geqq 1/2$ at each point b of B. First let $c(b, r)$ be a circle with center b

which does not contain all of R. The circumference of $c(b, r)$ then contains points of R, since R is connected, and points of the complement of R, since R is simply connected. Consequently there are points of B on this circumference and, moreover, on the circumference of every concentric circle $c(b, r')$ with $r' < r$.

Let $U_1, U_2, \cdots$ be a sequence of open convex sets whose union contains $Bc(b, r)$. If U_k contains b let I_k be the radius of the smallest circle with center b containing U_k. If U_k does not contain b let I_k be the difference of the radii of the two circles with centers b, one the smallest containing U_k, the other the largest containing no point of U_k. Since I_k is not greater than the diameter d_k of U_k and there is a point of B at every distance $\leqq r$ from b, $r \leqq \sum I_k \leqq \sum d_k$. Thus $r \leqq LBc(b, r)$,† so

$$\underline{D}(B, b) = \liminf_{r \to \infty} \frac{LBc(b, r)}{2r} \geqq \frac{1}{2}.$$

12. We shall also use the following property of normal sets.

THEOREM 4. *If a set B is normal and A is a linearly measurable subset of B with LA finite, then A is also normal.*‡

First, *any closed subset K of B is normal.* Let F_K be a family of circles covering K, and, since K is closed, F_{B-K} a family of circles none with a point in common with K, covering $B-K$. Then F_K together with F_{B-K} constitutes a family F_B of circles covering B. But B is normal, so there exists in F_B a sequence of mutually exclusive circles $c_1, c_2, \cdots$ whose union s contains almost all of B. However, those circles $c_1', c_2', \cdots$ of this sequence having points in common with K are circles of F_K and their union s' is such that $K - Ks' \subset B - Bs$. Thus $L(K-Ks') \leqq L(B-Bs) = 0$, so K is normal.

Next *A is normal.* For let F_A be a family of circles covering A and ϵ_n a sequence of decreasing numbers approaching zero. Since A is linearly measurable with LA finite, there exists a closed subset K_1 of A such that $LA < LK_1 + \epsilon_1/2$. But K_1, a closed subset of B, is normal, so there exists a finite number of mutually exclusive circles of F_A whose union s_1 is such that $LK_1 < LK_1 s_1 + \epsilon_1/2$. Thus $LA < LK_1 s_1 + \epsilon_1 \leqq LAs_1 + \epsilon_1$. But s_1 is closed, so the circles of F_A

† The inequality $r \leqq \Phi_0 Bc(b, r)$, plausible as it seems, has not been established. That caution is necessary with Φ_0 measure is indicated by the example of a set A in the ring $c(p, r') - c(p, r)$, $r' > r$, with a point on every radius, but with $\Phi_0 A = 0$ (and of course $LA \geqq 2\pi r$), given by Saks, Fundamenta Mathematicae, vol. 9, p. 16. If, however, the inequality were established, our results would hold for Φ_0, as well as for Carathéodory, linear measure.

‡ Results similar to those of this theorem and its corollary were indicated by Schauder. However, his theorem VIII does not follow from his previous work since he has not proved the closed subset theorem for the general measure Φ. A theorem that does follow would be obtained if Φ were replaced by Φ_0 throughout.

with no point in common with s_1 constitute a family F_{A-As_1} covering $A-As_1$. Hence in F_{A-As_1} there exists a finite number of circles whose union s_2 is such that $L(A-As_1)<LAs_2+\epsilon_2$, or, since s_1 and s_2 have no point in common, $L[A-A(s_1+s_2)]<\epsilon_2$. Consequently, there exists a sequence of mutually exclusive sets $s_1, s_2, \cdots$, each the union of a finite number of mutually exclusive circles of F_A, such that for each n, $L[A-A(s_1+s_2+\cdots)] \leqq L[A-A(s_1+\cdots+s_n)]<\epsilon_n$, so A is normal.

Corollary. *If LB is finite, F_A a family of circles covering A, and ϵ an arbitrary positive number, there exists in F_A a sequence of mutually exclusive circles whose union s is such that simultaneously*

$$LA = LAs \text{ and } L(B-A)s < \epsilon.$$

Since the set $B-A$ is linearly measurable with $L(B-A)$ finite, it contains a closed subset K such that $L[(B-A)-K]<\epsilon$. Then the circles of F_A without points in common with K again constitute a family F_A' covering A. Thus in F_A' there exists a sequence of mutually exclusive circles whose union s is such that $LA=LAs$. But the part of s in common with $B-A$ is at most $(B-A)-K$, so $L(B-A)s<\epsilon$.

IV. Projection of a set and further normality properties

13. Let B be a linearly measurable set with LB finite. From §5, the projection B_X of B on the x-axis is linearly measurable.

We shall show that *for a fixed point p the two functions $LBc(p, r)$ and $m[Bc(p, r)]_X$ of r are continuous from the right.* With r_n a decreasing sequence approaching $r_0>0$, temporarily let $B_n=Bc(p, r_n)$ and $B_0=Bc(p, r_0)$. The intersection of the sets B_n is B_0, so $\lim_{n\to\infty} LB_n=LB_0$. Also $(B_n)_X=(B_0)_X+(B_n-B_0)_X$, so $m(B_n)_X-m(B_0)_X\leqq m(B_n-B_0)_X\leqq L(B_n-B_0)$, and both functions have right hand continuity in r.

Next for r fixed the set of points P_λ where $LBc(p, r)\geqq\lambda>0$ is closed. For suppose $LBc(p_0, r)<\lambda$ where p_0 is a limit point of P_λ. We first choose a $\delta>0$ such that $LBc(p_0, r+\delta)<\lambda$, then a point $\bar{p}$ of P_λ such that $c(\bar{p}, r)\subset c(p_0, r+\delta)$, and thus obtain a contradiction. In the same way the set of points of the plane where $m[Bc(p, r)]_X\geqq\lambda>0$ is seen to be closed. One then sees that the set of points where these functions take on values between two constants is linearly measurable. Hence, following the terminology of the Lebesgue theory, we say *for r fixed $LBc(p, r)$ and $m[Bc(p, r)]_X$ are linearly measurable functions of p.*

14. We shall let B_0 represent the set of all points p of B such that $LBc(p, r)=0$ for some circle. By the Lindelöf-Young theorem† there is a

† Carathéodory, *Vorlesungen über Reelle Funktionen*, 1927, p. 46.

sequence of such circles whose union contains B_0. But the intersection of this union with B has linear measure zero, so $LB_0=0$.

Thus for every point β of $B-B_0$ the quotient

$$\frac{m[Bc(\beta, r)]_X}{LBc(\beta, r)} = Q(B, \beta, r)$$

is defined. Moreover, for r fixed, $Q(B, \beta, r)$ is the quotient of two linearly measurable functions, so is itself linearly measurable on $B-B_0$. Also for β fixed, $Q(B, \beta, r)$ has right hand continuity in r.

If $Q(B, \beta, r)$ has a limit as r approaches zero, we designate this limit by $C(B, \beta)$, otherwise by $\overline{C}(B, \beta)$ and $\underline{C}(B, \beta)$ its limit superior and limit inferior.† We extend these functions to all points of the plane by arbitrarily assigning the value zero at points not in $B-B_0$.

The function $\overline{C}(B, \beta)$ is linearly measurable on $B-B_0$. For let r_n be a decreasing sequence of numbers approaching zero and $q_1, q_2, \cdots$ the positive rational numbers in some ordering. Then let

$\overline{R}(\beta, r_n)=$least upper bound of $Q(B, \beta, q_m)$ for $0<q_m<r_n$,

$\overline{F}(\beta, r_n)=$least upper bound of $Q(B, \beta, r)$ for $0<r<r_n$.

Then $\overline{R}(\beta, r_n)\leqq F(\beta, r_n)$. But the right hand continuity in r of $Q(B, \beta, r)$ reveals that $\overline{R}(\beta, r_n)\geqq \overline{F}(\beta, r_n)$, so these two functions are equal. Both functions are then linearly measurable on $B-B_0$, since $\overline{R}(\beta, r_n)$ is the least upper bound of a sequence of functions linearly measurable on $B-B_0$. But it is seen that

$$\overline{C}(B, \beta) = \limsup_{r\to 0} Q(B, \beta, r) = \lim_{n\to\infty} F(\beta, r_n).$$

Thus $\overline{C}(B, \beta)$ is the limit of a sequence of functions linearly measurable on $B-B_0$, so is itself linearly measurable on $B-B_0$.

A similar procedure shows that $\underline{C}(B, \beta)$ is also linearly measurable on $B-B_0$. Consequently $\overline{C}(B, p)$ and $\underline{C}(B, p)$ are linearly measurable on the plane or on any linearly measurable subset E of the plane. But these functions are bounded, so if LE is finite, the integrals $\int_E\overline{C}(B, p)\,dL$ and $\int_E\underline{C}(B, p)\,dL$, taken over E in the sense of Lebesgue with respect to Carathéodory linear measure, exist.

15. We now let B be normal, in addition to being linearly measurable with LB finite, and prove two lemmas and an important integral theorem.

LEMMA 1. *If A is a linearly measurable subset of B that projects on the x-axis in a set of Lebesgue measure zero, then $\overline{C}(B, \alpha)=0$ at almost all points α of A.*

† While no notion of direction is involved here, yet if β were a point of an ordinary curve, $C(B, \beta)$ would be the absolute value of the cosine of the angle between the x-axis and the direction of B at β.

Since $\overline{C}(B, p)$ is a linearly measurable function, the subset A_λ of A where $\overline{C}(B, p) > \lambda > 0$ is linearly measurable, and hence, being a subset of a normal set B, is normal. Suppose the lemma is not true. There is then a $\lambda > 0$ such that also $LA_\lambda > 0$. Then the collection of all circles $c(\alpha, r)$, with centers at points of A_λ, such that

$$\frac{\lambda}{2} < \frac{m[Bc(\alpha, r)]_X}{LBc(\alpha, r)}, \tag{1}$$

is a family F_{A_λ} of circles covering A_λ.

Since A_λ projects on the x-axis in a set of Lebesgue measure zero,

$$m[Bc(\alpha, r)]_X = m[(B - A_\lambda)c(\alpha, r)]_X \leqq L(B - A_\lambda)c(\alpha, r).$$

Also $LA_\lambda c(\alpha, r) \leqq LBc(\alpha, r)$. Thus for each circle of F_{A_λ}, from (1),

$$\frac{\lambda}{2} LA_\lambda c(\alpha, r) \leqq L(B - A_\lambda)c(\alpha, r).$$

But, with ϵ an arbitrary positive number, from the corollary of §12, there exists a sequence of mutually exclusive circles of F_{A_λ} such that for their union s, $LA_\lambda s = LA_\lambda$ and $L(B - A_\lambda)s < \epsilon$. We thus have the contradiction, $(\lambda/2)LA_\lambda < \epsilon$, to our assumption.

LEMMA 2. *Let A be a linearly measurable subset of B with at most one point on each line perpendicular to the x-axis. If at each point α of A*

$$\lambda \leqq \dot{C}(B, \alpha), \text{ then } \lambda LA \leqq mA_X,$$

or if

$$\lambda > \dot{C}(B, \alpha), \text{ then } \lambda LA \geqq mA_X,$$

where $\dot{C}(B, \alpha)$ means either $\overline{C}(B, \alpha)$ or $\underline{C}(B, \alpha)$.

We shall prove only the first part of this lemma.

Let η be an arbitrary positive number. For either interpretation of $\dot{C}(B, \alpha)$, the collection of all circles $c(\alpha, r)$ with centers at points of A for which the inequality

$$\lambda - \eta < \frac{m[Bc(\alpha, r)]_X}{LBc(\alpha, r)}$$

holds, is a family F_A of circles covering A. But A is a subset of B, so for each circle of F_A, $(\lambda - \eta)LAc(\alpha, r) < m[Bc(\alpha, r)]_X \leqq m[Ac(\alpha, r)]_X + m[(B - A)c(\alpha, r)]_X \leqq m[Ac(\alpha, r)]_X + L(B - A)c(\alpha, r)$.

Since there is at most one point of A on any line perpendicular to the x-axis, mutually exclusive subsets of A project into mutually exclusive sets.

Thus if s is the union of any sequence of mutually exclusive circles of F_A, $(\lambda-\eta)LAs<m(As)_X+L(B-A)s$. But (corollary §12) with $\epsilon>0$ arbitrary there is an s such that $LAs=LA$ and $L(B-A)s<\epsilon$. For this s also $m(As)_X=mA_X$, so we have $(\lambda-\eta)LA<mA_X+\epsilon$. Thus $\lambda LA\leqq mA_X$.

Now let E be a linearly measurable subset of B with at most one point on each perpendicular to the x-axis. From Lemma 2 we see that

$$mE_X=\int_E\overline{C}(B,p)dL=\int_E\underline{C}(B,p)dL.$$

But $\overline{C}(B,p)\geqq\underline{C}(B,p)$ so $C(B,p)$ exists at almost all points of E.

Furthermore, *$C(B,p)$ exists at almost all points of B.* For there is a sequence of closed subsets of B whose union $\underline{B}$ has $L\underline{B}=LB$. But (Theorems 1 and 2) $\underline{B}$ is the union of a sequence of linearly measurable sets $B^1, B^2, \cdots$ (each with at most one point on any line perpendicular to the x-axis) and a linearly measurable set $\mathcal{B}^\infty$ (every point of whose projection is the image of an infinite number of points of $\underline{B}$). From consideration of the above integrals, $C(B,p)$ exists at almost all points of each set B^m, and consequently at almost all points of the union $\sum_{m=1}^{\infty}B^m$. Also, from §5, $m(\mathcal{B}^\infty)_X=0$, so, Lemma 1, $C(B,p)$ exists and is zero at almost all points of $\mathcal{B}^\infty$. Thus $C(B,p)$ exists at almost all points of $\underline{B}$ so finally at almost all points of B.

We later make direct use of

THEOREM 5. *Let E be a linearly measurable subset of B with at most one point on any line $x=x_0$, and let $F(p)$ be a function summable on B with respect to Carathéodory linear measure. Then the function of x defined for each value x_0 as*

$$f(x_0)=\begin{cases}0 \text{ if } x_0 \text{ is not a point of } E_X, \text{ otherwise}\\ F(p), \text{ where } p \text{ is the point of } E \text{ on the line } x=x_0,\end{cases}$$

is summable on E_X with respect to Lebesgue linear measure and

$$\int_{E_X}f(x)dx=\int_E F(p)\overline{C}(B,p)dL.$$

First $F(p)\overline{C}(B,p)$ is linearly measurable on E, since it is the product of two such functions. Then $f(x)$ is Lebesgue measurable on E_X, since the part of E_X where $f(x)>k$ is the projection of the linearly measurable subset of E where $F(p)>k$.

With M and N two non-negative numbers, define

$$F_{MN}(p)=\{-M \text{ if } F(p)<-M,\ F(p) \text{ if } -M\leqq F(p)<N,\ N \text{ if } F(p)\geqq N\}$$

and in a like manner, $f_{MN}(x)$. Let $-M=a_0, a_1, \cdots, a_n=N$ be a subdivision of the interval $(-M, N)$. Call E_k the subset of E where $a_{k-1}\leqq F_{MN}(p)<a_k$. Then $(E_k)_X$ is the subset of E_X where $a_{k-1}\leqq f_{MN}(x)<a_k$.

Consequently, from Lemma 2,

$$m(E_k)_X = \int_{E_k} \overline{C}(B, p)dL.$$

Then

$$a_{k-1}m(E_k)_X \leqq \int_{E_k} F_{MN}(p)\overline{C}(B, p)dL \leqq a_k m(E_k)_X,$$

so

$$\sum_{k=1}^{n} a_{k-1}m(E_k)_X \leqq \int_{E} F_{MN}(p)\overline{C}(B, p)dL \leqq \sum_{k=1}^{n} a_k m(E_k)_X.$$

But this is true of every subdivision of $(-M, N)$, so

$$\int_{E_X} f_{MN}(x)dx = \int_{E} F_{MN}(p)\overline{C}(B, p)dL.$$

However, $\int_E F(p)\overline{C}(B, p)dL$ exists since the integrand is the product of two functions summable on E, one of which is bounded. Consequently, the summability of $f(x)$ and the equality

$$\int_{E_X} f(x)dx = \int_{E} F(p)\overline{C}(B, p)dL$$

follows.

V. A replacement for direction and generalization of the Gauss-Green lemma

16. We divide the boundary B of a plane set G into three mutually exclusive subsets B_{I}, B_{II}, B_{III}; with B_{I} consisting of two parts. Let b be a point of B and Y the line through b perpendicular to the x-axis. Then b shall belong to

B_{I_1} if there is a segment $a\leqq y<b$ of Y below b of points of the complement of G and a segment $b<y\leqq c$ of Y above b of points of G.

B_{I_2} if there is a segment $a\leqq y<b$ of points of G and a segment $b<y\leqq c$ of points of the complement of G.

B_{II} if there exist two segments $a\leqq y<b$ and $b<y\leqq c$ which either both contain only points of G, or both contain only points of the complement of G.

B_{III} if either half of every segment of Y with mid point b contains both points of G and points of the complement of G.

One sees that a line perpendicular to the x-axis through a point of B_{III} contains an infinite number of points of B, i.e., $B_{\mathrm{III}} \subset \mathcal{B}^\infty$.

17. In the introduction we pointed out that Schauder made the material restriction that $m(B_{\mathrm{II}})_X = 0$. We are able to avoid this restriction by showing eventually that B_{II}, and also $\mathcal{B}^\infty$, contribute nothing to the boundary integral. The result is then obtained by integrating over the remaining part of B, i.e., over $B_{\mathrm{I}} - B_{\mathrm{I}}\mathcal{B}^\infty$. However, we find it convenient to integrate over $B_{\mathrm{I}_1} - B_{\mathrm{I}_1}\mathcal{B}^\infty$ and $B_{\mathrm{I}_2} - B_{\mathrm{I}_2}\mathcal{B}^\infty$ separately. It is thus necessary to know that these two sets are linearly measurable. We give the demonstration only for the first set.

First let $B_{\mathrm{I}_1 n}$ be all points b of B such that $b - 1/n \leqq y < b$ consists only of points of the complement of G while $b < y \leqq b + 1/n$ consists only of points of G. Let b_0 be a limit point of $B_{\mathrm{I}_1 n}$. Then b_0 is a point of B, since B is closed. If b_0 is a point of B_{I_2}, there is an interval $a \leqq y < b$, of points of G, each of which is a limit point of points of the complement of G, i.e., $a \leqq y \leqq b$ is an interval of boundary points, so b_0 belongs to $\mathcal{B}^\infty$. In a like manner one sees that each limit point of $B_{\mathrm{I}_1 n}$ which is a point of B_{II} is a point of $\mathcal{B}^\infty$.

Hence with $\overline{B}_{\mathrm{I}_1 n}$ the closure of $B_{\mathrm{I}_1 n}$, $\overline{B}_{\mathrm{I}_1 n} \subset B_{\mathrm{I}_1} + \mathcal{B}^\infty$ and $\sum_{n=1}^{\infty} \overline{B}_{\mathrm{I}_1 n} \supset B_{\mathrm{I}_1}$, so $\sum_{n=1}^{\infty}(\overline{B}_{\mathrm{I}_1 n} + \mathcal{B}^\infty) = B_{\mathrm{I}_1} + \mathcal{B}^\infty$. Thus $B_{\mathrm{I}_1} + \mathcal{B}^\infty$ is linearly measurable because each $\overline{B}_{\mathrm{I}_1 n}$ is linearly measurable (closed) and $\mathcal{B}^\infty$ is linearly measurable (Theorem 2). Consequently, *the set* $B_{\mathrm{I}_1} - B_{\mathrm{I}_1}\mathcal{B}^\infty$ *is linearly measurable* since it is the intersection $(B - \mathcal{B}^\infty)(B_{\mathrm{I}_1} + \mathcal{B}^\infty)$ of two linearly measurable sets.

Likewise, the set $B_{\mathrm{I}_2} - B_{\mathrm{I}_2}\mathcal{B}^\infty$ is linearly measurable. Thus $B_{\mathrm{I}} - B_{\mathrm{I}}\mathcal{B}^\infty$, and then $B_{\mathrm{II}} - B_{\mathrm{II}}\mathcal{B}^\infty$, is linearly measurable.

18. At all points b of B we now define the function

$$\cos(B, b) = \begin{cases} -\overline{C}(B, b) \text{ if } b \text{ is a point of } B_{\mathrm{I}_1} - B_{\mathrm{I}_1}\mathcal{B}^\infty \\ \overline{C}(B, b) \text{ if } b \text{ is a point of } B_{\mathrm{I}_2} - B_{\mathrm{I}_2}\mathcal{B}^\infty \\ 0 \text{ if } b \text{ is a point of } B_{\mathrm{II}} - B_{\mathrm{II}}\mathcal{B}^\infty \\ \overline{C}(B, b) \text{ if } b \text{ is a point of } \mathcal{B}^\infty. \end{cases}$$

If b is given by its coordinates (x, y) we designate

$$\cos(B, b) \text{ by } \cos[B, (x, y)].$$

Since, as we have seen, $C(B, b)$ exists at almost all points of B it would have been as well, in the following integral theorems, to define cos (B, b) in terms of $C(B, b)$. Also, since $\overline{C}(B, b) = 0$ at almost all points of $\mathcal{B}^\infty$, we could have defined cos $(B, b) = 0$ at all points of $\mathcal{B}^\infty$.

In the ordinary Green's theorem, the points of a crosscut correspond essentially to $B_{\mathrm{II}} - B_{\mathrm{II}}\mathcal{B}^\infty$. Since the integral around the complete boundary traverses a crosscut twice, and its value when taken in one direction is annulled by its value in the opposite direction, the same result would be ob-

tained if at each point of a crosscut the direction cosine of the external normal were arbitrarily replaced by zero. In an analogous manner, even though there is no notion of direction in our case, we have defined cos (B, b) to be zero at all points of $B_{\text{II}}-B_{\text{II}}\mathcal{B}^\infty$.

19. We now prove the theorem connecting a double integral over a plane set and a single integral over the boundary of the set.

THEOREM 6. *Let G be a bounded plane point set whose boundary B is normal with LB finite, and let $(G+B)x_0$ be the intersection of $G+B$ and the line $x=x_0$. If a function $F(x, y)$ is summable on B with respect to linear measure and is absolutely continuous in y on $(G+B)x_0$ for almost all values of x_0 in B_X, and if $\partial F(x, y)/\partial y$ is summable on $G+B$ with respect to Lebesgue plane measure, then $F(x, y)$ cos $[B, (x, y)]$ is summable on B with respect to linear measure, and*

$$\iint_G \frac{\partial F(x, y)}{\partial y} dxdy = \int_B F(x, y) \cos\, [B, (x, y)]dL.$$

The plane measure of B is zero since LB is finite,† hence

$$\iint_G \frac{\partial F(x, y)}{\partial y} dxdy = \iint_{G+B} \frac{\partial F(x, y)}{\partial y} dxdy = \int_{B_X} dx \int_{(G+B)x} \frac{\partial F(x, y)}{\partial y} dy.$$

But the set $\mathcal{B}^m$ (of all points of B on all lines perpendicular to the x-axis containing exactly m points of B) is linearly measurable, so

$$\int_{B_X} dx \int_{(G+B)x} \frac{\partial F(x, y)}{\partial y} dy = \int_{(\mathcal{B}^\infty)_X} dx \int_{(G+B)x} \frac{\partial F(x, y)}{\partial y} dy$$
$$+ \sum_{m=1}^{\infty} \int_{(\mathcal{B}^m)_X} dx \int_{(G+B)x} \frac{\partial F(x, y)}{\partial y} dy.$$

However, since $m(\mathcal{B}^\infty)_X=0$ and $\overline{C}[B, (x, y)]=0$ at almost all points of $\mathcal{B}^\infty$,

$$\int_{(\mathcal{B}^\infty)_X} dx \int_{(G+B)x} \frac{\partial F(x, y)}{\partial y} dy = \int_{\mathcal{B}^\infty} F(x, y) \cos\, [B, (x, y)]dL = 0.$$

Thus the proof of the theorem will be complete upon showing that $F(x, y)$ $\cdot$ cos $[B, (x, y)]$ is summable on $\mathcal{B}^m$ and

$$\text{(A)} \quad \int_{(\mathcal{B}^m)_X} dx \int_{(G+B)x} \frac{\partial F(x, y)}{\partial y} dy = \int_{\mathcal{B}^m} F(x, y) \cos\, [B, (x, y)]dL.$$

Toward proving (A), let x_0 be a point of $(\mathcal{B}^m)_X$. Then each of the m points of B on $x=x_0$ belongs to either B_{I} or B_{II}. If there are points of B_{I} on

† Göttinger Nachrichten, 1914, p. 425.

$x=x_0$, there are an even number of such points; the lowest belonging to B_{I_1}, the highest to B_{I_2}. With (x_0, b_1), (x_0, b_2), $\cdots$, (x_0, b_{2n}) the points of B_{I} in order of increasing ordinates, the set $(G+B)x_0$ consists of the non-abutting intervals $b_{2i-1} \leqq y \leqq b_{2i}$, $i=1, 2, \cdots, n$, and a finite number of points of B_{II}. Thus

$$(1) \qquad \int_{(G+B)x_0} \frac{\partial F(x_0, y)}{\partial y} dy = \sum_{i=1}^{n} \int_{b_{2i-1}}^{b_{2i}} \frac{\partial F(x_0, y)}{\partial y} dy = \sum_{i=1}^{2n} (-1)^i F(x_0, b_i),$$

except perhaps for a set of measure zero made up of the values of x_0 where the integral on the left fails to exist and those values of x_0 for which $F(x_0, y)$ is not absolutely continuous. The second equality follows from the fact that an absolutely continuous function is reproduced by the integral of its derivative.†

We recall that the set B^r consists of all points p of B such that exactly $r-1$ points of B lie below p on the same perpendicular to the x-axis. With (x_0, y) the point of B^r on the line $x=x_0$, define

$$f_r(x_0) \qquad (r = 1, 2, \cdots, m)$$

as $-F(x_0, y)$, $F(x_0, y)$ or zero according as (x_0, y) belongs to B_{I_1}, B_{I_2}, or B_{II}. Then since (x_0, b_i) belongs to B_{I_1} if i is odd or to B_{I_2} if i is even,

$$\sum_{r=1}^{m} f_r(x_0) = \sum_{i=1}^{2n} (-1)^i F(x_0, b_i).$$

Consequently, from (1), $\sum_{r=1}^{m} f_r(x)$ is summable on $(\mathcal{B}^m)_X$ and

$$(2) \qquad \int_{(\mathcal{B}^m)_X} dx \int_{(G+B)x} \frac{\partial F(x, y)}{\partial y} dy = \int_{(\mathcal{B}^m)_X} \sum_{r=1}^{m} f_r(x) dx.$$

Since $F(x, y)$ is summable on B and B^r has at most one point on any line perpendicular to the x-axis, while all three sets $B_{\mathrm{I}_1} - B_{\mathrm{I}_1}\mathcal{B}^\infty$, $\mathcal{B}^\infty$, and B^r are linearly measurable, it follows from Theorem 5 and the definition of $\cos [B, (x, y)]$ at points of B_{I_1} that

$$\int_{(B_{\mathrm{I}_1}\mathcal{B}^m B^r)_X} f_r(x) dx = \int_{B_{\mathrm{I}_1}\mathcal{B}^m B^r} - F(x, y) \overline{C}[B, (x, y)] dL$$

$$= \int_{B_{\mathrm{I}_1}\mathcal{B}^m B^r} F(x, y) \cos [B, (x, y)] dL.$$

In like manner

$$\int_{(B_{\mathrm{I}_2}\mathcal{B}^m B^r)_X} f_r(x) dx = \int_{B_{\mathrm{I}_2}\mathcal{B}^m B^r} F(x, y) \cos [B, (x, y)] dL.$$

† Hobson, *Real Variables*, p. 553.

But $f_r(x)=0$ at each point of $(B_{II}B^r)_X$ and $\cos[B, (x, y)]=0$ at each point of $B_{II}-B_{II}\mathcal{B}^\infty$, so

$$\int_{(B_{II}\mathcal{B}^m B^r)_X} f_r(x)dx = \int_{B_{II}\mathcal{B}^m B^r} F(x, y)\cos[B, (x, y)]dL = 0.$$

Consequently, from the last three equations, by addition,

$$\int_{(\mathcal{B}^m)_X} f_r(x)dx = \int_{\mathcal{B}^m B^r} F(x, y)\cos[B, (x, y)]dL.$$

Now, since $f_r(x)$ is summable on $(\mathcal{B}^m)_X$, we have

$$\int_{(\mathcal{B}^m)_X} \sum_{r=1}^{m} f_r(x)dx = \sum_{r=1}^{m}\int_{(\mathcal{B}^m)_X} f_r(x)dx.$$

But from the last two equations and (2)

$$\begin{aligned}\int_{(\mathcal{B}^m)_X} dx\int_{(G+B)_x} \frac{\partial F(x, y)}{\partial y}dy &= \sum_{r=1}^{m}\int_{\mathcal{B}^m B^r} F(x, y)\cos[B, (x, y)]dL \\ &= \int_{\mathcal{B}^m} F(x, y)\cos[B, (x, y)]dL,\end{aligned}$$

which is equation (A). The proof is thus complete.

In Theorem 6 the only condition on the boundary B of G is that it be normal and have LB finite, while there is no condition on G except that it be bounded. The boundedness of G and the finiteness of LB are inherent in the problem, whereas the normality of B is a less natural restriction introduced to fit the method of attack. That, however, the normality condition is not a very drastic restriction is shown by the fact that the boundary of every simply connected region is normal if this boundary has Carathéodory linear measure finite. Furthermore, it follows that a much larger class of sets G have normal boundaries. For let G be a set whose boundary B is contained in the union B' of the boundaries $B_1, B_2, \cdots, B_n$ of a finite number of simply connected regions, where each B_i has Carathéodory linear measure finite. Then B' is normal (lemma §10) since $\underline{D}(B', b)\geq 1/2$ at each point b of B'. Consequently, B is normal (Theorem 4) since it is a linearly measurable subset of B'. The usual crosscut scheme is included in this extension.

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EXISTENCE THEOREMS FOR DOUBLE INTEGRAL PROBLEMS OF THE CALCULUS OF VARIATIONS.†

BY

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For single-integral problems of the calculus of variations there are in the literature a number of existence theorems of considerable generality. Recently Tonelli has established several existence theorems for double integral problems of the form $\iint f(x, y, z, z_x, z_y)dxdy = \min$. But to the best of my knowledge, except for the several discussions of the problem of Plateau the literature contains no proof of any existence theorem for double-integral problems in parametric form, that is, for problems of the form $F(S) = \iint f(x, y, z, X, Y, Z)dudv = \min$, where the equations $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ represent a surface and X, Y, Z are the three jacobians of (x, y, z) with respect to (u, v).

The present paper gives the proof of two such theorems, in each of which the integrand function is permitted to be a function of (X, Y, Z) of quite general type, but is required to be independent of the coordinates (x, y, z). The theorems are based on a semi-continuity proof and a convergence theorem. The semi-continuity of quasi-regular functionals $F(S)$ I have already established under conditions of adequate generality. Here I develop the convergence theorem needed. The methods are extensions of those previously used in connection with the problem of Plateau.‡

1. **Preliminary remarks.** The word surface will always be used to mean a continuous surface of the type of the circle, represented by three equations $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, where (u, v) ranges over the interior and boundary of a Jordan region B (i.e., a region bounded by a simple closed curve). In case the six partial derivatives x_u, x_v, etc., all exist and are finite, we denote the three jacobians of x, y, z with respect to u, v by the symbols X, Y, Z:

$$X = \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}, \qquad Y = \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix}, \qquad Z = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}.$$

Let us suppose that $f(X, Y, Z)$ is a function positively homogeneous of degree 1 in (X, Y, Z) and continuous together with its first partial derivatives for all $(X, Y, Z) \neq (0, 0, 0)$. For all arguments (X, Y, Z) such that $X^2 + Y^2 + Z^2$

† Presented to the Society, October 28, 1933; received by the editors July 28, 1934.

‡ E. J. McShane, *Parametrizations of saddle surfaces*, etc., these Transactions, vol. 35 (1933), pp. 716–733. This paper will henceforth be cited as S.S.

=1 the function f is bounded, say $|f| \leqq M$; hence by homogeneity the inequality

$$|f(X, Y, Z)| \leqq M[X^2 + Y^2 + Z^2]^{1/2} \leqq M(|X| + |Y| + |Z|)$$

holds for all X, Y, Z. Consequently, if $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, (u, v) on B, is a representation of a surface such that the six partial derivatives x_u, etc., are defined almost everywhere in B, and the jacobians X, Y, Z are summable over the set on which they are defined, it follows that the integral

$$\int\int_B f(X, Y, Z)dudv$$

exists.†

As in the case of single integrals, the mere existence of this integral is inadequate for our purposes.‡ In the study of the parametric problem (single integrals) we restrict ourselves to representations $x = x(t)$, etc., in which the functions $x(t), \cdots$ are absolutely continuous. Lacking an adequate generalization of the notion of absolute continuity to the pairs of functions of two variables, we say that a surface S with finite Lebesgue area $L(S)$ is an *admissible* surface (for the integrand $f(X, Y, Z)$) if S has representation $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, (u, v) on B, for which the jacobians X, Y, Z are defined almost everywhere in B and for which the following approximation property holds: there exists a sequence $\{\pi_n\}$ of polyhedra $\pi_n: x = x_n(u, v)$, $y = y_n(u, v)$, $z = z_n(u, v)$, (u, v) on B_n, such that $\lim \pi_n = S$, $\lim L(\pi_n) < \infty$, and $\lim \int\int_{B_n} f(X_n, Y_n, Z_n)dudv = \int\int_B f(X, Y, Z)dudv$. The representation $x = x(u, v)$, etc., is then called an *admissible representation* of the surface S.

If for the corresponding single-integral problem we write the analogous definition of admissible curves, we find that for every integral $\int f(x, y, z, x', y', z')dt$ the class of admissible curves is the same as the class of rectifiable curves; and if in addition the integral is positive definite (i.e., $f > 0$ whenever $(x', y', z') \neq (0, 0, 0)$) and positive quasi-regular, the admissible representations are the same as the absolutely continuous representations. For double integrals no such simple characterization is at present known. But it can be stated that for every integrand $f(X, Y, Z)$ with the continuity and homogeneity properties above described the class of admissible surfaces

† The integrand may be undefined on a set of measure 0. Here and henceforth we agree that if a function $\phi(u)$ is defined at all points of a set E except those of a set N of measure 0 and is summable on $E - N$, the symbol $\int_B \phi(u)du$ shall mean the integral $\int_{B-N} \phi(u)du$.

‡ As has been shown by M. Lavrentieff, *Sur quelques problèmes du calcul des variations*, Annali di Matematica, (4), vol. 4 (1927), p. 7.

includes the class of surfaces of "type† C," which in turn includes the class of all continuous surfaces having representations $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, (u, v) on B, which satisfy the conditions

(1.1a) for almost all numbers K, the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ are absolutely continuous functions of u on every segment of the line $v=K$ lying interior to B, and are absolutely continuous functions of v on every segment of the line $u=K$ lying interior to B;

(1.1b) the six partial derivatives x_u, x_v, y_u, y_v, z_u, z_v (which by (1.1a) exist almost everywhere in B) are summable together with their squares over the region B.

For surfaces of type C typically represented (in particular, for representations satisfying conditions (1.1)) we already know‡ that the value of the integral is independent of the particular representation and is thus a functional of the surface alone. We then have the right to denote the integral by the symbol $F(S)$,

$$F(S) = \iint f(X, Y, Z)dudv. \tag{1.2}$$

But for admissible surfaces not of type C it is not known that this invariance property holds; hence for general admissible surfaces we shall always write the integral in full, avoiding the (possibly multiple-valued) symbol $F(S)$.

We define the Weierstrass $\mathcal{E}$-function as usual:

$$\begin{aligned} \mathcal{E}(X, Y, Z, \overline{X}, \overline{Y}, \overline{Z}) = f(\overline{X}, \overline{Y}, \overline{Z}) &- \overline{X}f_X(X, Y, Z) \\ &- \overline{Y}f_Y(X, Y, Z) - \overline{Z}f_Z(X, Y, Z); \end{aligned} \tag{1.3}$$

and as usual we call $F(S)$ positive quasi-regular if $\mathcal{E}(X, Y, Z, \overline{X}, \overline{Y}, \overline{Z}) \geqq 0$ for all $(\overline{X}, \overline{Y}, \overline{Z})$ and all $(X, Y, Z) \neq (0, 0, 0)$, and we call it positive definite if $f(X, Y, Z) > 0$ for all $(X, Y, Z) \neq (0, 0, 0)$.

We shall say that a surface S is of type L_2 if it possesses a representation

$$x = x(u, v), \ y = y(u, v), \ z = z(u, v), \ u^2 + v^2 \leqq 1, \tag{1.4}$$

in which the functions $x(u, v)$, etc., satisfy conditions (1); the representation (1.4) we shall correspondingly call a typical representation. It is known§ that if a surface S has a representation

† Defined and studied in *Integrals over surfaces in parametric form*, Annals of Mathematics, vol. 34 (1933), p. 815; cf. also C. B. Morrey, *A class of representations of manifolds*, American Journal of Mathematics, vol. 55 (1933), p. 701.

‡ E. J. McShane, loc. cit. in the preceding footnote.

§ E. J. McShane, *On the minimizing property of the harmonic function*, Bulletin of the American Mathematical Society, vol. 40 (1934), p. 593.

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad (u, v) \text{ on } B,$$

which satisfies conditions (1.1), it is necessarily of type L_2; but we shall not make use of this.

2. **A transformation of the integrand.** By change of coordinates we can bring the integrands under consideration into a special form, useful for later proofs.

LEMMA 2.1. *Let the inequality*

$$f(X, Y, Z) + f(-X, -Y, -Z) > 0 \tag{2.1}$$

hold for all $(X, Y, Z) \neq (0, 0, 0)$. *There exists a linear transformation*

$$\begin{aligned} X &= A_1X' + B_1Y' + C_1Z', \\ Y &= A_2X' + B_2Y' + C_2Z', \\ Z &= A_3X' + B_3Y' + C_3Z', \end{aligned} \tag{2.2}$$

of determinant 1, *such that the function*

$$\phi(X', Y', Z') = f(X, Y, Z) \tag{2.3}$$

satisfies the conditions

$$\begin{aligned} \phi_{Y'}(1, 0, 0) &= \phi_{Y'}(-1, 0, 0), \\ \phi_{Z'}(1, 0, 0) &= \phi_{Z'}(-1, 0, 0), \\ \phi_{Z'}(0, 1, 0) &= \phi_{Z'}(0, -1, 0), \\ \phi_{X'}(0, 1, 0) &= \phi_{X'}(0, -1, 0). \end{aligned} \tag{2.4}$$

Let us define

$$g(X, Y, Z) \equiv f(X, Y, Z) + f(-X, -Y, -Z). \tag{2.5}$$

The function g obviously has the same differentiability and homogeneity properties as f, and the surface S in XYZ-space defined by the equation

$$g(X, Y, Z) = 1 \tag{2.6}$$

is symmetrical with respect to the origin. (By (2.1) the surface exists, and in the direction of the unit vector X_u, Y_u, Z_u has the distance from the origin $r = [g(X_u, Y_u, Z_u)]^{-1}$.) Moreover, from the homogeneity relation

$$Xg_X + Yg_Y + Zg_Z = g = 1$$

we see that the three derivatives cannot vanish simultaneously, so that the surface S is continuously differentiable.

On S there is a point at maximum distance from the origin. By a rotation of axes we bring this point to the X-axis. Then for $Y = Z = 0$ the tangent plane to S is parallel to the YZ-plane. We now introduce polar coordinates,

r, θ, ϕ so that $X=r\sin\phi$, $Y=r\cos\phi\cos\theta$, $Z=r\cos\phi\sin\theta$. The surface S can then be represented in the form

$$r = r(\theta, \phi),$$

where $r(\pi+\theta, -\phi)=r(\theta,\phi)$. Since this implies

$$\frac{\partial}{\partial\phi} r(\pi+\theta, 0) = -\frac{\partial}{\partial\phi} r(\theta, 0)$$

and this derivative is continuous, there exists a point, with arguments $(\theta_0, 0)$, at which $\partial r/\partial\phi$ vanishes. By rotation about the X-axis we bring this point to the Y-axis, so that $\partial r/\partial\phi$ vanishes for $\theta=\phi=0$ (that is, for $X=Z=0$). The tangent plane at $\theta=\phi=0$ is then parallel to the X-axis, but not necessarily to the Z-axis. Let l_1 be the line through the origin parallel to the intersection of that tangent plane with the YZ-plane. By an affine transformation T of the form $X=\overline{X}$, $Y=\overline{Y}+K\overline{Z}$, $Z=\overline{Z}$, we bring l_1 to the Z-axis, leaving the X and Y axes unchanged. After this transformation the tangent plane at $X=Z=0$ is parallel to the XZ-plane; the tangent plane at $Y=Z=0$ and the YZ-plane are unchanged, hence remain parallel.

The two rotations and the affine transformation T can be combined into a single linear transformation of the form (2.2). In terms of the new coordinates, the surface S has the equation

$$\phi(X', Y', Z') + \phi(-X', -Y', -Z') = 1$$

where ϕ is defined by equation (2.3). The normal to S has the direction numbers (dropping primes)

$$\begin{aligned} &\phi_X(X, Y, Z) - \phi_X(-X, -Y, -Z), \\ &\phi_Y(X, Y, Z) - \phi_Y(-X, -Y, -Z), \\ &\phi_Z(X, Y, Z) - \phi_Z(-X, -Y, -Z). \end{aligned} \tag{2.7}$$

But for $Y=Z=0$ the normal has direction cosines $(\pm 1, 0, 0)$, so that the last two of the numbers (2.7) are 0 for $Y=Z=0$ whether the positive or negative value of X be chosen. Recalling that ϕ_Y and ϕ_Z are positively homogeneous of degree 0, this yields the first pair of equations (2.4). For $X=Z=0$ the normal has direction cosines $(0, \pm 1, 0)$; this likewise yields the second pair of equations (2.4).

Let us suppose that we are given an integrand $f(X, Y, Z)$ satisfying inequality (2.1), and let the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

be the reciprocal of the matrix of the transformation (2.2). We find readily that for every surface

$$S: \qquad x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ (u, v) \text{ on } B,$$

for which $f(X, Y, Z)$ is summable the transformation

$$\begin{aligned} x &= a_1x' + b_1y' + c_1z', \\ y &= a_2x' + b_2y' + c_2z', \\ z &= a_3x' + b_3y' + c_3z' \end{aligned} \tag{2.8}$$

induces transformation (2.7) on the jacobians X, Y, Z, and

$$\iint_B \phi(X', Y', Z')dudv = \iint_B f(X, Y, Z)dudv. \tag{2.9}$$

The surfaces admissible for f transform into surfaces admissible for ϕ; likewise the class of surfaces of type C transforms into itself and the class of surfaces of type L_2 also transforms into itself. Hence if we are given a Jordan curve Γ, transformed by (2.8) into a curve Γ' of $x'y'z'$-space, the problem of finding a minimizing surface for $\iint f(X, Y, Z)dudv$ in the class of all surfaces bounded by Γ and belonging to any one of the three analytic classes just mentioned is equivalent to the problem of finding a minimizing surface for $\iint \phi(X', Y', Z')dudv$ in the class of all surfaces bounded by Γ' and belonging to the corresponding analytic class. In other words, there is no loss of generality in assuming to begin with that $f(X, Y, Z)$ satisfies the equations

$$\begin{aligned} f_Y(1, 0, 0) &= f_Y(-1, 0, 0), \\ f_Z(1, 0, 0) &= f_Z(-1, 0, 0), \\ f_Z(0, 1, 0) &= f_Z(0, -1, 0), \\ f_X(0, 1, 0) &= f_X(0, -1, 0). \end{aligned} \tag{2.10}$$

3. **First existence theorem for positive definite integrals.** In this section we shall consider integrands $f(X, Y, Z)$ which satisfy the condition

$$f(X, Y, Z) > 0 \text{ for } (X, Y, Z) \neq (0, 0, 0). \tag{3.1}$$

Given a Jordan curve Γ, it is clear that the greatest lower bound i of $\iint f(X, Y, Z)dudv$ for all admissible surfaces bounded by Γ is non-negative. Another lower bound associated with Γ we define in the following way:

Let S_n: $x = x_n(u, v)$, etc., $n = 1, 2, \cdots$, be a sequence of admissible surfaces whose boundaries tend to Γ, and let $m(\{S_n\})$ be the lower limit of $\iint f(X_n, Y_n, Z_n)dudv$. We define m to be the greatest lower bound of the numbers $m(\{S_n\})$ for all such sequences $\{S_n\}$. Clearly

$$m \leqq i, \tag{3.2}$$

for we can construct a sequence $\{S_n\}$ of admissible surfaces bounded by Γ for which the integrals tend to i, and m is not greater than the limit of the integrals over the surfaces S_n.

We now proceed to the proof of

THEOREM 3.1. *Let the integral*

$$\iint f(X, Y, Z)dudv \tag{3.3}$$

be positive definite and positive quasi-regular, and let the curve

$$\Gamma: \qquad x = x(t), \ y = y(t), \ z = z(t)$$

be a Jordan curve in xyz-space, bounding at least one admissible surface.† *Then there exists a triple of functions* $x(u, v)$, $y(u, v)$, $z(u, v)$, *defined for* $u^2+v^2 \leqq 1$, *with the following properties*:

(1) *the surface*

$$S: \quad x = x(u, v), \ y = y(u, v), \ z = z(u, v), \ u^2 + v^2 \leqq 1, \tag{3.4}$$

is bounded by the curve Γ; *that is, the equations* $x=x(\cos\theta, \sin\theta)$, $y=y(\cos\theta, \sin\theta)$, $z=z(\cos\theta, \sin\theta)$, $0 \leqq \theta \leqq 2\pi$, *form a representation*‡ *of the curve* Γ;

(2) *the functions* $x(u, v)$, $y(u, v)$, $z(u, v)$ *satisfy conditions* (1.1);

(3) *the surface* (3.4) *minimizes the integral* (3.3) *in the class of all admissible surfaces bounded by* Γ, *and in fact*

$$F(S) = i = m. \tag{3.5}$$

In accordance with the remark at the end of §2, there is no loss of generality in assuming that equations (2.10) are satisfied. By the homogeneity of f we have

$$f_Z(0, 0, 1) = f(0, 0, 1), \ f_Z(0, 0, -1) = -f(0, 0, -1). \tag{3.6}$$

By hypothesis inequality (3.1) is valid; and from (3.6) and (3.1) we see that there exist numbers a, b such that

$$\begin{aligned} af_Z(0, 0, 1) + f_X(0, 0, 1) &= af_Z(0, 0, -1) + f_X(0, 0, -1), \\ bf_Z(0, 0, 1) + f_Y(0, 0, 1) &= bf_Z(0, 0, -1) + f_Y(0, 0, -1). \end{aligned} \tag{3.7}$$

† From the results of S.S. (Lemma 3 and Theorem I) this is equivalent to requiring that Γ bound at least one surface of finite area.

‡ But in this representation it is possible that two distinct points $(\cos\theta_1, \sin\theta_1)$ and $(\cos\theta_2, \sin\theta_2)$ might yield the same point (x, y, z).

We define the number k by the relation

$$k = 2(1 + a^2 + b^2). \tag{3.8}$$

Let us now select a sequence of admissible surfaces

$$S_n: \qquad x = x_n^*(u, v), \ y = y_n^*(u, v), \ z = z_n^*(u, v), \ (u, v) \text{ on } B_n,$$

bounded by curves Γ_n^* such that $\Gamma_n^* \to \Gamma$ and such that

$$\lim_{n \to \infty} \iint_{B_n} f(X_n^*, Y_n^*, Z_n^*) du dv = m. \tag{3.9}$$

Since the S_n are admissible surfaces, we can for each S_n find a polyhedron π_n, which we assume to have non-degenerate triangles for faces, such that

$$\text{dist } (\pi_n, S_n) < 1/2^n \tag{3.10}$$

and

$$\left| F(\pi_n) - \iint_{B_n} f(X_n^*, Y_n^*, Z_n^*) du dv \right| < \frac{1}{2^n}. \tag{3.11}$$

From (3.10) and the relation $\Gamma_n^* \to \Gamma$ we see that the boundary curves Γ_n of the polyhedra π_n satisfy

$$\Gamma_n \to \Gamma; \tag{3.12}$$

and from (3.11) and (3.9) we see that

$$F(\pi_n) \to m. \tag{3.13}$$

It is known† that every polyhedron π with non-degenerate faces admits of a parametric representation of the following kind.

(a) The functions representing π are defined in the unit circle; that is, π is represented by equations

$$x = x(u, v), \ y = y(u, v), \ z = z(u, v), \ u^2 + v^2 \leqq 1. \tag{3.14}$$

(b) The unit circle is subdivided by arcs into a finite number of curvilinear triangles $\delta_1, \cdots, \delta_k$ and equations (3.14) carry each triangle into a rectilinear triangle in xyz-space.

(c) The triangles δ_i are bounded by arcs which are analytic, including end points.

(d) Interior to each triangle δ_i the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ are analytic and satisfy the relations

† See, e.g., Carathéodory, *Conformal Representation* (No. 28 of the Cambridge Tracts in Mathematics and Physics), chapter VII.

(3.15) $$E = G, \qquad F = 0.$$

(e) Three arbitrarily given distinct points A_1, A_2, A_3 on the boundary curve of π correspond to three arbitrarily given distinct points A_1^*, A_2^*, A_3^* on the unit circle $u^2+v^2=1$.

Accordingly, we choose on the curve Γ three distinct points A_1, A_2, A_3, and on each Γ_n we choose three distinct points $A_1^{(n)}$, $A_2^{(n)}$, $A_3^{(n)}$ such that

$$\lim_{n\to\infty} A_i^{(n)} = A_i \qquad (i = 1, 2, 3).$$

On the circumference of the unit circle $u^2+v^2\leqq 1$ (which circle we shall henceforth denote by K) we choose three distinct points A_1^*, A_2^*, A_3^*, and we represent each polyhedron π_n by equations

(3.16) $$\pi_n\colon\ x = x_n(u, v),\ y = y_n(u, v),\ z = z_n(u, v),\ u^2 + v^2 \leqq 1,$$

such that conditions (a), (b), (c), (d) are satisfied and the points $A_1^{(n)}$, $A_2^{(n)}$, $A_3^{(n)}$ correspond to A_1^*, A_2^*, A_3^* respectively.

Let $\mu>0$ be the lower bound of $f(X, Y, Z)$ on the bounded closed set $X^2+Y^2+Z^2=1$; then for all X, Y, Z we have

(3.17) $$f(X, Y, Z) \geqq \mu[X^2 + Y^2 + Z^2]^{1/2}.$$

We may assume without loss of generality that $F(\pi_n)<m+1$ for all n; whence

$$m + 1 > \iint_K f(X_n, Y_n, Z_n)dudv \geqq \iint_K \mu[X_n^2 + Y_n^2 + Z_n^2]^{1/2}dudv$$
$$= \mu \iint_K [E_nG_n - F_n^2]^{1/2}dudv = \frac{\mu}{2}\iint_K (E_n + G_n)dudv,$$

so that

(3.18) $$\iint_K (E_n + G_n)dudv \leqq H, \qquad H = 2(m + 1)/\mu.$$

On the functions (3.16) we now operate to reduce their monotonic deficiency.† We choose a cube $d\leqq x\leqq d+h$, $d\leqq y\leqq d+h$, $d\leqq z\leqq d+h$ large enough to include the whole curve Γ and all the curves Γ_n in its interior (as is possible, since $\Gamma_n\to\Gamma$). The set of points (u, v) such that $z_n(u, v)>d$ is an open set, except that it may contain limit points on the circumference of the unit circle K, and it consists of a finite number of maximal connected subsets. We reject those subsets which have points in common with the circumference of K, and name the rest $R_1, \cdots, R_p$. We proceed similarly with the set $z_n(u, v)<d$;

† S.S., p. 717.

the maximal connected portions of this set which have no point in common with the circumference of the unit circle we call $R_{p+1}, R_{p+2}, \cdots, R_q$.

On each R_i we define the functions $\xi^{(1)}(u, v)$, $\eta^{(1)}(u, v)$, $\zeta^{(1)}(u, v)$ by the relations

$$\xi^{(1)} = x_n + a(z_n - d), \quad \eta^{(1)} = y_n + b(z_n - d), \quad \zeta^{(1)} = d; \tag{3.19}$$

on the remainder $K - \sum R_i$ of the unit circle we set

$$\xi^{(1)} \equiv x_n, \quad \eta^{(1)} \equiv y_n, \quad \zeta^{(1)} \equiv z_n. \tag{3.20}$$

The functions $\xi^{(1)}, \eta^{(1)}, \zeta^{(1)}$ clearly retain properties (a), (b), (c), (e), and the surface

$$\Sigma^{(1)}: \quad x = \xi^{(1)}(u, v), \quad y = \eta^{(1)}(u, v), \quad z = \zeta^{(1)}(u, v) \tag{3.21}$$

is bounded by Γ_n. Moreover, if we denote the jacobians of $\xi^{(1)}, \eta^{(1)}, \zeta^{(1)}$ by $\Xi^{(1)}, \mathrm{H}^{(1)}, \mathrm{Z}^{(1)}$, we find

$$\Xi^{(1)} = 0, \quad \mathrm{H}^{(1)} = 0, \quad \mathrm{Z}^{(1)} = Z_n - aX_n - bY_n, \quad (u, v) \text{ on } \sum R_i, \tag{3.22}$$

so that

$$\begin{aligned} &\iint_K [f(X_n, Y_n, Z_n) - f(\Xi^{(1)}, \mathrm{H}^{(1)}, \mathrm{Z}^{(1)})]dudv \\ &= \iint_{\Sigma R_i} [\mathcal{E}(0, 0, \mathrm{Z}^{(1)}, X_n, Y_n, Z_n) - Y_n f_Y(0, 0, \mathrm{Z}^{(1)}) \\ &\qquad - (aX_n + bY_n) f_Z(0, 0, \mathrm{Z}^{(1)})]dudv \\ &\geqq -\sum \iint_{R_i} \{X_n[f_X(0, 0, \mathrm{Z}^{(1)}) + af_Z(0, 0, \mathrm{Z}^{(1)})] \\ &\qquad + Y_n[f_Y(0, 0, \mathrm{Z}^{(1)}) + bf_Z(0, 0, \mathrm{Z}^{(1)})]\}dudv. \end{aligned} \tag{3.23}$$

Since the derivatives f_X, etc., are positively homogeneous of degree 0 in (X, Y, Z), it follows from (3.7) that for all $\mathrm{Z}^{(1)} \neq 0$ the equations

$$\begin{aligned} f_X(0, 0, \mathrm{Z}^{(1)}) + af_Z(0, 0, \mathrm{Z}^{(1)}) &= f_X(0, 0, 1) = af_Z(0, 0, 1) = c_1, \\ f_Y(0, 0, \mathrm{Z}^{(1)}) + bf_Z(0, 0, \mathrm{Z}^{(1)}) &= f_X(0, 0, 1) + af_Z(0, 0, 1) = c_2 \end{aligned} \tag{3.24}$$

hold. If $\mathrm{Z}^{(1)} = 0$ we assign f_X the value $f_X(0, 0, 1)$, and likewise for f_Y and f_Z; (3.23) continues to hold, and also the equations (3.24). But for each of the regions R_i we have

$$\iint_{R_i} X_n dudv = \int y_n z_n' ds, \tag{3.25}$$

the single integral being taken around the boundary of R_i. On the boundary of R_i we have $z_n = d$, by the definition of R_i; hence $z_n' = 0$, and

$$\iint_{R_i} X_n du dv = 0. \tag{3.26}$$

From this and (3.23) follows

$$F(\Sigma^{(1)}) \leqq F(\pi_n). \tag{3.27}$$

To the surface $\Sigma^{(1)}$, in its representation (3.21), we apply a similar process, the number d being replaced by $d + h/n$ in defining the sets R_i and in equations (3.19). We thus obtain a surface

$$\Sigma^{(2)}: \quad x = \xi^{(2)}(u, v), \; y = \eta^{(2)}(u, v), \; z = \zeta^{(2)}(u, v). \tag{3.28}$$

As before, the functions $\xi^{(2)}$, $\eta^{(2)}$, $\zeta^{(2)}$ continue to satisfy conditions (a), (b), (c), (e), and also

$$F(\Sigma^{(2)}) \leqq F(\Sigma^{(1)}) \leqq F(\pi_n), \tag{3.29}$$

where $\Sigma^{(2)}$ is the surface $x = \xi^{(2)}(u, v), \cdots, (u, v)$ on K. We repeat the process with $d + 2h/n$ in place of $d + 1/n$, obtaining the functions $\xi^{(3)}$, $\eta^{(3)}$, $\zeta^{(3)}$, and continue $n+1$ times, using the numbers $d + ih/n$ $(i = 3, 4, \cdots, n)$ successively to obtain functions $\xi^{(n+1)}$, $\eta^{(n+1)}$, $\zeta^{(n+1)}$. We re-name these functions, calling them $\bar{\xi}_n$, $\bar{\eta}_n$, $\bar{z}_n$ respectively. They satisfy conditions (a), (b), (c), (e), and also

$$F(\bar{\Sigma}_n) \leqq F(\pi_n), \tag{3.30}$$

where $\bar{\Sigma}_n$ is the surface $x = \bar{\xi}_n(u, v)$, etc.

The set $\sum R_i$ is an open set, and its boundary, which consists of a finite number of analytic arcs, is of measure zero. Hence, neglecting a set of measure 0, we have for (u, v) in $K - \sum R_i$ the equality

$$\bar{E}_n = E_n, \quad \bar{G}_n = G_n.$$

At each point of $\sum R_i$ the functions $\bar{\xi}_n$, etc., are defined by equations (3.19) or their analogues, so that by the use of elementary inequalities we find that

$$\bar{E}_n \leqq kE_n, \quad \bar{G}_n \leqq kG_n,$$

where k is defined in (3.8). Hence, recalling inequality (3.18),

$$\iint_K (\bar{E}_n + \bar{G}_n) du dv \leqq kH. \tag{3.31}$$

We readily see that the function $\bar{z}_n$ has a monotonic deficiency not greater than h/n.

To the functions $\bar{\eta}_n$ of equations (3.3) we now apply a similar process. The points (u, v) such that $\bar{\eta}_n(u, v) > d$ fall into a finite number of maximal connected sets; we reject those which have points in common with the circumference of K, and name the others $R_1, R_2, \cdots, R_p$. We treat the points (u, v) for which $\bar{\eta}_n(u, v) < d$ similarly, obtaining sets $R_{p+1}, \cdots, R_q$. We now define

$$\bar{\eta}^{(1)} = d, \quad (u, v) \text{ on } \sum R_i,$$
$$\bar{\eta}^{(1)} = \bar{\eta}_n, \quad (u, v) \text{ on } K - \sum R_i.$$

The surface

$$(3.32) \quad \bar{\Sigma}^{(1)}: \quad x = \bar{\xi}_n(u, v), \ y = \bar{\eta}^{(1)}(u, v), \ z = \bar{z}_n(u, v), \ (u, v) \text{ on } K,$$

is easily seen to satisfy the inequality

$$(3.33) \qquad \iint_K (\bar{E}^{(1)} + \bar{G}^{(1)})dudv \leqq \iint_K (\bar{E}_n + \bar{G}_n)dudv,$$

where the functions $\bar{E}^{(1)}, \bar{G}^{(1)}$ correspond to $\bar{\Sigma}^{(1)}$ and $\bar{E}_n, \bar{G}_n$ to $\bar{\Sigma}_n$; for

$$\bar{E}_n - \bar{E}^{(1)}$$

has the value 0 on $K - \sum R_i$ and the value $(\partial\bar{\eta}_n/\partial u)^2$ on $\sum R_i$. Moreover, an argument similar to the above proves that

$$F(\bar{\Sigma}^{(1)}) \leqq F(\bar{\Sigma}_n) \leqq F(\pi_n);$$

we need only to permute X, Y, Z cyclically and set $a = b = 0$ in (3.23), recalling equations (2.10).

Applying the same process to the sets R_i on which $\bar{\eta}^{(1)} > d + h/n$ or $\bar{\eta}^{(1)} < d + h/n$ gives $\bar{\eta}^{(2)}$; and continuing the process we obtain successively $\bar{\eta}^{(3)}, \cdots, \bar{\eta}^{(n+1)}$. The function $\bar{\eta}^{(n+1)}$ we re-name $\bar{y}_n$. Each alteration reduces (or leaves unchanged) the value of $\iint f dudv$ and of $\iint (E+G)dudv$, and leaves $\bar{z}_n(u, v)$ and $\bar{\xi}_n(u, v)$ unaltered.

Finally, we apply to the function $\bar{\xi}_n(u, v)$ the same process as we have just applied to $\bar{\eta}_n(u, v)$, arriving at a function $\bar{x}_n(u, v)$. We define the surface $\bar{S}_n$ by the equations

$$(3.34) \qquad \bar{S}_n: \quad x = \bar{x}_n(u, v), \ y = \bar{y}_n(u, v), \ z = \bar{z}_n(u, v), \ (u, v) \text{ on } K.$$

The following relations then hold:

$$(3.35) \qquad F(\bar{S}_n) \leqq F(\pi_n),$$

$$(3.36) \qquad \iint_K (\bar{E}_n + \bar{G}_n)dudv \leqq k^2H.$$

Moreover, the functions $x_n(u, v)$, etc., have monotonic deficiency not greater than h/n, and they satisfy conditions (a), (b), (c), and hence satisfy conditions (1.1). Since the functional values on the boundary have been left unaltered, it remains true that the points $A_i^{(n)}$ of Γ_n correspond under (3.34) to the points A_i^* of the circumference of K.

The hypotheses of Lemma 2 of S.S. are satisfied by the surfaces (3.34). Hence there exists a representation

$$x = x(\theta),\ y = y(\theta),\ z = z(\theta),\ 0 \leqq \theta \leqq 2\pi,$$

of the curve Γ and a subsequence $\{\overline{S}_\alpha\}$ of the $\{\overline{S}_n\}$, the subscript α ranging over a subset of the positive integers, such that†

$$\lim \bar{x}_\alpha(\theta) = x(\theta),\ \lim \bar{y}_\alpha(\theta) = y(\theta),\ \lim \bar{z}_\alpha(\theta) = z(\theta) \tag{3.37}$$

uniformly in θ.

Now by Lemma 1 of S.S. we can select a subsequence $\{\overline{S}_\beta\}$ of the sequence $\{\overline{S}_\alpha\}$ such that the functions $\bar{x}_\beta$ converge uniformly over the whole circle K to a limit function $x(u, v)$. From the sequence $\{\overline{S}_\beta\}$ we can select a subsequence $\{\overline{S}_\gamma\}$ such that $\bar{y}_\gamma(u, v)$ converges uniformly on K to a limit function $y(u, v)$. Finally, we can select a subsequence $\{\overline{S}_\delta\}$ of the sequence $\{\overline{S}_\gamma\}$ such that $\bar{z}_\delta$ converges uniformly on K to a limit function $z(u, v)$. Moreover, by Lemma 1 of S.S. these limit functions are monotonic and the surface

$$S:\quad x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ u^2 + v^2 \leqq 1, \tag{3.38}$$

satisfies conditions (1.1). From (3.37) we see that S is bounded by Γ; hence

$$F(S) \geqq i. \tag{3.39}$$

By hypothesis the integral $F(S)$ is positive quasi-regular and positive definite; and we have just seen that the surfaces $\overline{S}_n$, S all satisfy conditions (1.1), and by (3.36) their areas are uniformly bounded. Under these conditions it is known‡ that $F(S)$ is lower semi-continuous, so that

$$F(S) \leqq \liminf F(\overline{S}_n). \tag{3.40}$$

This, in conjunction with inequality (3.35) and equation (3.13), implies

$$F(S) \leqq m \leqq i. \tag{3.41}$$

Comparing inequalities (3.39) and (3.41) we find

$$F(S) = i = m,$$

and the theorem is proved.

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† We here use $\bar{x}_\alpha(\theta)$ to denote $\bar{x}_\alpha(\cos\theta, \sin\theta)$, etc.

‡ E. J. McShane, *Integrals over surfaces in parametric form*, Annals of Mathematics, vol. 34 (1933); in particular, Theorem III.

4. Second existence theorem: non-definite integrals. If we restrict our attention to rectifiable curves Γ and admit only comparison surfaces of type† L_2, the hypothesis that $F(S)$ is positive definite can be omitted, and we have

THEOREM 4.1. *Let the integral*

$$F(S) = \iint f(X, Y, Z)dudv \tag{4.1}$$

be positive quasi-regular, and let Γ be a rectifiable Jordan curve in xyz-space. Then in the class of all surfaces of type L_2 bounded‡ by Γ there exists a surface S which minimizes $F(S)$.

We first show that for any constants a, b, c the integral

$$\iint_K (aX + bY + cZ)dudv \tag{4.2}$$

has the same value for all surfaces

$$x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ (u, v) \text{ on } K,$$

bounded by Γ and such that the functions $x(u, v)$, etc., satisfy conditions (1.1). (As before, K is the unit circle.) Since the Dirichlet integrals of x, y and z are finite over K, the same is true of the harmonic functions ξ, η, ζ having the same boundary values as x, y, z respectively. Reflecting these functions in the unit circumference yields harmonic functions $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$, defined outside of K, having finite Dirichlet integrals over the region $u^2+v^2 \geqq 1$. Hence if we set $\bar{x}(u, v) = x(u, v)$ for $u^2+v^2 \leqq 1$ and $\bar{x}(u, v) = \bar{\xi}(u, v)$ for $1 < u^2+v^2 \leqq 2$, and define $\bar{y}$, $\bar{z}$ analogously, the functions $\bar{x}$, $\bar{y}$, $\bar{z}$ satisfy conditions (1.1) over the whole circle $u^2+v^2 \leqq 2$ and coincide with x, y, z respectively for $u^2+v^2 \leqq 1$.

We can now apply the proof of Lemma 4 of the last-cited paper, with the trivial change that the integrals $\int yz'dt$, etc., are replaced by Lebesgue-Stieltjes integrals $\int ydz$, etc.; we thus find that

$$\iint_K Xdudv = \int ydz,\quad \iint_K Ydudv = \int zdx,\quad \iint_K Zdudv = \int xdy.$$

The single integrals are taken around Γ and are independent of the particular representation of Γ, and the invariance of the integral (4.2) follows at once.

† Defined at end of §1.

‡ Cf. footnote to Theorem 3.1.

We treat separately the cases in which the $\mathcal{E}$-function $\mathcal{E}(X, Y, Z, \overline{X}, \overline{Y}, \overline{Z})$ is identically zero and that in which it is not identically zero. If it is identically zero, the integrand has the form

$$f(X, Y, Z) = aX + bY + cZ;$$

for

$$\begin{aligned} 0 = (0, 0, 1, X, Y, Z) &= f(X, Y, Z) - f(0, 0, 1) \\ &\quad - Xf_X(0, 0, 1) - Yf_Y(0, 0, 1) - (Z - 1)f_Z(0, 0, 1) \\ &= f(X, Y, Z) - Xf_X(0, 0, 1) - Yf_Y(0, 0, 1) - Zf_Z(0, 0, 1). \end{aligned}$$

Hence the integral $F(S)$ has the same value for all surfaces under consideration, and if we choose any surface S of type L_2 bounded by Γ (the existence of such surfaces being obvious), it serves as a minimizing surface for $F(S)$.

If the $\mathcal{E}$-function is not identically zero, it is possible to find three constants a, b, c such that†

$$\phi(X, Y, Z) \equiv f(X, Y, Z) + aX + bY + cZ > 0 \tag{4.3}$$

for all $(X, Y, Z) \neq (0, 0, 0)$. For all surfaces of the type under consideration which are bounded by Γ, the integrals $F(S)$ and

$$\Phi(S) \equiv \iint \phi(X, Y, Z)\,du\,dv$$

differ by a constant, hence a minimizing surface for $\Phi(S)$ is simultaneously a minimizing surface for $F(S)$. But because of inequality (4.3), Theorem 3.1 guarantees the existence of a minimizing surface of type L_2 for $\Phi(S)$, and our theorem is established.

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† E. J. McShane, *Remark concerning Mr. Graves' paper*, etc., Monatshefte für Mathematik und Physik, vol. 39 (1932), p. 105. The proof applies without change to the present case.

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THE CHARACTERIZATION OF PLANE COLLINEATIONS IN TERMS OF HOMOLOGOUS FAMILIES OF LINES*

BY

WALTER PRENOWITZ

Introduction. This paper is concerned with the problem of specifying minimal conditions that a transformation be a plane collineation.†

Let Γ represent any region‡ of the euclidean plane. A set of line intervals§ contained in Γ is called a *family of lines* or a *family of lines in* Γ, if the end points of *each* interval of the set are *not* in Γ, and each point of Γ is on exactly *one* interval of the set. The region Γ is said to *contain* the family of lines. If n families of lines are contained in Γ, and no two have a common line, they are said to constitute an *n-web of lines in* Γ.

We may now state our principal result.

THEOREM V. *Any topological transformation of region* Γ *which carries a* 4-*web of lines in* Γ *into a* 4-*web of lines is a projective collineation.*

This theorem was proved by E. Kasner‖ on the assumption that the transformation is differentiable twice. Other results characterizing projective transformations have been derived by E. Gourin¶ and by W. Blaschke and his co-workers at Hamburg in connection with their study of webs of curves.** The theorem is related to the work of the Hamburg geometers on transformations of webs of curves since it is equivalent to the assertion that *if two* 4-*webs of lines are topologically equivalent, they are projectively equivalent.*

The proof of this theorem is given in §III of the paper. It is preceded by a set of lemmas in §II, which establish the requisite properties of families and

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† We use the term *collineation* in the sense of a one-to-one transformation of a point set which carries collinear points into collinear points. Any projective transformation which carries points into points is a collineation.

‡ The word *region* is used in the sense of an open set of points, such that any two points of the set can be joined by a broken line wholly contained in the set.

§ The term *line interval* is used in the general sense to include half lines and infinite lines as well as finite intervals.

‖ Bulletin of the American Mathematical Society, vol. 9 (1903), pp. 545–546.

¶ In an unpublished manuscript he proved that *any one-to-one transformation on the general projective plane, which carries four independent pencils of lines into four pencils of lines, carries a net of rationality into a net of rationality.* This result first attracted the writer to the present problem.

** See Mayrhofer, Mathematische Zeitschrift, vol. 28 (1928), p. 733; Reidemeister, ibid., vol. 29 (1929), p. 433; Mayrhofer, ibid., vol. 30 (1929), p. 142, and Abhandlungen Hamburg Seminar, vol. 7 (1929), pp. 9, 10; Blaschke, Abhandlungen Hamburg Seminar, vol. 7, p. 69; Podehl, ibid., vol. 7, p. 397.

webs of lines, and of transformations which carry webs of lines into webs of lines. The most important of these lemmas are the following:

LEMMA X. *If a topological transformation of region* Γ *carries a regular* 3-web of lines in* Γ *into a 3-web of lines, it is differentiable and its Jacobian is nowhere zero.*

LEMMA XI. *If a topological transformation* T *carries* w, *a 3-web of lines, into a 3-web of lines, and a family of* w *is not regular* at one of its lines, then* T *is a projectivity on this line.*

LEMMA XII. *The slope function of a regular* family of lines is differentiable.*

The proof of the main theorem consists of two parts in which different methods are used. If the given 4-web contains a regular 3-web we show that the transformation can be extended twice by the application of Lemmas X and XII. It is then easy to prove that the transformation is projective in the neighborhood of a point by a method essentially that of Kasner's paper. On the other hand, if the given 4-web does not contain a regular 3-web, Lemma XI makes it possible to show that the transformation is projective in the neighborhood of a point, without reference to questions of differentiability. In either case, the result then follows by Theorem IV, which asserts that *any one-to-one transformation of region* Γ *which carries a 3-web of lines in* Γ *into a 3-web of lines, is projective, if it is projective in the neighborhood of one point.*

In §I we derive characterizations of collineations on the projective plane in terms of homologous pencils of lines, with and without the assumption of the continuity of the transformation. Likewise §III contains characterizations of projective transformations of a region of the euclidean plane in terms of pencils rather than arbitrary families of lines, without the assumption of continuity.

I. COLLINEATIONS ON THE PROJECTIVE PLANE

We assume only the postulates of alignment and extension, and the fundamental theorem of projective geometry† in the following theorem.

THEOREM I. *A one-to-one point transformation on the projective plane is a collineation, if it carries three independent‡ pencils and a line not of these pencils into three pencils and a line respectively.*

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* The sense in which we use this term is given on p. 579.

† See Veblen and Young, *Projective Geometry*, vol. I, assumptions A, E, P.

‡ Three pencils are called *independent*, if their vertices are not collinear, otherwise they are *dependent*.

Let A, B, C (Figure 1) be the vertices of the three given pencils, and l the line which is not in the pencils. Let D, E be the intersections of l with AB, AC respectively, and F the intersection of BE and CD. Let A', B', C', F' be the images of A, B, C, F respectively, under the given transformation T. Then no three of the points A', B', C', F' are collinear and there is a projective collineation, S, which carries A', B', C', F' into A, B, C, F respectively. Let R be the resultant of T and S. Transformation R preserves points A, B, C, D, E, F, *pencils* A, B, C, and line l.

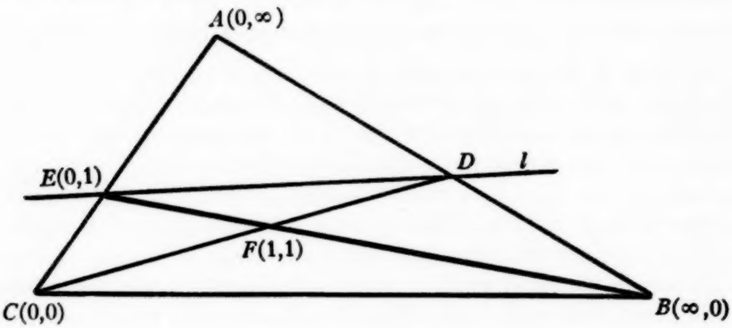

FIGURE 1

Now we introduce non-homogeneous coordinates in the plane so that A, B, C, F are designated by $(0, \infty)$, $(\infty, 0)$, $(0, 0)$, $(1, 1)$ respectively. Then any point, not on the line at infinity, is representable by (x, y) where x and y are elements of some field.

Thus, R is represented by

$$x' = \phi(x, y), \qquad y' = \psi(x, y),$$

and pencils A, B, C by the equations $x=t$, $y=t$, $y=tx$ respectively, where t is a variable element of the field. The invariance of the pencils $x=t$, $y=t$ implies that ϕ is independent of y and ψ of x. Hence R may be expressed by the equations

$$x' = \phi(x), \qquad y' = \psi(y).$$

Moreover, since the line $y=x$ is invariant, R may be represented in the more simple form

$$x' = \phi(x), \qquad y' = \phi(y).$$

The invariance of the pencil $y=tx$ yields

$$\phi(tx) = p(t)\phi(x). \tag{1}$$

Since R leaves (0, 0) and (1, 1) invariant, $\phi(0)=0$ and $\phi(1)=1$. Substituting $x=1$ in (1) and eliminating $p(t)$ from (1), we have

$$\phi(tx) = \phi(t)\phi(x). \tag{2}$$

Since the equation of line l is $y=x+1$, its invariance implies

$$\phi(x+1) = \phi(x) + 1. \tag{3}$$

We have from (2) and (3),

$$\phi[t(y+1)] = \phi(t)\phi(y) + \phi(t)$$

and

$$\phi(ty+t) = \phi(ty) + \phi(t). \tag{4}$$

Substituting x for ty in (4), we may assert

$$\phi(x+t) = \phi(x) + \phi(t) \tag{5}$$

for all x, t in the field.

The relations (2) and (5) are sufficient to prove that R is a collineation. For the finite points of any line not in pencil A are represented by the equation $y=mx+n$. Then

$$y' = \phi(mx+n) = \phi(m)\phi(x) + \phi(n) = \phi(m)x' + \phi(n),$$

so that collinear points not on AB have collinear images, and parallel lines have parallel images. From this it is easy to show that R is a collineation and T is likewise.

For the case of the *real* projective plane, we have the following result.

Corollary. *A one-to-one transformation on the real projective plane is a projective collineation, if it carries three independent pencils and a line not of these pencils into three pencils and a line respectively.*

In this case, the functional equations (2) and (5)* hold for the real field and it is known that the only common solution is $\phi(x)=x$. This may easily be proved. From (2) we have

$$\phi(x^2) = [\phi(x)]^2$$

so that $\phi(x)>0$, if $x>0$. This implies, in virtue of (5), that $\phi(x)$ is monotonic. By means of iteration on (5), we can show that

$$\phi(rx) = r\phi(x)$$

* Darboux, Mathematische Annalen, vol. 17 (1880), pp. 55–61, derives essentially these equations, in proving the fundamental theorem of projective geometry. We use his method of solution.

and hence that

$$\phi(r) = r \tag{6}$$

for all rational r. Since ϕ is monotonic, (6) holds for all real r. It easily follows that R is the identical transformation. Thus T is S^{-1}, a *projective* collineation.

The hypothesis of Theorem I cannot be lessened with respect to the additional line. For there exist transformations on the real projective plane which are not collineations and which carry three independent pencils of lines into three pencils. For example, the transformation

$$x' = x^3, \qquad y' = y^3, \qquad t' = t^3$$

preserves the pencils $x = pt$, $y = pt$, $y = px$, where p is a parameter. The condition of independence of the three pencils also is essential. In fact, if $\phi(x)$ is a discontinuous solution of (5)* which assumes each real value exactly once, the equations

$$\begin{array}{llll} x' = t\phi(x/t), & y' = t\phi(y/t), & t' = t & (t \neq 0), \\ x' = x, & y' = y, & t' = t & (t = 0) \end{array}$$

define a one-to-one transformation on the real projective plane which preserves the infinitude of pencils of the form $y = rx + pt$, where r is rational and p is a parameter.

However, if we assume that the transformation is *continuous*, we can lessen the remainder of the hypothesis a good deal. This is shown in the following theorem.

THEOREM II. *A topological transformation† on the real projective plane is a projective collineation, if it carries into lines, the lines of two pencils with vertices A, B and three additional lines which concur on AB, provided that these three lines and AB are not in a net of rationality.*

Let C be the intersection of the three lines. Let D be a point, distinct from C, on one of these lines, E and F, the intersections of AD with the other two lines, and G, the intersection of CD and BE. (See Figure 2.) Then, as in the preceding theorem, apply to the given transformation a projective collineation, such that the resultant transformation leaves A, B, D, G invariant. Introduce coordinates so that A, B, D, G are represented by $(0, \infty)$, $(\infty, 0)$, $(0, 0)$, $(1, 1)$ respectively.

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* See Hamel, Mathematische Annalen, vol. 60 (1905), p. 459.

† A topological transformation is a uniform continuous transformation which has a uniform continuous inverse.

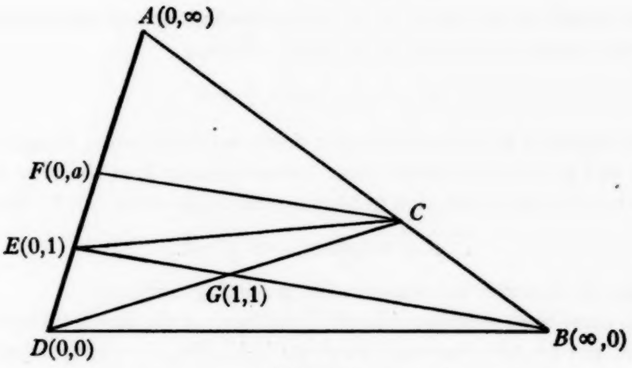

FIGURE 2

The equations of the three lines through C, and of their images, are, respectively, $y=x$, $y=x+1$, $y=x+a$ and $y=x$, $y=x+1$, $y=x+b$. Since, by hypothesis, CF is not in the net of rationality determined by CD, CE and CA, its intercept a, on the axis DA, is not in the domain of rationality determined by their intercepts on DA, namely 0, 1, ∞. Hence a is an *irrational* number.

By the same argument as in the preceding theorem, the transformation takes the analytic form

$$x' = \phi(x), \qquad y' = \phi(y).$$

The functional equations characterizing ϕ are

$$\phi(x+1) = \phi(x) + 1$$

and

$$\phi(x+a) = \phi(x) + b.$$

From these we get by iteration

$$\phi(x + m + na) = \phi(x) + m + nb \tag{1}$$

where m and n are arbitrary integers.

Now if $x=0$ in (1), we have

$$m + nb = \phi(m + na),$$

which with (1) yields

$$\phi(x + m + na) = \phi(x) + \phi(m + na).$$

Since a is irrational, m and n can be chosen so that $m+na$ approximates any given real number as closely as we please. Thus,

$$\phi(x + y) = \phi(x) + \phi(y) \tag{2}$$

holds for all real x and an everywhere dense set of values y. Hence (2) holds for all x and y, since ϕ is continuous. Therefore $\phi(x)$ is of the form cx, which is the only continuous solution of the functional equation (2). In fact

$$\phi(x) = x \text{ since } c = \phi(1) = 1.$$

From this the theorem follows as in the preceding corollary.

If the three lines mentioned in the hypothesis of the above theorem concur at a point not on AB, the result does not hold. This is evident from the first example given at the end of Theorem I.

If the three lines are *not* concurrent, the result holds with a slight modification as is shown in the following theorem.

THEOREM III. *A topological transformation on the real projective plane is a projective collineation, if it carries into lines, the lines of two pencils A, B and three additional non-concurrent lines, provided that no two of these three lines intersect on the line AB.*

Let C, D, E (Figure 3) be the intersections of pairs of the three non-concurrent lines. Consider ΔCDE. It may be that the lines joining a vertex of ΔCDE to A and B are harmonically separated by the sides of ΔCDE which contain this vertex. But such an harmonic relationship cannot hold at each of two vertices of ΔCDE. For suppose it does hold at C and D. Then we have the harmonic sets of lines

$$H[CA, CB; CD, CE]$$

and

$$H[DA, DB; DC, DE],$$

three pairs of corresponding lines of which meet on AB. Hence the fourth pair, CE, DE, meet on AB. Thus, E is on AB, contrary to the hypothesis.

Therefore, there is a vertex of ΔCDE, such that the lines joining it to A and B are *not* harmonically separated by the sides of ΔCDE which contain this vertex. Let us suppose that C is such a vertex.

Now, apply to the given transformation a projective collineation, such that the resultant transformation leaves A, B, C, D invariant; and introduce coordinates so that A, B, C, D are represented by $(0, \infty)$, $(\infty, 0)$, $(0, 0)$, $(1, 1)$, respectively.

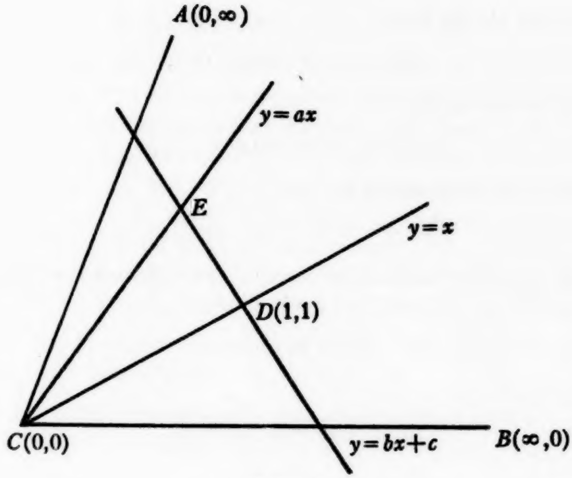

FIGURE 3

The equations of CD, CE, DE are, respectively,

$$y = x,$$
$$y = ax \qquad (a \neq 0, \pm 1),$$
$$y = bx + c \qquad (b \neq 0, c \neq 0),$$

and those of their images are, respectively,

$$y = x,$$
$$y = dx \qquad (d \neq 0, 1),$$
$$y = ex + f \qquad (e \neq 0, f \neq 0).$$

The transformation takes the form

$$x' = \phi(x), \qquad y' = \phi(y),$$

where ϕ is characterized by the equations

$$\phi(ax) = d\phi(x), \tag{1}$$
$$\phi(bx + c) = e\phi(x) + f. \tag{2}$$

We need only consider the case where $|a| > 1$, since if $|a| < 1$, we may replace (1) by

$$\phi[(1/a)x] = (1/d)\phi(x) \tag{3}$$

and proceed in the same manner.

From (1) and (2), we have

$$\phi(abx + c) = de\phi(x) + f,$$

which yields, in view of (3),

$$\phi(bx + c/a) = e\phi(x) + f/d.$$

By repetition of this procedure, we have

$$\phi(bx + c/a^m) = e\phi(x) + f/d^m \tag{4}$$

where m is any positive integer. Let $m \to \infty$. Then, since ϕ is continuous, f/d^m has a limit, which must be zero, as $d \neq 1$. Thus,

$$\phi(bx) = e\phi(x),$$

which, with (2), gives

$$\phi(bx + c) = \phi(bx) + f$$

and

$$\phi(x + c) = \phi(x) + f.$$

Thus

$$\phi(x + nc) = \phi(x) + nf \tag{5}$$

easily follows, where n is *any* integer.

Applying to (5) the procedure used in deriving (4), we have

$$\phi(x + nc/a^p) = \phi(x) + nf/d^p, \tag{6}$$

where p is any *non-negative* integer. By iteration, (6) yields

$$\begin{aligned} &\phi[x + c(n_0 + n_1/a^1 + \cdots + n_q/a^q)] \\ &\quad = \phi(x) + f(n_0 + n_1/d^1 + \cdots + n_q/d^q), \end{aligned} \tag{7}$$

where the n's are arbitrary integers and q is an arbitrary *positive* integer. Letting $x=0$ in (7), we may get

$$\phi(x + y) = \phi(x) + \phi(y) \tag{8}$$

where x is any real number and

$$y = c(n_0 + n_1/a^1 + \cdots + n_q/a^q).$$

Since we can choose q and the n's so that y approximates any given real number as closely as we please, (8) holds for all x, y and the desired result follows as in the preceding theorem.

II. Lemmas on families of lines

We employ a series of lemmas on families and webs of lines, in proving the later theorems. Families of lines are indicated by the letters f, g, h, etc., and lines of a given family by the letter which represents the family, usually with subscripts affixed. Sometimes, the letter f will be used to represent an arbitrary line of the family f, in which case the context will indicate the sense intended. Let family f be contained in region Γ. Then if P is any point of Γ, f_P represents the line of the family f which contains P. A line interval l, which is in Γ, and the end points of which are not in Γ, is called a *transversal* to f_P at Q, if it intersects* f_P at Q.

Lemma I. *If a transversal intersects f_P, it also intersects f_X, where X is any point of some neighborhood of P.*

Let Γ be the region containing f, l the transversal, and Q the intersection of l with f_P. Choose A and B on f_P, so that P and Q are between A and B. (See Figure 4.)

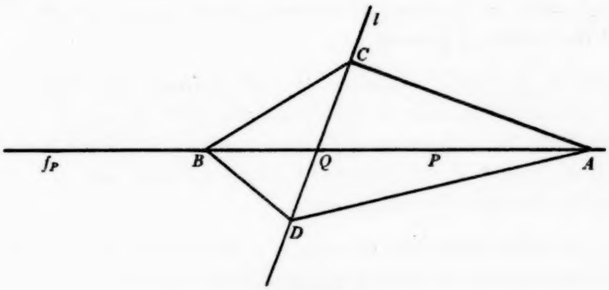

Figure 4

If Γ has a boundary, the closed interval AB has a positive distance ϵ from this boundary. Choose points C and D on l, so that Q is between them and the distance of each from Q is less than ϵ. Then the quadrilateral $ACBD$ is contained in Γ, since the distance of each of its points from the closed interval AB is less than ϵ. If X is any point inside the quadrilateral $ACBD$, f_X intersects segment† CD and hence intersects l. Thus, since the interior of the quadrilateral $ACBD$ contains P, it may be taken as the required neighborhood and the lemma is proved.

Corollary. *If Q is a point of f_P and N is any neighborhood of Q, there exists $\overline{N}$, a neighborhood of P, such that if X is in $\overline{N}$, f_X contains a point of N.*

* We say that lines l and m *intersect*, if they have exactly *one* common point.

† A segment is an open finite interval.

Let l be a transversal to f_P at Q. Choose A, B, C, D as in the lemma with the additional condition that segment CD be in N. Then the interior of quadrilateral $ACBD$ can be chosen as $\overline{N}$.

DEFINITION. *If in region Γ, containing a family of lines f, the sequence of points $\{P_n\}$ converges on point P, we say the sequence of lines $\{f_{P_n}\}$ approaches or converges on f_P, and we write $\{f_{P_n}\}\to f_P$.*

LEMMA II. *If $\{f_n\}\to f$ and l is a transversal to f at Q, then l intersects $\{f_n\}$ in a sequence of points $\{Q_n\}$ and $\{Q_n\}\to Q$.**

Since $\{f_n\}\to f$, there exists, for each n, a point P_n on f_n such that $\{P_n\}\to P$, where P is some point on f. Thus, for all sufficiently large n, l intersects f_n, by Lemma I, so that $\{Q_n\}$ exists.

Now, if we identify P, Q and l with the objects denoted by those symbols in Lemma I, we have that, for CD any segment on l containing Q, there exists U, a neighborhood of P, such that f_{P_n} intersects l in segment CD for *all* P_n contained in U. Thus, Q_n lies in segment CD, any assigned neighborhood of Q, for all $n>p$, where p is a natural number depending on the given neighborhood, and the lemma is proved.

DEFINITION. *Let l be a transversal to f, an arbitrary line of family f, and l', the infinite line which contains l. Let a fixed side of l' and a fixed direction on l' be specified. Then the inclination of f with regard to l is $\angle PQR$,† where Q is the intersection of f and l, P is a point of f on the specified side of l', and R is a point of l in the specified direction from Q.*

LEMMA III. *The inclination of a line of a family with regard to a transversal is a continuous function of its intersection with the transversal.*

Let the family be f in region Γ and the transversal l. Let P_1, P (Figure 5) be, respectively, fixed and variable points of l, and θ_1, θ, the inclinations with regard to l of f_{P_1}, f_P respectively. We shall show that as $P\to P_1$, $\theta\to\theta_1$. If the extensions of f_{P_1}, f_P outside of Γ intersect, let the intersection be Q. Let the lengths PP_1 and PQ be s and t respectively.

Then if Q exists, we have

$$\frac{|\sin(\theta-\theta_1)|}{\sin\theta_1}=\frac{s}{t}, \tag{1}$$

---

* We use the lemma frequently to infer convergence of a sequence of points from that of a sequence of lines. It is interesting to note that it guarantees that a convergent sequence of lines converges on a unique line.

† We use this symbol in two senses, viz., the *angle PQR* and the *measure* of the angle *PQR*.

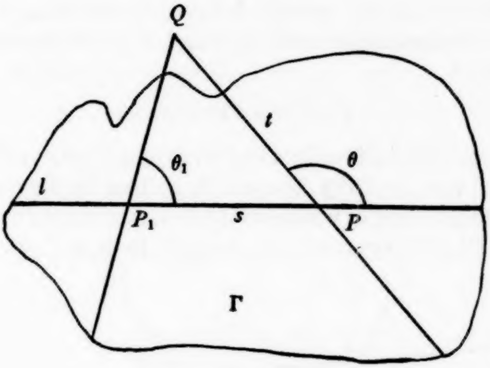

FIGURE 5

and if Q does not exist, we have

$$\sin(\theta - \theta_1) = 0. \tag{2}$$

As $P \to P_1$, t is always greater than a positive constant a, since Q is not in Γ and P is in some closed neighborhood of P_1. Thus, by (1) and (2), as $P \to P_1$, $\sin(\theta - \theta_1) \to 0$. We have either

$$|\theta - \theta_1| < \pi - \theta_1$$

or

$$|\theta - \theta_1| < \theta_1,$$

since $0 < \theta < \pi$ and $0 < \theta_1 < \pi$. Thus

$$|\theta - \theta_1| < b < \pi,$$

so that as $P \to P_1$,

$$(\theta - \theta_1) \to 0 \text{ and } \theta \to \theta_1.$$

LEMMA IV. *If f' and f are two lines of a family f, and l, m are transversals to f' at the same point, the ratio of the distances intercepted on l and m by f and f' converges to a non-zero limit as $f \to f'$.**

* That is, the ratio approaches the *same* limit for all sequences $\{f_n\}$ which converge on f'.

Let P (Figure 6) be the common intersection of l and m with f', and L, M be the respective intersections of l, m with f. If $L=M$, the theorem is trivial. If $L \neq M$, we have

$$PL/PM = \sin\beta/\sin\alpha,$$

where α and β are the inclinations of f with regard to l and m respectively. As $f \to f'$, $M \to P$ and $L \to P$ by Lemma II, so that by Lemma III, α and β converge to limits neither of which is zero or π, since both l and m are distinct from f'. Thus PL/PM approaches a non-zero limit as $f \to f'$.

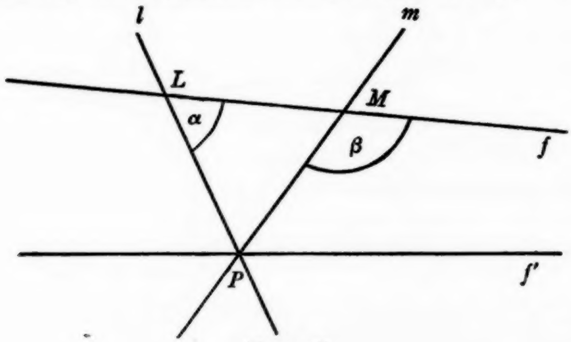

FIGURE 6

LEMMA V. *If f' and f are two lines of a family f and l, m are transversals to f', the ratio of the distances intercepted on l and m by f and f' has an upper bound and a positive lower bound as $f \to f'$.**

Let family f be contained in region Γ and let l, m intersect f in L, M and f' in L', M' respectively. If $L'=M'$, the result is immediate, by Lemma IV.

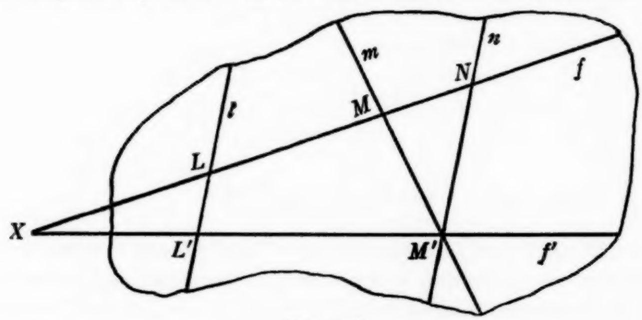

FIGURE 7

* That is, the ratio has fixed bounds for all sequences $\{f_n\}$ which converge on f'.

If $L' \neq M'$ (see Figure 7), consider the line n through M' parallel to l. Since n intersects f', it intersects all f sufficiently close to f' by Lemma I. Let N be the intersection of n with f. Then, we have

$$L'L/M'M = (L'L/M'N)(M'N/M'M),$$

and $M'N/M'M$ converges as $f \to f'$, by Lemma IV. Consider $L'L/M'N$. Let the intersection of the infinite lines LN and $L'M'$ be X, if it exists. Then, if X exists,

$$L'L/M'N = XL'/XM' = (XM' \pm L'M')/XM' = 1 \pm (L'M'/XM'),$$

and if X does not exist,

$$L'L/M'N = 1.$$

But $L'M'$ is fixed and $XM' > a > 0$ since X is not in Γ. Thus $L'L/M'N$ is bounded above and the same is true of $L'L/M'M$. By the identical argument $M'M/L'L$ is bounded above, so that $L'L/M'M$ has a positive lower bound.

DEFINITION. *Let P be a point of the region of definition of a uniform transformation T, l a line containing P, and Q a point on l. Let P', Q' be the images under T of P, Q respectively. Then if* $\lim_{Q\to P} P'Q'/PQ$ *exists, it is called the directional derivative* (*abbreviated* D.D.) *of T at P in the direction l.* The* D.D. *of T at P in the direction f_P is called the* D.D. *of T at P in the direction of family f.*

LEMMA VI. *If a topological transformation T carries w, a 3-web of lines, into a 3-web of lines, then*

(a) *the* D.D. *of T exists at almost all points of l, any line of w,† in the direction l;*

(b) *if the* D.D. *of T exists at a point in the direction of one family of w, it exists at that point in the direction of each family of w;*

(c) *the* D.D. *of T is not zero at any point of w† in the direction of a line of w.*

(a) Since the transformation is topological, betweenness is preserved for points on l, any line of w. Thus, if we establish scales of ordinates on l and on its image, the ordinate x' of the image point is a *monotonic* function of x, the ordinate of the given point. Therefore, by a theorem of Lebesgue, the derivative of x' with regard to x exists and is finite *almost everywhere* on l. Since the absolute value of the derivative dx'/dx at a point of l is the D.D. of T at this point in the direction l, conclusion (a) is true.

---

* Note that the phrase is defined *only* if l contains P.

† The lines of the families which constitute a web are called *lines of the web*, and the points of these lines, *points of the web*.

(b) Let the three families which form w be f, g, and h. Then it will suffice to show that if the D.D. of T exists at P (Figure 8) in the direction f_P, it also exists at P in the direction g_P. Let Q be any point on g_P, close to P. Then h_Q intersects f_P at R by Lemma I. Let the images of P, Q, R be P', Q', R' respectively. Then we have

$$(1) \qquad P'Q'/PQ = (P'Q'/P'R')(P'R'/PR)(PR/PQ).$$

The fractions $P'Q'/P'R'$ and PR/PQ converge as $Q \to P$ in view of Lemma IV. As $Q \to P$, $R \to P$ also, by Lemma II. Thus $P'R'/PR$ converges as $Q \to P$. Therefore, the desired D.D. which is $\lim_{Q \to P} P'Q'/PQ$ exists.

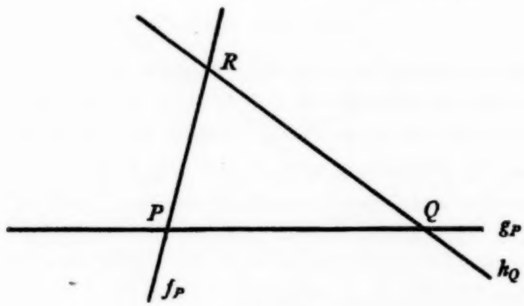

FIGURE 8

(c) Suppose that the D.D. of T is zero at some point in the direction of a line of the web, let us say at point P in the direction f_P. We notice from (1) in the proof of (b), that if $P'R'/PR \to 0$, $P'Q'/PQ \to 0$ likewise. In other words, if the D.D. of T is zero at a point in the direction of one line of the web, it is zero at that point in the direction of each line of the web through that point. We shall use this to show that the D.D. of T is zero in the direction f_P, at each point of f_P.

Let Q (Figure 9) be any point on f_P distinct from P. Then if R is an arbitrary point on g_Q, sufficiently close to Q, f_R intersects g_P at S. Let the corresponding image points be P', Q', R', and S'. We have

$$(1) \qquad Q'R'/QR = (Q'R'/P'S')(P'S'/PS)(PS/QR).$$

The first and third factors of (1) are bounded as $R \to Q$ in view of Lemma V, and the second factor approaches zero. Thus, the D.D. of T at Q in the direction g_Q is zero, whence it is also zero at Q in the direction f_P. But since Q is *arbitrary*, the D.D. of T is zero *everywhere* on f_P in the direction f_P, which implies that the image of f_P is a single point. Thus our original supposition is false and (c) is proved.

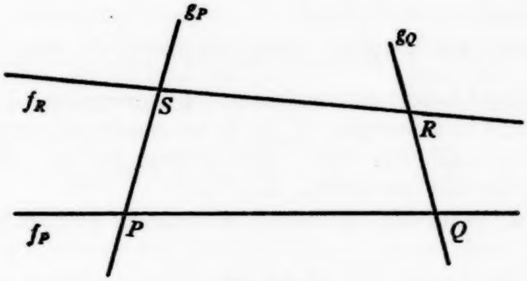

FIGURE 9

DEFINITION. *If f_1 is a line of family f, such that the ratio of the distances intercepted by f and f_1 on each pair of transversals to f_1 converges as $f \to f_1$, the family f is said to be regular at f_1. A regular family is one which is regular at each of its lines, and a web is called regular if each of its families is regular.*

Regular families are important in this paper mainly because their slope functions are differentiable, as is shown in Lemma XII. An example of a regular family is a set of parallel lines in a circular region; the family consisting of the lines

$$y = k \qquad (1 \leqq k < 2), \qquad\qquad y = l(x-2) + 1 \qquad (0 < l < \infty)$$

contained in the triangular region whose vertices are (0, 0), (2, 0), (2, 2) is *not* regular at the line $y = 1$.

LEMMA VII. *A sufficient condition that family f be regular at f_1, is that the ratio of the distances intercepted by f and f_1 on a single pair of transversals which intersect f_1 at different points, converge as $f \to f_1$.*

Let m, n (Figure 10) be the two given transversals, and r, s any two transversals to f_1, and let their respective intersections with f_1, f be M_1, M, N_1, N, R_1, R, S_1, S. We have to derive the existence of $\lim_{f \to f_1} RR_1/SS_1$ from that of $\lim_{f \to f_1} MM_1/NN_1$. Draw parallel transversals to f_1 at M_1, N_1, R_1, S_1, which intersect f at $\overline{M}$, $\overline{N}$, $\overline{R}$, $\overline{S}$ respectively.

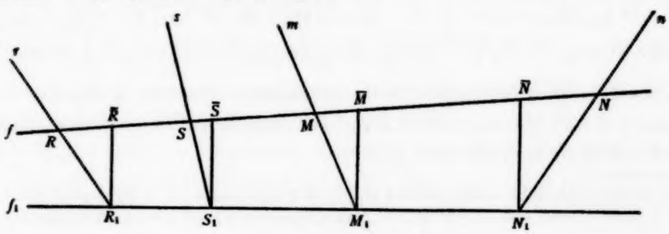

FIGURE 10

Then we have

$$(1) \qquad \overline{M}M_1/\overline{N}N_1 = (\overline{M}M_1/MM_1)(MM_1/NN_1)(NN_1/\overline{N}N_1).$$

The first and third factors on the right in (1) converge as $f\to f_1$, by Lemma IV; thus $\overline{M}M_1/\overline{N}N_1$ converges as $f\to f_1$. In the same way, we show that the existence of $\lim_{f\to f_1}\overline{R}R_1/\overline{S}S_1$ implies that of $\lim_{f\to f_1}RR_1/SS_1$. Thus we have only to prove the existence of $\lim_{f\to f_1}\overline{R}R_1/\overline{S}S_1$.

By the point of division formula of elementary analytic geometry, we have

$$(2) \qquad \frac{\overline{R}R_1}{\overline{S}S_1} = \frac{\overline{M}M_1 + a\overline{N}N_1}{1+a} \div \frac{\overline{M}M_1 + b\overline{N}N_1}{1+b} = \frac{\overline{M}M_1/\overline{N}N_1 + a}{\overline{M}M_1/\overline{N}N_1 + b}\cdot\frac{1+b}{1+a}$$

$$(a \neq -1,\ b \neq -1).$$

In addition

$$\left| (\overline{M}M_1/\overline{N}N_1) + b \right| = \left| (\overline{S}S_1/\overline{N}N_1)(1+b) \right| > c > 0$$

where c is constant, by Lemma V. Thus, by (2) the convergence of $\overline{M}M_1/\overline{N}N_1$ as $f\to f_1$ implies that of $\overline{R}R_1/\overline{S}S_1$ and the lemma is true.

LEMMA VIII. *If a topological transformation carries a regular 3-web of lines into a 3-web of lines, the latter is regular also.**

Let f and g be any two families of the given web, and f' the image of f. It will be sufficient to show that f' is regular at f_1', any one of its lines. Let f_1' be the image of f_1. Choose distinct points P, Q on f_1, at each of which the D. D. of the transformation exists in the direction f_1. Let g_P, g_Q intersect f, an arbitrary line of family f, at R, S respectively. Let f' be the image of line f, and P', Q', R', S', the images of P, Q, R, S, respectively.

Then

$$\frac{P'R'}{Q'S'} = \left(\frac{P'R'}{PR}\cdot\frac{PR}{QS}\right) \div \frac{Q'S'}{QS}.$$

We know that $\lim_{R\to P}P'R'/PR$ and $\lim_{S\to Q}Q'S'/QS$ exist and the latter is not zero, by Lemma VI (b), (c), and $\lim_{f\to f_1}PR/QS$ exists, by hypothesis. However, $f'\to f_1'$ implies that $f\to f_1$ and hence that $R\to P$ and $S\to Q$ by Lemma II. Therefore $\lim_{f'\to f_1'}P'R'/Q'S'$ exists and the result follows by Lemma VII.

LEMMA IX. *If a topological transformation carries w, a regular 3-web of lines, into a 3-web of lines, the* D.D. *of the transformation exists at each point of w, in the directions of the lines of w.*

* The proof which follows also justifies the more general result: *If a topological transformation carries w, a 3-web of lines, into a 3-web of lines, and a family of w is regular at a line, then the image of the family is regular at the image of the line.*

Let f and g be any two families of w. We shall show that the D.D. exists at P, any point of w, in the direction of f. There is a point Q, distinct from P, on f_P at which the D.D. exists in the direction of f. Choose an arbitrary point R on g_P, distinct from P and close enough to P so that f_R intersects g_Q at S. Denoting image points in the usual way, we have

$$(P'R'/PR) = (P'R'/Q'S')(Q'S'/QS)(QS/PR).$$

Now we can show that the three factors on the right converge as $R \to P$, the first by means of Lemma VIII, the second by Lemma VI(b), and the third directly by the hypothesis. Thus $\lim_{R\to P} P'R'/PR$, i.e., the D.D. of the transformation at P in the direction of g, exists, which implies its existence at P in the direction of f, and the lemma is proved.

Lemma X. *If a topological transformation of region* Γ *carries a regular 3-web of lines in* Γ *into a 3-web of lines, it is differentiable and its Jacobian is nowhere zero.*

We shall show that if there are established arbitrary rectangular coordinate systems in the given and image planes, the functions representing the transformation are differentiable and have a *nowhere* vanishing Jacobian. However, if we prove that the transformation is differentiable and its Jacobian is not zero at P, an arbitrary point of Γ, for special cartesian axes dependent on P, the above result will follow. For the change of coordinates which must be applied to shift from the special axes to those originally chosen is a non-singular linear transformation, and hence is differentiable with a non-vanishing Jacobian.

Let f, g be two families of the given web and let their images be u, v respectively. Take f_P, g_P (Figure 11) as the special cartesian axes and set up a coordinate system using an arbitrary unit distance and arbitrary positive directions on f_P and g_P. We represent image points as heretofore. In the image plane, we establish a similar coordinate system using $u_{P'}$ and $v_{P'}$ as coordinate axes and choosing an arbitrary unit distance. However, the positive directions on $u_{P'}$ and $v_{P'}$ are taken to be the images of the positive directions on f_P and g_P respectively.

Let $Q(x, y)$ be any point in the neighborhood of P and $Q'(x', y')$ its image. We shall show that x' and y' are differentiable functions of x and y, at $(0, 0)$. Let g_Q intersect f_P in R, and $v_{Q'}$ intersect $u_{P'}$ in R'.

If R is in the positive (negative) direction on f_P from P, then R' is in the positive (negative) direction on $u_{P'}$ from P'. Moreover, if R is in the positive (negative) direction from P, x is positive (negative) and a similar relation holds for R' and x'. Thus, x and x' have the same sign.

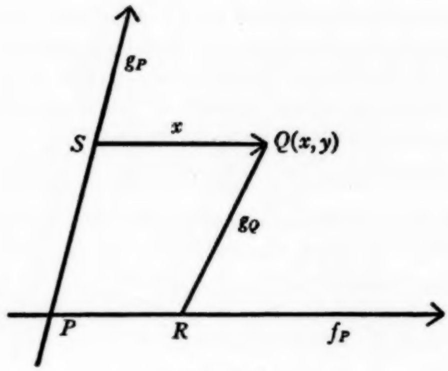

FIGURE 11

Now suppose that Q is not on g_P, so that $x \neq 0$. Then we have

$$x' = (|x'|/P'R')(P'R'/PR)(PR/|x|)x. \tag{1}$$

As $Q \to P$, $R \to P$ and $P'R'/PR \to a$, the D.D. of the transformation at P in the direction of f. Also, as $Q \to P$, the first and third factors of the right member of (1) converge to unity. We shall prove this for $PR/|x|$; the proof is similar for $|x'|/P'R'$.

Let S be the intersection of g_P and the line through Q parallel to f_P. Then either

$$PR = |x|$$

or

$$PR/|x| = PT/ST = (ST \pm PS)/ST = 1 \pm (PS/ST),$$

where T is the intersection of the extensions of g_P and g_Q. As $Q \to P$, $PS \to 0$ and $ST > k > 0$, so that $PR/|x| \to 1$.

Therefore for Q not on g_P, we have

$$x' = (1 + \epsilon_1)(a + \epsilon_2)(1 + \epsilon_3)x = ax + \epsilon x, \tag{2}$$

where $a \neq 0$ by Lemma VI (c) and $\epsilon \to 0$ as $Q \to P$. If Q is on g_P, $x = x' = 0$, so that (2) holds in this case with $\epsilon = 0$. Therefore (2) holds for all (x, y) in the neighborhood of $(0, 0)$, and ϵ approaches zero with x and y. Similarly

$$y' = by + \eta y$$

where b is the D.D. of the transformation at P in the direction g_P and hence is not zero, and η vanishes with x and y. Thus by definition x' and y' are

differentiable functions of x and y at $(0, 0)$. Moreover, the Jacobian at $(0, 0)$ is

$$\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} \neq 0,$$

and the desired result follows.

LEMMA XI. *If a topological transformation T carries w, a 3-web of lines, into a 3-web of lines, and a family of w is not regular at one of its lines, then T is a projectivity on this line.*

Let two families of the given web be f and g and suppose that f is not regular at f_1. Let P_1, P_2, P (Figure 12) be distinct points on f_1 at which the D.D. of T in the direction f_1 exists; P_1 and P_2 are to be fixed and P is a variable point. Consider three parallel transversals to f_1 at P_1, P_2, P respectively, and let their intersections with f, a variable line of f distinct from f_1, be R_1, R_2, R respectively. Let f_1', f', P_1', P_2', P' be the images of f_1, f, P_1, P_2, P respectively. Consider three parallel transversals to f_1' at P_1', P_2', P' meeting f' in $\overline{R}_1$, $\overline{R}_2$, $\overline{R}$. Let s_1, s_2, s, $\bar{s}_1$, $\bar{s}_2$, $\bar{s}$ represent the respective distances P_1R_1, P_2R_2, PR, $P_1'\overline{R}_1$, $P_2'\overline{R}_2$, $P'\overline{R}$.

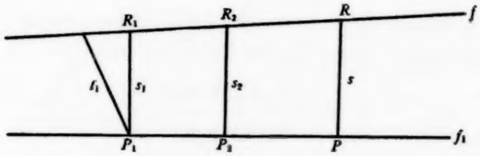

FIGURE 12

Then, as $f \to f_1$, the ratios $\bar{s}_1/s_1$, $\bar{s}_2/s_2$, $\bar{s}/s$ approach non-zero limits. We shall show this for $\bar{s}_1/s_1$. Let t_1 be the distance intercepted on g_P by f_1 and f, and t_1' the image distance. Then $\lim_{f\to f_1} t_1'/t_1$ exists and is not zero by Lemma VI (b), (c), since the D.D. of T exists at P_1 in the direction f_1. By Lemma IV, as $f \to f_1$, $\bar{s}_1/t_1'$ and t_1/s_1 converge to values different from zero. Thus, since

$$\bar{s}_1/s_1 = (\bar{s}_1/t_1')(t_1'/t_1)(t_1/s_1),$$

$\bar{s}_1/s_1$ converges to a non-zero value as $f \to f_1$.

We have

$$s = (s_1 + \lambda s_2)/(1 + \lambda) \qquad (\lambda \neq -1),$$

$$\bar{s} = (\bar{s}_1 + \lambda' \bar{s}_2)/(1 + \lambda') \qquad (\lambda' \neq -1),$$

where λ, λ' are the ratios in which P, P' divide the *directed* segments P_1P_2, $P_1'P_2'$ respectively. By division, we obtain

$$\frac{\bar{s}}{s} = b\,\frac{\bar{s}_1 + \lambda'\bar{s}_2}{s_1 + \lambda s_2}, \tag{1}$$

where

$$b = (1+\lambda)/(1+\lambda') \neq 0.$$

Then if a_1, a_2 are the respective limits of $\bar{s}_1/s_1$, $\bar{s}_2/s_2$ as $f \to f_1$,

$$\bar{s}_1 = a_1 s_1 + \epsilon_1 s_1 \qquad (a_1 \neq 0)$$

and

$$\bar{s}_2 = a_2 s_2 + \epsilon_2 s_2 \qquad (a_2 \neq 0),$$

where $\epsilon_1 \to 0$ with s_1 and $\epsilon_2 \to 0$ with s_2. Thus, substituting in (1), we have

$$\frac{\bar{s}}{s} = b\,\frac{a_1 s_1 + \epsilon_1 s_1 + \lambda' a_2 s_2 + \lambda' \epsilon_2 s_2}{s_1 + \lambda s_2}$$

and

$$\frac{\bar{s}}{s} = b a_1 + b\,\frac{\epsilon_1 s_1 + \lambda'\epsilon_2 s_2}{s_1 + \lambda s_2} + b\,\frac{(\lambda' a_2 - \lambda a_1) s_2}{s_1 + \lambda s_2}. \tag{2}$$

The second term of the right member of (2) converges as $f \to f_1$. For, if we divide its numerator and denominator by s_2, we have

$$b\,\frac{\epsilon_1 (s_1/s_2) + \lambda'\epsilon_2}{(s_1/s_2) + \lambda},$$

the numerator of which approaches zero as $f \to f_1$, since, by Lemma V, s_1/s_2 is bounded; and the absolute value of the denominator

$$|(s_1/s_2) + \lambda| = |(s_1 + \lambda s_2)/s_2| = |cs/s_2| > d > 0$$

where c and d are constants, by Lemma V. Thus, the third term on the right in (2) converges as $f \to f_1$. But this term is

$$b\,\frac{(\lambda' a_2 - \lambda a_1) s_2}{s_1 + \lambda s_2} = b\,\frac{\lambda' a_2 - \lambda a_1}{(s_1/s_2) + \lambda},$$

and s_1/s_2 does *not* converge as $f \to f_1$, since family f is *not* regular at f_1. Thus

$$\lambda' a_2 - \lambda a_1 = 0$$

and

$$\lambda' = (a_1/a_2)\lambda. \tag{3}$$

Now we may consider λ, λ' to be the projective ordinates of P, P' on

the lines f_1, f_1' respectively. Thus by (3), the transformation is a projectivity on f_1 for the *everywhere dense* set of points P, and hence for *all* points.

DEFINITION. *If a family of lines f is referred to rectangular coordinate axes, the slope of f_P, considered as a function of the coordinates of P, is called the slope function of the family f. The slope function is not defined at any point of a vertical line of the family.*

LEMMA XII. *The slope function of a regular family of lines is differentiable.**

Let the family be f, contained in Γ, and let $P(x_1, y_1)$ be any point of Γ at which $m(x, y)$, the slope function of f, exists. We shall show that $m(x, y)$ is differentiable at (x_1, y_1). Let $Q(x, y)$ (Figure 13) be any point in the neighborhood of P. Then the line $x = x_1$ intersects f_Q at (x_1, y_1+d_1). If x_2 is distinct from but close to x_1, the line $x = x_2$ intersects f_P and f_Q at (x_2, y_2) and (x_2, y_2+d_2), respectively. We note that d_1 and d_2 have the *same* algebraic sign.

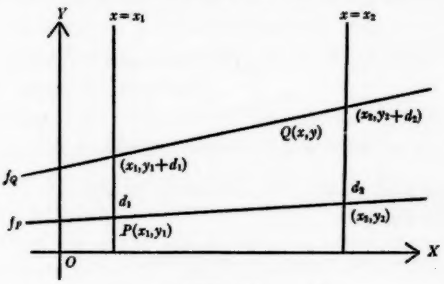

FIGURE 13

By Lemma I, f_Q intersects $x = x_1$, and hence has a slope, $m(x, y)$. We write m_1, m for $m(x_1, y_1)$, $m(x, y)$ respectively. Suppose that Q is *not* on f_P. We may assert that

$$m = \frac{y_2 + d_2 - (y_1 + d_1)}{x_2 - x_1}$$

and

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1}.$$

Hence, we have

$$(1) \qquad m - m_1 = a(d_2 - d_1)$$

* The proof which follows also justifies the more general result: *The slope function of family f is differentiable at $P(x_1, y_1)$, if f is regular at f_P.*

and

$$m = m_1 + a(d_2 - d_1), \tag{2}$$

where $a = 1/(x_2 - x_1)$. We may easily get

$$d_1 = \Delta y - m\Delta x, \tag{3}$$

where $\Delta y = y - y_1$ and $\Delta x = x - x_1$. Substituting for m in (3) its value in (2), we have

$$d_1 = \Delta y - m_1\Delta x - a(d_2 - d_1)\Delta x. \tag{4}$$

Since $d_1 \neq 0$, we may obtain from (1) and (4)

$$m - m_1 = \frac{a(d_2 - d_1)}{d_1}[\Delta y - m_1\Delta x - a(d_2 - d_1)\Delta x].$$

Since family f is *regular*, $\lim_{Q\to P} d_2/d_1$ exists and hence $\lim_{Q\to P}(d_2 - d_1)/d_1$ exists.

Therefore

$$m - m_1 = (b + \epsilon_1)(\Delta y - m_1\Delta x + \epsilon_2\Delta x),$$

and we have

$$m - m_1 = b(\Delta y - m_1\Delta x) + \epsilon\Delta x + \eta\Delta y, \tag{5}$$

where b is constant and ϵ and η vanish with Δx and Δy.

If Q is on f_P, we have

$$m - m_1 = b(\Delta y - m_1\Delta x) = 0.$$

Consequently (5) holds for all Q, and ϵ and η vanish with Δx and Δy regardless of how the latter approach zero. Thus, by definition, $m(x, y)$ is differentiable at (x_1, y_1).

III. Collineations of regions of the euclidean plane

We shall first establish the following result, which is used as a lemma to the succeeding theorems.

Theorem IV. *Any one-to-one transformation of region* Γ *which carries a 3-web of lines in* Γ *into a 3-web of lines is projective, if it is projective in the neighborhood of one point.*

Let the given web w consist of families f, g, and h. Let P be a point of Γ in a neighborhood of which the given transformation T is projective. Choose four points, no three of which are collinear in this neighborhood of P, and let S be the projective transformation which carries the images of these four points under T into the original points, respectively. Transformation S is de-

fined for all points of the image web, with the possible exception of points of one line, the so-called *vanishing line* of the projective transformation.

Let R be the resultant of T and S in that order. Transformation R is defined for all points of Γ, with the exception of those whose images under T are on the vanishing line of S, if such exist; and R carries points on any line of w into collinear points. Moreover, if R is not defined for a point of Γ, the lines of w which contain this point are carried into sub-sets of parallel lines by R. Hence, if R carries any two of f_M, g_M, h_M into sub-sets of lines which intersect at N, R is defined for point M and carries M into N.

It is evident that R is the identical transformation in the neighborhood of P. We shall obtain the result by showing that R is the identity over the whole region Γ.

Consider points X, contained in Γ, such that all points in the neighborhood of X are invariant under R, and X can be joined to P by a broken line, *each* point of which has a neighborhood composed of points invariant under R. Let Γ_1 be the set consisting of P and all points X. It can easily be shown that Γ_1 is a sub-region of Γ. We shall show that Γ_1 is identical with Γ, thus proving that R leaves each point of Γ invariant.

First we show that *if A is a point of Γ on the boundary of Γ_1, not more than one of the lines f_A, g_A, h_A contains a point of* Γ_1. We shall call this assertion (1). Let us suppose that this is not so. Then there is no loss in generality in assuming that f_A and g_A each contains a point of Γ_1. It follows that each contains *several* points of Γ_1. Therefore R carries f_A and g_A into sub-sets of the respective infinite lines which contain f_A and g_A. This implies that A is an invariant point of R.

If Y is any point in a sufficiently small neighborhood of A, f_Y and g_Y each contains a point of Γ_1, by the corollary to Lemma I. Thus the invariance of Y follows by the argument just used to prove the invariance of A. In other words, we have shown that A has a neighborhood consisting of points invariant under R. It follows that A can be joined to P by a broken line, with the property *that each of its points has a neighborhood composed of invariant points*. Because if Q, distinct from P, is a point of Γ_1 in the neighborhood of A, A can be joined to Q, and Q to P by broken lines which have the same property. Thus, by definition, A belongs to Γ_1, which is impossible because A is a boundary point of Γ_1. This contradiction establishes our original assertion concerning A.

Now, in order to prove that Γ_1 and Γ are identical, let us suppose that they are distinct. Then we can show there is a line of the web, which contains a point of Γ_1 and a point of its boundary. For, since Γ_1 is a *proper part* of Γ, there is, in Γ, a point B, of the boundary of Γ_1. Then, if f_B contains a point

of Γ_1, it is the desired line. If f_B contains no point of Γ_1, we can find C, a point of Γ_1, so close to B, that g_C intersects f_B at a point which we call D. Since g_C contains C, a point of Γ_1, and D, a point *not* of Γ_1, it must contain a boundary point of Γ_1. Hence in this case, g_C is the desired line.

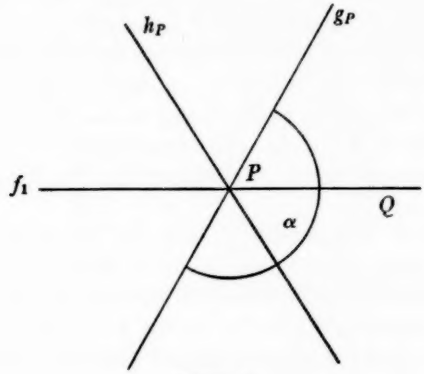

FIGURE 14

Let f_1 (Figure 14) be a line which contains a point of Γ_1 and a point of its boundary.* Then f_1 contains point P, on the boundary of Γ_1, and point Q, such that the segment PQ consists of points of Γ_1. Let α be a semicircular region, with P as center, with diameter on g_P, and containing a point of segment PQ. If *each* such region α contains a boundary point of Γ_1, we can find E, a boundary point of Γ_1, in a region α which is so small that g_E intersects f_P at a point of segment PQ, and f_E contains a point of Γ_1. Thus f_E and g_E each contains a point of Γ_1, which contradicts assertion (1) proved above.

Therefore, there exists a semicircular region α, which contains no boundary point of Γ_1. Since this region contains one point of Γ_1, it can contain only points of Γ_1. Hence h_P, which contains a point of *each* region α, contains a point of Γ_1. But this also is inconsistent with assertion (1), since f_P contains a point of Γ_1, and P is on the boundary of Γ_1.

Hence, the assumption that Γ and Γ_1 are distinct is false, and the truth of the theorem follows.

Now we shall prove the principal theorem.

THEOREM V. *Any topological transformation of region* Γ *which carries a 4-web of lines in* Γ *into a 4-web of lines is a projective collineation.*

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* There is no loss of generality in supposing that family f contains a line of this type.

We need merely show, in view of Theorem IV, that the transformation is a projective collineation in the neighborhood of one point of Γ.

First, we shall prove the theorem on the supposition that Γ contains a sub-region Γ_1, in which three families of the given web are regular.* Let P be any point of Γ_1. Choose rectangular coordinate axes in the given and image planes so that the lines of the web through P and their image lines have slopes. Then, in view of Lemma I, Γ_1 contains V, a circular neighborhood of P, such that the lines of the given and image webs have slopes, if they contain points of V or of V', its image.

By Lemma X, we can extend the transformation in V, and get

$$\frac{dy'}{dx'} = \frac{p_1 + p_2(dy/dx)}{p_3 + p_4(dy/dx)} \tag{1}$$

where the p's are the partial derivatives of x', y', the coordinates of the image of (x, y), and the Jacobian

$$J = \begin{vmatrix} p_1 & p_2 \\ p_3 & p_4 \end{vmatrix}$$

does not vanish in V. Let $m_i\,(x, y)$ be the slope functions of the three given *regular* families in V, and $m_i'\,(x', y')$ the slope functions in V' of their respective image families, which also are *regular* by Lemma VIII. These six slope functions are differentiable by Lemma XII. Moreover, m_i' is differentiable with respect to x and y, since it is differentiable with respect to x' and y' which are differentiable with respect to x and y.

Then

$$m_i'(x', y') = \frac{p_1 + p_2 m_i(x, y)}{p_3 + p_4 m_i(x, y)} \qquad (i = 1, 2, 3),$$

the denominator of which cannot vanish since $m_i' \neq \infty$ and J does not vanish. Thus, we have

$$p_1 + p_2 m_i - p_3 m_i' - p_4 m_i m_i' = 0 \qquad (i = 1, 2, 3), \tag{2}$$

which we shall solve for the p's.

The matrix of the system (2),

$$\begin{Vmatrix} 1 & m_1 & -m_1' & -m_1 m_1' \\ 1 & m_2 & -m_2' & -m_2 m_2' \\ 1 & m_3 & -m_3' & -m_3 m_3' \end{Vmatrix}, \tag{3}$$

* That is, three families of the web are regular at those of their respective lines which contain points of Γ_1.

is of rank three for each (x, y). For, if we assume the contrary, we have

$$\begin{vmatrix} 1 & m_1 & m_1' \\ 1 & m_2 & m_2' \\ 1 & m_3 & m_3' \end{vmatrix} = \begin{vmatrix} 1 & m_1 & m_1 m_1' \\ 1 & m_2 & m_2 m_2' \\ 1 & m_3 & m_3 m_3' \end{vmatrix} = 0,$$

which yields upon eliminating m_i'

$$(m_1 - m_2)(m_1 - m_3)(m_2' - m_3') = 0.$$

But this is impossible, since there are exactly four lines at each point of a 4-web. Thus (2) is a set of three linear homogeneous equations of rank three in four quantities p_j. Therefore, all solutions may be put in the form

$$\lambda L_j \qquad (j = 1, 2, 3, 4),$$

where λ is a parameter, and the L's, which are certain minor determinants in the matrix (3), are polynomials in the m's and m''s. For each (x, y) in V, the p's are uniquely determined, and are solutions of (2). Hence for each (x, y) in V, there is a $\lambda(x, y)$ such that

$$p_j = \lambda(x, y)L_j \qquad (j = 1, 2, 3, 4),$$

where $\lambda(x, y)$ cannot vanish, since J does not.

Thus, substituting for the p's in (1), we have

$$\frac{dy'}{dx'} = \frac{L_1 + L_2(dy/dx)}{L_3 + L_4(dy/dx)} \tag{4}$$

and

$$K = \begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} \neq 0$$

where the L's are *differentiable* functions of x and y. Therefore we may extend the transformation once more. We obtain

$$\frac{d^2y'}{dx'^2} = \frac{\alpha + \beta \dfrac{dy}{dx} + \gamma \left(\dfrac{dy}{dx}\right)^2 + \delta \left(\dfrac{dy}{dx}\right)^3 + K \dfrac{d^2y}{dx^2}}{\lambda \left(L_3 + L_4 \dfrac{dy}{dx}\right)^3}.* \tag{5}$$

Since four non-vertical lines through each point of V go into non-vertical lines, there are at each point of V four directions at which d^2y'/dx'^2 vanishes with d^2y/dx^2. Thus the equation in m

$$\alpha + \beta m + \gamma m^2 + \delta m^3 = 0$$

* We are following the method used by Kasner, loc. cit.

has four roots for each (x, y) and

$$\alpha = \beta = \gamma = \delta = 0.$$

Therefore, we have from (5)

$$\frac{d^2y'}{dx'^2} = \frac{K\dfrac{d^2y}{dx^2}}{\lambda\left(L_3 + L_4\dfrac{dy}{dx}\right)^3}.$$

Thus, if at a point of V, dy/dx has a value such that $L_3+L_4(dy/dx)$ is not zero, then for this value, d^2y'/dx'^2 vanishes with d^2y/dx^2.

Now consider l, any non-vertical line interval in V. If $L_3+L_4(dy/dx)$ vanishes identically on l, its image l' is a vertical interval. Suppose that $L_3+L_4(dy/dx)$ does not vanish at A, a point on l. Then, since L_3 and L_4 are continuous, there is an interval on l, containing A, at each point of which $L_3+L_4(dy/dx)$ is not zero. If $L_3+L_4(dy/dx)$ equals zero anywhere on l, there is, on l, a point B such that this expression vanishes at B, but not on the segment AB. Thus, on the image of segment AB, d^2y'/dx'^2 is zero and dy'/dx' is a finite constant. But dy'/dx' is infinite at B', the image of B, which is absurd since the L's in (4) are continuous. Thus it is impossible that $L_3+L_4(dy/dx)$ vanish on l. Hence d^2y'/dx'^2 is zero everywhere on l', and l' is a line interval.

We have thus shown that all non-vertical line intervals in V go into line intervals. This implies that vertical intervals, also, go into line intervals. Thus (4) defines a projectivity on each pencil of lines whose vertex is in V, and it easily follows that the transformation is projective on V. This completes the proof of the theorem under the original supposition.

Now let us suppose that in *no* sub-region of Γ are three families of the given web regular. Then there are at least two families of the web which are not regular in *any* neighborhood of a given point of Γ.

Let N be a point of Γ, and f and g two families of the web which are not regular in any neighborhood of N. Let h and k be the other two families of the web. Then in *each* neighborhood of N, there is a line of f and a line of g at which f and g respectively are not regular. Hence in a sufficiently small neighborhood of N, we can find a line f_1, at which f is not regular, such that f_1 intersects g_N. Similarly, we can find a line g_1 at which g is not regular, such that g_1 intersects f_1. Let f_1 and g_1 intersect at Q.

We can easily prove that the transformation is projective in the neighborhood of Q. Let C, D be two points on f_1 distinct from Q, and E, F two points

on g_1, also distinct from Q. Consider S, the projective transformation which carries C, D, E, F into their respective images under the given transformation. The given transformation and S coincide at C, D, E, F, Q. By Lemma XI, the transformation is a *projectivity* on f_1 and g_1. Hence the transformation and S are identical on f_1 and g_1, by the fundamental theorem of projective geometry. Let W be a neighborhood of Q, such that if X is in W, h_X and k_X intersect *both* f_1 and g_1. Then, if X is in W and not on h_Q or k_Q, the images of h_X under the given transformation and S are sub-sets of the *same* infinite line. And the same is true for k_X. Thus the transformations are identical for all *such* points X. Since they are continuous, they coincide for all points of W. Thus, the transformation is projective on W, which establishes the theorem.

DEFINITION. *A family of lines is called a pencil if its lines concur when extended.** *The point of concurrence is the vertex of the pencil.*

Now we may prove the following analogue of Theorem I, for a region of the euclidean plane.

THEOREM VI. *A one-to-one transformation of region* Γ *is projective, if it carries a line and a web of three independent pencils in* Γ *into a line and a web of three pencils, respectively.*

Let T be the transformation and π, π' the planes which contain the given and image webs respectively. Let f and g be two pencils of the given web. Let S_1 be a projective transformation on π which carries f and g into *parallel* pencils. Since the vanishing line of S_1, if it exists, contains no point of Γ, S_1 is a topological transformation over Γ. Thus S_1 carries Γ into a region Γ_1, and the given web into a web, in Γ_1, consisting of two parallel pencils and an ordinary pencil.

In the same way, we apply to π' a projective transformation S_2, which carries the images of f and g under T into parallel pencils. Thus R, the resultant of S_1^{-1}, T and S_2, is a one-to-one transformation of Γ_1, which carries a web composed of two parallel pencils f, g and an ordinary pencil h into a web composed of two parallel pencils p, q and a third pencil r. R also carries an additional line l into a line l'. We shall prove the theorem by showing that R is *projective* in the neighborhood of a point.

Set up a cartesian coordinate system in π, with the vertex of pencil h as origin, and with coordinate axes parallel to lines of pencils f, g respectively. Let A, any point on l, have coordinates $(1, 1)$. Similarly, establish a coordinate system in π' so that the point with coordinates $(1, 1)$ is D, the image of A under R, and the origin is an arbitrary point on the infinite line which

* We make the usual agreements about "*ideal points*" so that our results may apply to *parallel* pencils.

contains r_D. Choose the coordinate axes parallel to lines of pencils p, q respectively.

Let (x, y) represent any point of Γ_1, and (x', y'), its image. Then R carries line $y=x$ and pencils $x=t$, $y=t$ into $y'=x'$ and $x'=t'$, $y'=t'$ respectively. It follows that, in the neighborhood of $(1, 1)$, R can be represented by the equations

$$x' = \phi(x), \qquad y' = \phi(y), \tag{1}$$

where $\phi(1)=1$.

Let $y=ax+1-a$, $y'=bx'+1-b$ be the equations of l, l' respectively. Then we have

$$\phi(ax + 1 - a) = b\phi(x) + 1 - b \tag{2}$$

for all values of x sufficiently close to 1.

Now we shall show that r is an *ordinary* pencil. Suppose that it is not. Then since r contains the line $y'=x'$, it can be represented by $y'=x'+t$. Thus, since h is of the form $y=mx$, we have

$$\phi(mx) = \phi(x) + p(m) \tag{3}$$

for all x and m such that (x, mx) is in the neighborhood of $(1, 1)$. It follows that (3) holds for all x and m in the neighborhood of 1, i.e., for all x and m between $1-\epsilon$ and $1+\epsilon$, where ϵ is a real number.

In (3) we assign to x the value 1, and easily derive

$$\phi(mx) = \phi(m) + \phi(x) - 1. \tag{4}$$

Applying (4) to (2) we get

$$\begin{aligned}\phi[amx + (1 - a)m] &= \phi(m) + b\phi(x) - b \\ &= b[\phi(m) + \phi(x)] + (1 - b)\phi(m) - b.\end{aligned}$$

Thus

$$\phi[amx + (1 - a)m] = b\phi(mx) + (1 - b)\phi(m) \tag{5}$$

holds for all m and x in the neighborhood of 1. If in (5) we replace mx by x, and m by y, we may assert

$$\phi[ax + (1 - a)y] = b\phi(x) + (1 - b)\phi(y) \tag{6}$$

for all x and y in the neighborhood of 1.

If we substitute 1 for x in (6), then 1 for y in (6), and eliminate $b\phi(x)$ and $(1-b)\phi(y)$ between the resulting equations and (6), we get

$$\phi[ax + (1 - a)y] = \phi(ax + 1 - a) + \phi[a + (1 - a)y] - 1,$$

and hence

$$\phi[(ax + 1 - a) + \{a + (1 - a)y\} - 1]$$
$$= \phi(ax + 1 - a) + \phi[a + (1 - a)y] - 1.$$

Thus

$$\phi(u + v - 1) = \phi(u) + \phi(v) - 1 \tag{7}$$

holds for all u, v in the neighborhood of 1. Expressing (4) with u and v as variables, we have in view of (7)

$$\phi(uv) = \phi(u + v - 1). \tag{8}$$

Substituting $1/u$ for v in (8), we have

$$\phi[u + (1/u) - 1] = \phi(1)$$

for u close to 1. This contradicts the fact that R is a *one-to-one* transformation. Thus the assumption that r is *not* an ordinary pencil is false.

In the discussion thus far, the origin of coordinates in plane π' has not been uniquely determined. Now we choose the vertex of the pencil r as the origin in π'. Relations (1) and (2) hold as before, and in addition, since $y=mx$ goes into $y'=m'x'$, we have

$$\phi(mx) = \phi(m)\phi(x) \tag{9}$$

for m and x in the neighborhood of 1.

Applying (9) to (2), we have

$$\phi[amx + (1 - a)m] = b\phi(mx) + (1 - b)\phi(m) \tag{10}$$

for m and x near 1. But (10) is identical with (5). Thus (7), which follows directly from (5), holds for all u, v near 1.

For convenience, we write (9) as

$$\phi(uv) = \phi(u)\phi(v). \tag{11}$$

The result follows from the functional equations (7) and (11). We apply to these equations the substitution

$$u = 1 + U, \qquad v = 1 + V,$$

and change the function from ϕ to ψ where

$$\psi(Z) = \phi(1 + Z) - 1.$$

As a result, we have for U, V near zero

$$\psi(U + V) = \psi(U) + \psi(V) \tag{12}$$

and

$$\psi(UV + U + V) = \psi(U)\psi(V) + \psi(U) + \psi(V). \tag{13}$$

Applying (12) to (13), we have

$$\psi(UV) = \psi(U)\psi(V). \tag{14}$$

Relations (12) and (14) enable us to show that $\psi(x) = x$ in the neighborhood of zero. The method is essentially the same as that used in the corollary to Theorem I, where (12) and (14) hold with U and V as unrestricted real variables. Thus $\phi(x) = x$ in the neighborhood of $x = 1$, and transformation R is projective in the neighborhood of (1, 1), from which the desired result follows by Theorem IV.

The following theorem and corollary are concerned with the case of three *dependent* pencils.

THEOREM VII. *A one-to-one transformation of a region* Γ, *which is not the entire euclidean plane, is affine, if it carries three parallel pencils in* Γ *into three parallel pencils.*

Let f, g, h be the three pencils contained in Γ, and p, q, r their respective images under the given transformation T. Let Γ' be the image of Γ, and π, π' the planes which contain Γ, Γ', respectively.

Since Γ is a *proper* sub-set of the euclidean plane, one of the pencils f, g, h contains a line which is not an infinite line, i.e., it contains a *half line* or a *finite interval*. Let us suppose that f_A is such a line. Let D be the image of A under T.

We shall show that the inverse transformation T^{-1} is projective in the neighborhood of D. We establish cartesian coordinate systems in π and π', locating the respective origins at A and D, and the points (1, 1) on h_A, r_D respectively, and choosing the coordinate axes from the pencils f, g, p, q respectively.

It follows, by the method used in the preceding theorem, that T^{-1} may be represented in the neighborhood of D by

$$x = \phi(u), \qquad y = \phi(v), \tag{1}$$

where

$$\phi(u + v) = \phi(u) + \phi(v) \tag{2}$$

for all u and v near zero. Moreover, the fact that T^{-1} carries p_D into a half line or a finite interval implies that $\phi(u)$ has an *upper* or *lower bound*. From

this we can show that $\phi(u) = cu$ in the neighborhood of zero, where c is a constant.*

Hence T^{-1} is projective, by Theorem IV. It obviously is affine since (1) defines an affinity in the neighborhood of (0, 0). Thus T is affine also.

COROLLARY. *A one-to-one transformation of a region* Γ, *which is not the entire euclidean plane, is projective, if it carries a web of three dependent pencils in* Γ *into a similar web.*

By the application of projective transformations to the given and image figures, we can reduce the given transformation to one which carries three parallel pencils into three parallel pencils. Thus the resultant transformation is *affine*, and the original is *projective*.

It is evident that the condition in the above theorem and corollary, that Γ be a proper sub-set of the euclidean plane, may be dispensed with, if we assume that the transformation is topological.

Finally, we prove the following theorem.

THEOREM VIII. *A one-to-one transformation of a region is projective, if it carries a 3-web of lines composed of two pencils and a family not a pencil, into a similar 3-web, families of the same type corresponding.*†

By the application of projective transformations as in Theorem VI, the given transformation can be reduced to a transformation T, which carries a web of lines composed of two parallel pencils f, g and a family h, not a pencil, into a similar web. It suffices to show that T is projective in the neighborhood of a point.

Let Γ be the region containing f, g, h and let A be any point of Γ. If there exists a positive number ϵ such that all the lines of h whose distance to A is less than ϵ concur at B, when extended, we shall say that *h is a pencil with vertex B in the neighborhood of A.*

We shall show that there exists a point M in Γ, such that h is *not* a pencil in the neighborhood of M. Suppose that this is not so. Then h is a pencil in the neighborhood of each point of Γ, and we shall prove that h is a pencil. Let C be a fixed point of Γ, and X, a variable point of Γ, distinct from C. Let h be a pencil with vertex D, in the neighborhood of C. We shall show that h_X, when extended, contains D.

* This can be proved, essentially, by the method which Darboux (loc. cit., pp. 56, 57) employs to derive the analogous result when (2) holds for all real u, v.

† Dubourdieu, Abhandlungen Hamburg Seminar, vol. 7 (1929), p. 219, derives this result on the assumption that the transformation and the families are *differentiable*. Kasner, in his studies of near-collineations, also has derived a related result. See Bulletin of the American Mathematical Society, vol. 36 (1930), p. 796, abstract No. 388.

Let l be a broken line joining C and X in Γ. Separate the points of l into two sets, the first of which contains all points P such that for *each* Q on l between C and P, h_Q, when extended, passes through D. Let the second set contain all other points of l. If the second set is not empty, there is a point E on l such that we can find two lines of h as close to E as we please, one and only one of which passes through D, when extended. But this is impossible since h is a pencil in the neighborhood of E. Thus the second set is empty, and h_X, when extended, passes through D. Therefore, h is a pencil since X is an *arbitrary* point of Γ. This absurdity implies the existence of a point M, such that h is not a pencil in the neighborhood of M.

Now establish a cartesian coordinate system in the given plane with M as the origin, N any other point on h_M as $(1, 1)$, and f_M, g_M as the coordinate axes. In the image plane introduce coordinates similarly, with the images of M, N as $(0, 0)$, $(1, 1)$, respectively, and the images of f_M, g_M as coordinate axes. As in previous theorems, T may be represented, in the neighborhood of M, by

$$x' = \phi(x), \qquad y' = \phi(y).$$

In view of Lemma I, the lines of h which are sufficiently close to M are represented by

$$y = ax + b$$

where $\delta_1 < b < \delta_2$, $\delta_1 < 0 < \delta_2$, $a = a(b)$ and $a(0) = 1$. It follows from Lemma III that $a(b)$ is a *continuous* function.

Since each line of h goes into a non-vertical straight line,

$$\phi(ax + b) = c\phi(x) + d \tag{1}$$

holds for all x and b near zero, where $c = c(b)$ and $d = d(b)$. If b_1 is an arbitrary value of b, we may similarly assert

$$\phi(a_1x + b_1) = c_1\phi(x) + d_1, \tag{2}$$

where $a_1 = a(b_1)$, $c_1 = c(b_1)$, $d_1 = d(b_1)$. Substituting $a_1x + b$ for x in (1) and $ax + b$ for x in (2), we have, for x, b, b_1 sufficiently close to zero,

$$\phi(aa_1x + ab_1 + b) = cc_1\phi(x) + cd_1 + d \tag{3}$$

and

$$\phi(aa_1x + a_1b + b_1) = cc_1\phi(x) + c_1d + d_1. \tag{4}$$

Eliminating $\phi(x)$ between (3) and (4), we have

$$\phi(aa_1x + ab_1 + b) = \phi(aa_1x + a_1b + b_1) + cd_1 + d - c_1d - d_1. \tag{5}$$

Substituting $(u-a_1b-b_1)/(aa_1)$ for x in (5), we have

$$\phi(u + ab_1 - a_1b + b - b_1) = \phi(u) + cd_1 + d - c_1d - d_1, \tag{6}$$

for all u, b, b_1 sufficiently close to zero. Letting u be zero in (6), we easily derive

$$\phi[u + p(b, b_1)] = \phi(u) + \phi[p(b, b_1)], \tag{7}$$

where

$$p(b, b_1) = ab_1 - a_1b + b - b_1.$$

The function p is continuous and $p(0, 0) = 0$. Thus in the neighborhood of (0, 0) either p is identically zero, or p assumes all values in an interval which terminates at zero. The former is impossible since it implies that

$$p(b, b_1) \equiv \begin{vmatrix} 1 & 1 & 0 \\ 1 & a & b \\ 1 & a_1 & b_1 \end{vmatrix} = 0 \tag{8}$$

for all b, b_1 near zero. And the vanishing of the determinant in (8) is a sufficient condition that the lines $y=x$, $y=ax+b$, $y=a_1x+b_1$ concur. Thus family h is a pencil in the neighborhood of M, which is absurd. Therefore $p(b, b_1)$ is not identically zero in any neighborhood of (0, 0), and assumes all values on an interval terminating at zero, in such a neighborhood.

Hence, if we substitute v for $p(b, b_1)$ in (7) we may assert that

$$\phi(u + v) = \phi(u) + \phi(v) \tag{9}$$

holds for all u, v on the respective intervals $(-\delta, \delta)$, $(0, \delta)$, where δ is a real number sufficiently close to zero. Letting $v = -u$ in (9) we have

$$\phi(-u) = -\phi(u)$$

for u on the interval $(0, -\delta)$. Then if u and v are on $(-\delta/2, \delta/2)$ and $(0, -\delta/2)$ respectively, we have

$$\phi[u + v + (-v)] = \phi(u + v) - \phi(v),$$

and

$$\phi(u + v) = \phi(u) + \phi(v).$$

Thus (9) holds for *all* u and v sufficiently close to zero.

From (1) we easily derive

$$\phi[a(b)x + b] = c(b)\phi(x) + \phi(b). \tag{10}$$

From (9) and (10) we get

$$\phi[a(b)x] = c(b)\phi(x). \tag{11}$$

The function $a(b)$ is continuous, is not constant in the neighborhood of zero, and $a(0)=1$. Thus, if we substitute t for $a(b)$ in (11), we may assert that

$$\phi(tx) = \psi(t)\phi(x) \tag{12}$$

holds for all x and t on the respective intervals $(-\eta, \eta)$ and $(1, 1+\eta)$. Substituting $1+z$ for t in (12), we have

$$\phi(x + xz) = \phi(x)\psi(1 + z) \tag{13}$$

for x and z on $(-\eta, \eta)$ and $(0, \eta)$ respectively. Substituting z for x in (13) and eliminating $\psi(1+z)$, we have

$$\phi(x + xz) = \frac{\phi(x)\phi(z + z^2)}{\phi^2(z)},$$

and hence, by (9), we have

$$\phi(xz) = \frac{\phi(x)\phi(z^2)}{\phi(z)}. \tag{14}$$

Interchanging x and z in (14) and eliminating $\phi(xz)$, we have

$$\phi(x^2) = \frac{\phi^2(x)\phi(z^2)}{\phi^2(z)}$$

and

$$\phi(x^2) = k\phi^2(x) \tag{15}$$

for x on $(0, \eta)$.

By an argument similar to that of the corollary to Theorem I, we can show from relations (9) and (15) that $\phi(x)=x/k$ in the neighborhood of zero. Thus T is projective in the neighborhood of $(0, 0)$ and the proof is complete.

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CORRECTIONS TO THE PAPER "INTEGRATION IN GENERAL ANALYSIS"*

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There are two tacit assumptions in this paper which need clarification. The author is indebted to Professor M. H. Stone for the observation of the first of these.

In the definition of the integral $\int_E f(P)d\alpha$ of a summable function $f(P)$ it is necessary to show that any two Cauchy sequences $\{f_n\}$ and $\{g_n\}$ of functions in $S_0(E)$ which define $f(P)$ have the property that

$$\lim_n \int_E f_n(P)d\alpha = \lim_n \int_E g_n(P)d\alpha.$$

This follows from Lemma 5, for we may assume that $f_n(P)-g_n(P)\to 0$ almost uniformly with respect to α on E so that

$$\lim_n \int_{E-} (f_n(P) - g_n(P))d\alpha = \lim_n \int_e (f_n(P) - g_n(P))d\alpha,$$

where $\beta(e)$ is arbitrarily small. Since the limit on the right of this equation exists uniformly with respect to e in $A(E)$, it follows from Lemma 5 that

$$\lim_{\beta(e)=0} \lim_n \int_e (f_n(P) - g_n(P))d\alpha = 0.$$

This gives the desired result. It might be pointed out that Theorem 4 shows that $\|f_n-g_n\|\to 0$.

In §4 it is tacitly assumed that the measurable set E can be partitioned into measurable sets E_η. This is always the case in separable spaces. To proceed without this assumption it will not be necessary to assume that J is metric. The class $S_0(E)$ is defined as the class of functions finitely valued on E. Such a function is one for which there is a decomposition of E into a finite number of disjoint measurable subsets on each of which it is constant. This basis necessitates only a slight rewording in a few places. In Lemma 1 the set E should be taken as a set in A. In Theorem 2 the words "functions uniformly continuous" should be replaced by "functions finitely valued." In the proof of Theorem 11 the sentence "Fix . . . continuous on e" should be worded "Fix e with $\beta(E-e)<\delta$ and so that for some f_0 in $S_0(E)$,

* These Transactions, vol. 37 (1935), pp. 441–453. These corrections were received by the editors August 24, 1935.

$$\|f_0(P) - f(P)\| \leqq \epsilon/(3 \sup_n \beta_n(E))$$

for P in e." Theorem 10 takes on a trivial form. Without a metric in J all reference to continuity is meaningless and consequently Theorem 3 drops out. All other theorems remain as stated. If J is metric and every set E in A contains a closed set e for which $\beta(E-e)<\epsilon$ then Theorem 3 holds.

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